A SYSTEMATIC APPROACH FOR SOLVING THE GREAT CIRCLE TRACK PROBLEMS BASED ON VECTOR ALGEBRA

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ABSTRACT

A systematic approach, based on multiple products of the vector algebra (S-VA), is proposed to derive the spherical triangle formulae for solving the great circle track (GCT) problems. Because the mathematical properties of the geometry and algebra are both embedded in the S-VA approach, derivations of the spherical triangle formulae become more understandable and more straightforward as compared with those approaches which use the complex linear combination of a vector basis. In addition, the S-VA approach can handle all given initial conditions for solving the GCT problems simpler, clearer and avoid redundant formulae existing in the conventional approaches. With the technique of transforming the Earth coordinates system of latitudes and longitudes into the Cartesian one and adopting the relative longitude concept, the concise governing equations of the S-VA approach can be easily and directly derived. Owing to the advantage of the S-VA approach, it makes the practical navigator quickly adjust to solve the GCT problems. Based on the S-VA approach, a program namely GCTPro_VA is developed for friendly use of the navigator. Several validation examples are provided to show the S-VA approach is simple and versatile to solve the GCT problems.

Keywords: great circle; vector algebra; multiple products; spherical trigonometry

INTRODUCTION

The great circle arc is the shortest distance between two non-antipodal points on the Earth's surface. Therefore, without considering the wind and tide effects, the most economical route is the great circle track (GCT). As known, a vessel sailing on the GCT in a constantly varying course is impossible and thus, a series of rhumb lines is necessary to be adopted for approximating the GCT in practice [2, 3, 9, 18]. Before doing so, the waypoints must be located. However, to determine the waypoints on the GCT, the navigator has to provide some initial conditions. Theoretically, these given initial conditions include: (1) giving the great circle distances to obtain the latitudes and longitudes of the waypoints (hereafter is called "condition 1"); (2) giving the longitudes of the waypoints (hereafter is called "condition 1"); (2) giving the

longitudes of the waypoints to yield the latitudes (hereafter is called "condition 2"); (3) giving latitudes of the waypoints to reach the longitudes (hereafter is called "condition 3"). Then, locating these obtained waypoints on the Mercator chart or ECDIS (Electronic Chart Display and Information System) for every rhumb line sailing is called the great circle sailings in practical navigation [2, 3, 9, 18]. For dealing with GCT problems under all initial conditions, a systematic approach to tackle the problems is necessary and becomes one of our research motivations.

Spherical trigonometry (ST) has been used for solving every navigational triangle. Since the Napier's rules was proposed by John Napier (1550-1617), the navigator has adopted the rules in conjunction with the spherical triangle formulae to solve

the GCT problems. While using the Napier's rules of the right-angled spherical triangles, one needs to take the vertex as the reference point (RP-V). This method has lasted for several centuries and is adopted in the current international maritime education [2, 3, 9, 12, 14, 18]. In addition, because finding the equator crossing point of the GCT is easier than finding the vertex, a method of taking the equator crossing point as the reference point (RP-E) was proposed [4, 7]. Both the RP-V and RP-E have to decide the reference point in advance when solving the GCT problems; they can be categorized as a kind of indirect methods. In contrast to the indirect methods, some researchers took the departure point as the reference point (RP-F) because such a method can directly obtain the waypoint on the GCT. Due to this advantage, Jofeh [13] used a linear equation of the GCT to represent a straight line on the polar gnomonic chart. According to his method, the waypoints are determined only under given condition 2. Thereafter, the linear combination of a vector basis (VB) to derive the used formula was also proposed [5, 15, 16]. Although the VB was claimed a method of the operation of the vector, it is actually an essence of geometric analysis, which make the derivation of this method hard to understand and results in redundant formulae. In addition, the VB cannot determine the waypoints under all given initial condition. Apart from the VB, a method using multiple products of the vector algebra (VA) appeared lately [4, 6, 7]. A disadvantage of this method is that all given initial conditions cannot be covered when solving the waypoints on the GCT. Therefore, this paper systematically adopts multiple products of the vector algebra (S-VA) to derive the spherical triangle formulae for solving the GCT problems. Advantages of the S-VA approach are that it can handle all given initial conditions more straightforward and avoid redundant formulae existing in the conventional approaches. A comparison of those mentioned methods are listed in Table 1 for convenience.

As compared with the geometric analysis, the dot or cross products of vector algebra are more convenient to describe the relationships between the sides and the angles of the spherical triangle [11, 19]. Because the S-VA approach transforms the Earth coordinates system of the latitude and longitude into the Cartesian one and adopts the relative longitude concept, the derivation of the spherical triangle formulae becomes simple and clear. In addition, the mathematical properties of the geometry and algebra are both embedded in the S-VA approach; therefore, they are adopted for constructing the theoretical background of the current research. Based on the S-VA approach, a program namely GCTPro_VA is developed for the use of the navigator. The GCTPro_VA with a friendly interface offers a quick and practical tool for the choice of three given initial conditions.

Apart from the current section, the theoretical backgrounds are presented in Section 2. Computation procedures including of all the given initial conditions and its numerical program are illustrated in Section 3. Several validated examples and discussions are given in Section 4. Finally, Section 5 draws concrete conclusions after recasting this research work.

THEORETICAL BACKGROUNDS

In the spherical geometry, the spherical triangle is the most concise and simplest form for all the geometrical figures. The fundamental structure of the spherical triangle consists in the relationships between the sides (great circle arcs) and angles (dihedral angles). Among the spherical triangle formulae, the two most important laws are the sine law and side cosine law, which are called the "genetic codes" for developing all the spherical triangle formulae [8].

There are two ways to derive the spherical triangle formulae. One is based on the geometry analysis and the other depends on the vector algebra. Theoretically speaking, the most straightforward and understandable operations are using systematic approach with multiple products of the vector algebra (S-VA) [8, 11, 19]. Since the S-VA implies both the mathematical properties of the geometry and algebra, all the spherical triangle formulae can be easily and directly derived. In this regard, three stages are necessary to derive the used spherical triangle formulae and governing equations with respect to the GCT problems. First, the Earth coordinates system of the latitude and longitude is transformed into the Cartesian coordinates system to yield the vector representation. Then, the S-VA is introduced to derive the used spherical triangle formulae. Finally, aimed at the GCT problems, the concise governing equations are yielded by simplifying the related spherical triangle formulae. All symbols used in this article are listed in NOMENCLATURE.

Tab. 1. *A comparison of different methods for solving the GCT problem*

Authors	Category	Formulae derivation	Available condition
Holm, 1972	RP-V	ST	2
Jofeh, 1981	RP-F	Linear eqn.	2
Bowditch, 1981; 2002	RP-V	ST	all
Keys, 1983	RP-V	ST	all
Miller et al., 1991	RP-F	VB	1
Chen, 2003	RP-V, RP-E	ST	all
	RP-F	VA (GCEM)	2 and 3
Cutler, 2004	RP-V	ST	all
Chen et al., 2004	RP-V	ST	all
	RP-F	VA (GCEM)	2 and 3
Nastro and Tancredi, 2010	RP-F	VB	1
Chen et al., 2014	RP-E	ST	all
	RP-F	VA (COFI)	1
Chen et al., 2015	RP-F	VB (RT)	1 and 2
Chen, 2016 (the current paper)	RP-F	S-VA	all

COORDINATES TRANSFORMATION

First, the Earth is treated as a unitary sphere for practical navigation. Then, the Earth coordinates system of the latitude and longitude is transformed into the Cartesian coordinates system. As shown in Fig. 1, the vector of any point P on the Earth's surface can be represented by the latitude and the
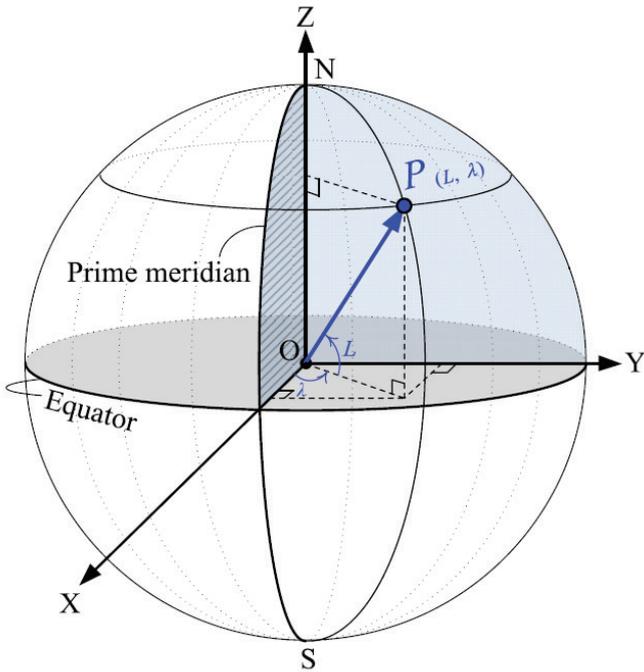
Fig. 1. *An illustration of transforming the Earth coordinates system into the Cartesian one*

longitude in the Cartesian coordinates system as:

$$\bar{P} = [\cos L \cos\lambda, \ \cos L \sin\lambda, \ \sin L], L = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \lambda = [0, 2\pi).$$

(1)

To avoid an additional judgment of sign convention, the concept of the fixed coordinates system is first considered, that is, the north latitude is treated as a positive value and the south latitude is treated as a negative one. To reduce the number of the variables, the relative longitude concept that replaces the Greenwich meridian by the meridian of the departure point is introduced. As shown in Fig. 2, the unit vectors of the North Pole (N), the departure (F), the destination (T) and the waypoints (X) along the GCT located on the Earth's surface can be expressed as:

$$\overline{N} = [0, \ 0, \ 1],$$

(2a)

$$\overrightarrow{F} = [\cos L_F, \ 0, \ \sin L_F],$$

(2b)

$$\overrightarrow{T} = [\cos L_T \cos DLo, \ \cos L_T \sin DLo, \ \sin L_T],$$

(2c)

$$\vec{X} = [\cos L_X \cos DLo_{FX}, \ \cos L_X \sin DLo_{FX}, \ \sin L_X].$$

(2d)
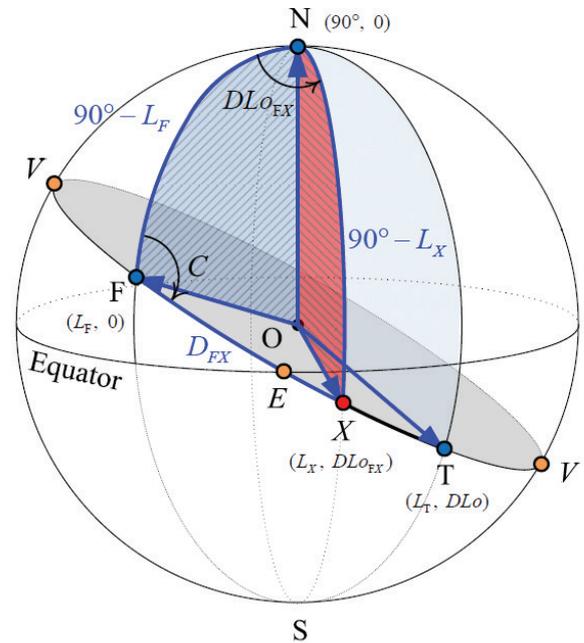
Fig. 2. An illustration of the unit vectors of four points located on the Earth's surface

MULTIPLE PRODUCTS OF VECTOR ALGEBRA

1. To obtain a great circle arc (side of spherical triangle), we can adopt the dot and cross product of two vectors. From the viewpoint of geometry, the dot product of two vectors is defined as:

$$\overrightarrow{N} \bullet \overrightarrow{F} = \overrightarrow{F} \bullet \overrightarrow{N} = \sin L_F, \ \overrightarrow{N} \bullet \overrightarrow{T} = \sin L_T, \ \overrightarrow{F} \bullet \overrightarrow{T} = \cos D,$$
$$\overrightarrow{N} \bullet \vec{X} = \sin L_X, \ \overrightarrow{F} \bullet \vec{X} = \cos D_{FX}.$$

(3)

Similarly, the cross product of two vectors is defined as:

$$\left|\overrightarrow{N} \times \overrightarrow{F}\right| = \left|\overrightarrow{F} \times \overrightarrow{N}\right| = \cos L_F, \ \left|\overrightarrow{N} \times \overrightarrow{T}\right| = \cos L_T,$$
$$\left|\overrightarrow{F} \times \overrightarrow{T}\right| = \sin D,$$

(4)

$$\left|\overrightarrow{N} \times \vec{X}\right| = \cos L_X, \ \left|\overrightarrow{F} \times \vec{X}\right| = \sin D_{FX}.$$

2. To construct a great circle equation (GCE), we can use the scalar triple products.

A necessary and sufficient condition for the vectors $\overrightarrow{F}$, $\overrightarrow{T}$, and $\vec{X}$ to be coplanar, the volume of the parallelepiped form by three vectors must be zero, that is,

$$\left(\overrightarrow{F} \times \overrightarrow{T}\right) \bullet \vec{X} \equiv \left(\overrightarrow{F}\ \overrightarrow{T}\ \vec{X}\right) = 0.$$

(5)

Let the coefficient vector be:

$$\left(\vec{\mathbf{F}} \times \vec{\mathbf{T}}\right)=[k_1, \ k_2, \ k_3],$$

in which the components of the coefficient vector are as follows:

$$k_1 = -\sin L_F \cos L_T \sin DLo, \qquad \textbf{(6a)}$$

$$k_2 = \sin L_F \cos L_T \cos DLo - \cos L_F \sin L_T, \qquad \textbf{(6b)}$$

$$k_3 = \cos L_F \cos L_T \sin DLo. \qquad \textbf{(6c)}$$

Substituting equations (6a), (6b) and (6c) into equation (5) yields the great circle equation (GCE) as [4, 6, 7]:

$$k_1 \cos L_X \cos DLo_{FX} + k_2 \cos L_X \sin DLo_{FX} + k_3 \sin L_X = 0. \qquad \textbf{(7)}$$

Equation (7) has two variables, that is L_x, and DLo_{FX}; therefore, we can rearrange equation (7) to obtain two forms in the following. One is

$$\tan L_X = \frac{k_1 \cos DLo_{FX} + k_2 \sin DLo_{FX}}{-k_3}, \qquad \textbf{(8)}$$

which can be used to yield latitudes of the waypoints by giving their longitudes.

When we assume

$$\sin \alpha = \frac{k_1}{\sqrt{k_1{}^2 + k_2{}^2}}, \ \cos \alpha = \frac{k_2}{\sqrt{k_1{}^2 + k_2{}^2}},$$

and substituting them into equation (7), the other form can be yielded as:

$$\tan \alpha = \frac{k_1}{k_2}, \qquad \textbf{(9a)}$$

$$\sin(DLo_{FX} + \alpha) = \frac{-k_3}{\sqrt{k_1{}^2 + k_2{}^2}} \tan L_X. \qquad \textbf{(9b)}$$

The above two equations are a set governing equations. They can be used to obtain longitudes of the waypoints by giving their latitudes.

3. To obtain a dihedral angle (angle of spherical triangle), we can adopt the dot and cross products of two normal vectors of two planes. Therefore the sine law, the side cosine law and the four-part formula can be yielded in the following:

1. Derive the sine law of the spherical triangle by using the cross product of two normal vectors of two planes
To obtain the dihedral angle (C), we have

$$\left(\vec{\mathbf{F}} \times \vec{\mathbf{N}}\right) \times \left(\vec{\mathbf{F}} \times \vec{\mathbf{X}}\right) = \left(\cos L_F \sin D_{FX} \sin C\right) \vec{\mathbf{F}}$$
$$\left(\vec{\mathbf{F}} \vec{\mathbf{N}} \vec{\mathbf{X}}\right)\vec{\mathbf{F}} \qquad \textbf{(10a)}$$

in which the first row of the above equation is the definition of geometry and the second row is the operation result of vector algebra [11, 19]. Then, from equation (10a), we have

$$\cos L_F \sin D_{FX} \sin C = \left(\overrightarrow{\mathbf{FNX}}\right). \qquad \textbf{(10b)}$$

Similarly, to obtain the dihedral angle (DLo_{FX}), we have

$$\left(\vec{\mathbf{N}} \times \vec{\mathbf{X}}\right) \times \left(\vec{\mathbf{N}} \times \vec{\mathbf{F}}\right) = \left(\cos L_X \cos L_F \sin DLo_{FX}\right)\vec{\mathbf{N}}$$
$$= \left(\vec{\mathbf{N}} \vec{\mathbf{X}} \vec{\mathbf{F}}\right)\vec{\mathbf{N}}, \qquad \textbf{(11a)}$$

in which the first row of the above equation is the definition of geometry and the second row is the operation result of vector algebra [11, 19]. Then, we have:

$$\cos L_X \cos L_F \sin DLo_{FX} = \left(\overrightarrow{\mathbf{NXF}}\right). \qquad \textbf{(11b)}$$

Since the right-hand sides of equations of (10b) and (11b) are equal, we achieve:

$$\sin D_{FX} \sin C = \cos L_X \sin DLo_{FX}, \qquad \textbf{(12)}$$

which is the sine law of the spherical triangle.

2. Derive the side cosine law of the spherical triangle by using the dot product of two normal vectors of two planes
To obtain the dihedral angle (C), we have:

$$\left(\vec{\mathbf{F}} \times \vec{\mathbf{N}}\right) \bullet \left(\vec{\mathbf{F}} \times \vec{\mathbf{X}}\right) = \cos L_F \sin D_{FX} \cos C$$

$$= \begin{vmatrix} \left(\vec{\mathbf{F}} \bullet \vec{\mathbf{F}}\right) & \left(\vec{\mathbf{F}} \bullet \vec{\mathbf{X}}\right) \\ \left(\vec{\mathbf{N}} \bullet \vec{\mathbf{F}}\right) & \left(\vec{\mathbf{N}} \bullet \vec{\mathbf{X}}\right) \end{vmatrix} = \sin L_X - \sin L_F \cos D_{FX},$$

$$\textbf{(13a)}$$

in which the first row of the above equation is the definition of geometry and the second row is the operation result of vector algebra [11, 19]. Then, from equation (13a), we have:

$$\cos L_F \sin D_{FX} \cos C = \sin L_X - \sin L_F \cos D_{FX}. \qquad \textbf{(13b)}$$

Similarly, to obtain the dihedral angle (DLo_{FX}), we have:

$$\left(\vec{\mathbf{N}} \times \vec{\mathbf{X}}\right) \bullet \left(\vec{\mathbf{N}} \times \vec{\mathbf{F}}\right) = \cos L_X \cos L_F \cos DLo_{FX}$$

$$= \begin{vmatrix} \left(\vec{\mathbf{N}} \bullet \vec{\mathbf{N}}\right) & \left(\vec{\mathbf{N}} \bullet \vec{\mathbf{F}}\right) \\ \left(\vec{\mathbf{X}} \bullet \vec{\mathbf{N}}\right) & \left(\vec{\mathbf{X}} \bullet \vec{\mathbf{F}}\right) \end{vmatrix} = \cos D_{FX} - \sin L_F \sin L_X,$$

$$\textbf{(14a)}$$

in which the first row of the above equation is the definition of geometry and the second row is the operation result of vector algebra [11, 19]. Then, we have:

$$\cos L_X \cos L_F \cos DLo_{FX} = \cos D_{FX} - \sin L_F \sin L_X. \tag{14b}$$

It should be noticed that both equation (13b) and equation (14b) are the side cosine laws.

3. Derive the four-part formula of the spherical triangle by using a combination of the sine and side cosine laws

Dividing equations (10a) by equation (13a) yields:

$$\frac{(\vec{F} \times \vec{N}) \times (\vec{F} \times \vec{X})}{(\vec{F} \times \vec{N}) \bullet (\vec{F} \times \vec{X})} = \tan C$$

$$= \frac{\cos L_F \sin D_{FX} \sin C}{\sin L_X - \sin L_F \cos D_{FX}}. \tag{15a}$$

From the above equation, we have:

$$\tan C = \frac{\cos L_F \sin D_{FX} \sin C}{\sin L_X - \sin L_F \cos D_{FX}}. \tag{15b}$$

To yield the four-part formula, we have to substitute equations (12) and (14b) into equation (15b) for eliminating D_{FX} Then, we have the four-part formula as:

$$\cos L_F \tan C \tan L_X = \sin DLo_{FX} + \sin L_F \tan C \cos DLo_{FX}. \tag{16}$$

DERIVATIONS OF THE GOVERNING EQUATIONS

1. The great circle distance
There are two ways to yield the great circle distance. One is:

$$\vec{F} \bullet \vec{T} = \cos D$$

$$= \cos L_F \cos L_T \cos DLo + \sin L_F \sin L_T. \tag{17a}$$

The other is to replace $\vec{X}$ by $\vec{T}$ in the equation (14a). Then, we have:

$$(\vec{N} \times \vec{T}) \bullet (\vec{N} \times \vec{F}) = \cos L_T \cos L_F \cos DLo$$

$$= \begin{vmatrix} (\vec{N} \bullet \vec{N}) & (\vec{N} \bullet \vec{F}) \\ (\vec{T} \bullet \vec{N}) & (\vec{T} \bullet \vec{F}) \end{vmatrix} = \cos D - \sin L_F \sin L_T. \tag{17b}$$

It is found that equations (17a) and (17b) are the same. Rewriting the equation (17a) yields:

$$\cos D = \sin L_F \sin L_T + \cos L_F \cos L_T \cos DLo. \tag{18}$$

It should be noticed that equation (18) is the governing equation to obtain the great circle distance.

2. The great circle initial course angle
There are also two ways to yield the great circle initial course angle. One is to rearrange equation (12), which is the sine law of the spherical triangle. Then, we have:

$$\sin C = \frac{\cos L_T \sin DLo}{\sin D}. \tag{19}$$

The other is to replace $\vec{X}$ by $\vec{T}$ in equation (13a), which is the side cosine law of the spherical triangle. Then, we have:

$$(\vec{F} \times \vec{N}) \bullet (\vec{F} \times \vec{T}) = \cos L_F \sin D \cos C$$

$$= \begin{vmatrix} (\vec{F} \bullet \vec{F}) & (\vec{F} \bullet \vec{T}) \\ (\vec{N} \bullet \vec{F}) & (\vec{N} \bullet \vec{T}) \end{vmatrix} = \sin L_T - \sin L_F \cos D. \tag{20a}$$

Either equation (19) or equation (20a) can be used to yield the great circle initial course angle; however, for avoiding an additional sign judgment, we suggest the navigator use equation (20a). Rearranging equation (20a) obtains:

$$\cos C = \frac{\sin L_T - \sin L_F \cos D}{\cos L_F \sin D}. \tag{20b}$$

It should be noticed that equation (20b) is the governing equation to obtain the great circle initial course angle.

3. The waypoints on the GCT
Condition 1: Giving the great circle distances to obtain the latitudes and the longitudes of the waypoints on the GCT
Rearranging equations (13b) and (14b), respectively yields:

$$\sin L_X = \sin L_F \cos D_{FX} + \cos L_F \sin D_{FX} \cos C, \tag{21}$$

and

$$\cos DLo_{FX} = \frac{\cos D_{FX} - \sin L_F \sin L_X}{\cos L_F \cos L_X}. \tag{22}$$

It should be noticed that equations (21) and (22) are a set of the governing equations to obtain the latitudes and the longitudes of the waypoints on the GCT under condition 1.

Condition 2: Giving the longitudes of the waypoints to obtain their latitudes
There are two ways to obtain the latitudes of the waypoints by giving the longitudes. Either rearranging equation (16) or substituting equations (18), (19), and (20b) into equation (8) to eliminate DLo yields the same result, which can be expressed as:

$$\tan L_X = \frac{\cos C \sin DLo_{FX} + \sin L_F \sin C \cos DLo_{FX}}{\cos L_F \sin C}. \tag{23a}$$

It is noticed, that equation (23a) is the governing equation to yield latitudes of the waypoints on the GCT under condition 2. However, this equation needs to find the great circle initial course angle (C) in advance. To avoid this unnecessary step, we first rearrange equation (16) to obtain:

$$\tan L_X \cos L_F = \cot C \sin DLo_{FX} + \sin L_F \cos DLo_{FX}. \quad \textbf{(23b)}$$

Then replacing X by T in equation (23b) yields:

$$\tan L_T \cos L_F = \cot C \sin DLo + \sin L_F \cos DLo. \quad \textbf{(23c)}$$

Finally, substituting equation (23c) into equation (23b) can eliminate C and introducing additional formula of trigonometric functions to obtain:

$$\tan L_X = \frac{\tan L_F \sin(\lambda_X - \lambda_T) + \sin L_T \sin(\lambda_X - \lambda_F)}{\sin(\lambda_T - \lambda_F)}. \quad \textbf{(23d)}$$

It is noticed, that equation (23d) is the most concise form of the governing equation (or so called meridian method) to yield latitudes of the waypoints on the GCT under condition 2 [18].

Condition 3: Giving the latitudes of the waypoints to obtain their longitudes

There are also two ways to obtain the longitudes of the waypoints by giving the latitudes. One way is to use a set of governing equations, equations (9a) and (9b), which are derived from the GCE. The other one is yielded from the four-part formula of spherical triangle, that is, equation (16) and derived as follows. First we set

$$\tan\alpha = \sin L_F \tan C, \quad \textbf{(24a)}$$

and then, introduce the additional formula of trigonometric functions into equation (16). Therefore, we have:

$$\sin(DLo_{FX} + \alpha) = (\cos\alpha \cos L_F \tan C)\tan L_X. \quad \textbf{(24b)}$$

It should be noticed, that equations (24a) and (24b) are a set of the most concise form of the governing equations to yield longitudes of the waypoints on the GCT under condition 3.

4. The equator crossing point

The latitude of the equator crossing point must be zero. Substituting it into equation (23) and rearranging it can yield the equation:

$$\tan DLo_{FE} = -\sin L_F \tan C. \quad \textbf{(25)}$$

It should be noticed, that equation (25) is the governing equation to obtain the longitude of the equator crossing point.

5. The vertex

The vertices of a great circle are the points nearest the

poles. Accordingly, let the first derivative of equation (23) be zero and rearrange it, we get equation:

$$\tan DLo_{FV} = \frac{1}{\sin L_F \tan C}. \quad \textbf{(26a)}$$

While using the sine law of the right-angled spherical triangles, we have:

$$\cos L_V = \cos L_F \sin C. \quad \textbf{(26b)}$$

It is noticed that equations (26a) and (26b) are a set of the governing equations to yield the vertex.

COMPUTATION PROCEDURES AND NUMERICAL PROGRAM

As mentioned above, the main task of the GCT problems is to obtain the waypoints on the GCT. These waypoints are then transferred to a Mercator chart or the ECDIS. Then, the course and distance of the rhumb line between two adjacent waypoints are determined by using the spherical Mercator sailing. In this regard, we first introduce all needed formulae used for the spherical Mercator sailing. Thereafter, all the governing equations derived for solving the GCT problems are summarized in computation procedures. Finally, a program GCTPro_VA is developed for navigator friendly use.

THE COURSES AND DISTANCES OF THE RHUMB LINES

Basically, there are two kinds of sailings to obtain the course and distance of the rhumb line between two adjacent waypoints on the GCT. One is the spheroid rhumb line sailing and the other is the spherical Mercator sailing [1, 3, 5, 7, 17, 18]. Since the spherical Mercator sailing meets most of the needs of navigational practice and the distance for a rhumb line on the navigation sphere is within 0.5% of the distance on the rhumb line on the spheroid [10], we adopt the spherical Mercator sailing to yield the course and distance of the rhumb line. For convenience, all needed formulae and related constants in the numerical program are listed as follows [3, 7, 17, 18]:

$$M = a_e \ln\left[\tan\left(45° + \frac{L}{2}\right) \times \left(\frac{1 - e\sin L}{1 + e\sin L}\right)^{\frac{e}{2}}\right], \quad \textbf{(27)}$$

in which nautical miles (nm) [3] $e=0.081819190842622$, for WGS-84 [17]. And then,

$$l = 60' \times (L_{X_{i+1}} - L_{X_i}), \quad dlo = 60' \times (\lambda_{X_{i+1}} - \lambda_{X_i}),$$

$$\Delta M = M_{X_{i+1}} - M_{X_i}, \quad \textbf{(28)}$$

$$\tan cm = \frac{dlo}{\Delta M}, \quad \textbf{(29)}$$

$$dm = \begin{cases} lseccm & , \ cm \neq 90° \\ dlo \cos L_{X_i} & , \ cm = 90°. \end{cases} \quad \textbf{(30)}$$

rhumb line between two adjacent waypoints are calculated by using equations (27), (28), (29) and (30) of spherical Mercator sailings.

CONSTRUCTING COMPUTATION PROCEDURES

To completely illustrate the derived governing equations used in each step of the computation procedures, they are summarized as follows.

Step 1. Calculating the great circle information:
The great circle distance and initial course angle are yielded by using equations (18) and (20b), respectively.
The equator crossing points of the great circle are obtained by using equation (25).
The vertices of the great circle are reached by using equations (26a) and (26b).

Step 2. Calculating the waypoints on the GCT under different given initial conditions:
Condition 1: Giving the great circle distances to obtain the latitudes and the longitudes of the waypoints on the GCT. The latitudes and longitudes of the waypoints are calculated by using equations (21) and (22), respectively.
Condition 2: Giving the longitudes of the waypoints to yield the latitudes. The latitudes of the waypoints are calculated by using equation (23d).
Condition 3: Giving the latitudes of the waypoints to reach the longitudes. The longitudes of the waypoints are calculated by using equations (24a) and (24b).

Step 3. Calculating the course and distance between two adjacent waypoints: The Mercator course and distance of every

DEVELOPING THE NUMERICAL PROGRAM

To deal with GCT problems under all given initial conditions, a program, namely GCTPro_VA, is developed based on the S-VA approach. In addition, for selecting the reasonable number of the waypoints, a diagram of total Mercator distance versus waypoints number (hereafter is called tMd-n diagram) is provided for the navigator. The GCTPro_VA program is coded in the Visual Basic 6.0 (VB6) with the graphical user interface (GUI) for popular and friendly use of the navigator.

VALIDATED EXAMPLES AND DISCUSSIONS

*Example 1. A vessel is proceeding from **San Francisco** (USA) to **Sydney** (AUSTRALIA). The navigator desires to use great circle sailing from L37°47.5'N, λ122°27.8'W to L33°51.7'S, λ151°12.7'E (Bowditch, 1981, pp. 616-618).*

Required: Using the GCTPro_VA program to calculate the great circle information and the latitudes and longitudes of the waypoints along the GCT at equal interval of great circle distance, 360 nm (6°) apart from the departure (Condition 1). In addition, the relationships of total Mercator distance and numbers of the waypoints are acquired.

Solution: The GCTPro_VA program is run to solve the great circle information and the waypoints on the GCT under
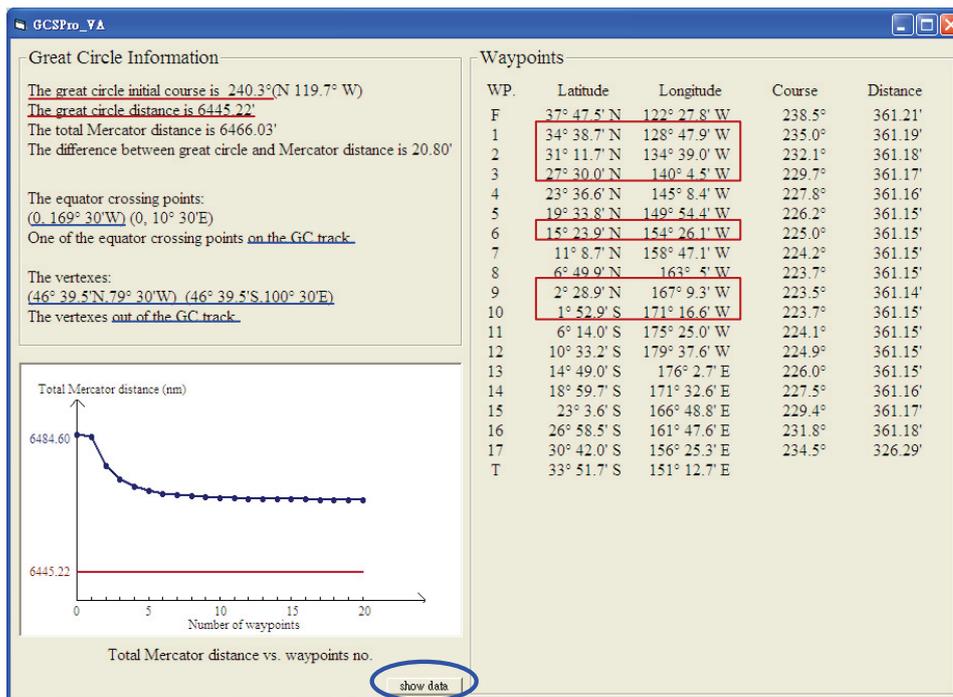
Fig. 3. Results of condition 1 by running the GCTPro_VA program in example 1

condition 1 successfully. Results of the great circle information, the waypoints on the GCT and a tMd-n diagram are all shown in Fig. 3. The comparison of results obtained by the S-VA approach and those by tabular method (Ageton method) [2] is shown in Table 2. In addition, when the button of "show data" in Fig. 3 is clicked, the relationships between the total Mercator distance and numbers of the waypoints on the GCT will be shown in Table 3.

Remark: With respect to the waypoints on the GCT under condition 1, the S-VA approach has been validated by this example. It is noticed that the computerized solution is always more accurate than the tabular method because it is free of rounding errors, which was also reported in Bowditch book [3]. As shown in Table 3 the total Mercator distance of 16 waypoints is nearly equal to that of 8 waypoints and their difference of the distance is less than 1 nm. A reasonable number of waypoints can be considered from this table for the navigator. It is an interesting discovery here.

Example 2. *A vessel is proceeding from* **Sydney** *(AUSTRALIA) to* **Balboa** *(PANAMA). The master desires to use the great circle sailing from* L33°51.5'S, λ151°13.0'E *to* L08°53.0'N, λ079°31.0'W *(Chen et al, 2004, pp. 317-319).*

Required:
1. Using the GCTPro_VA program to calculate the great circle information and the latitudes and longitudes of the waypoints along the GCT at longitude 170°E and

Tab. 3. Relationships between waypoints number on the GCT and total Mercator distance (nm) in example 1

waypoints number	total Mercator distance	waypoints number	total Mercator distance
0	6484.60	9	6466.74
1	6483.89	10	6466.56
2	6475.60	11	6466.43
3	6471.67	12	6466.32
4	6469.65	13	6466.24
5	6468.49	14	6466.17
6	6467.77	15	6466.12
7	6467.30	16	6466.07
8	6466.97	17	6466.03

at each 20 degrees of longitude thereafter to longitude 90°W (Condition 2).
2. Using the GCTPro_VA program to calculate the great circle information and the latitudes and longitudes of the waypoints on the GCT at latitude 35°S and at each 10 degrees of latitude northward to latitude 05°N (Condition 3).

Solution:
1. The GCTPro_VA program is used to solve the great circle information and the waypoints on the GCT under condition 2 successfully. Results of the great circle information, the waypoints on the GCT and a tMd-n diagram are all shown in Fig. 4.
2. The GCTPro_VA program is used to solve the great circle information and the waypoints on the GCT under condition 3 successfully. Results of the great circle information, the waypoints on the GCT and a tMd-n diagram are all shown in Fig. 5.

Remark:
1. With respect to the waypoints on the GCT under conditions 2 or 3, the S-VA approach has been validated by this example. A comparison of the results of the GCE method [6] and those of the S-VA approach shows that the proposed approach is as accurate as the GCE method. However, the derived governing equations of the S-VA approach are simpler and clearer than those of the GCE method.
2. The developed GCTPro_VA program has also been verified by examples 1 and 2. It is found, that the GCTPro_VA program shows the advantages of completeness and wide applications for navigator.

CONCLUSIONS

The proposed S-VA approach has systematically derived the spherical triangle formulae for solving the GCT problems and it can handle the waypoints on the GCT

Tab. 2. A comparison of results obtained by tabular method and those by the S-VA approach in example 1

Item		Tabular method*	GCTPro_VA
D		6445.5'	6445.22'
C		240°17.5'	240.3°
V		L46°39.5'S, λ100°29.7'E	L46°39.5'S, λ100°30.0'E L46°39.5'N, λ79°30.0'W
E		-	L00°00.0', λ169°30.0'W L00°00.0', λ010°30.0'E
D_{FX}	6°	L34°39.0'N, λ128°48.3'W	L34°38.7'N, λ128°47.9'W
	12°	L31°12.0'N, λ134°39.3'W	L31°11.7'N, λ134°39.0'W
	18°	L27°30.0'N, λ140°04.3'W	L27°30.0'N, λ140°04.5'W
	36°	L15°24.0'N, λ154°26.3'W	L15°23.9'N, λ154°26.1'W
	54°	L02°29.0'N, λ167°09.3'W	L02°28.9'N, λ167°09.3'W
	60°	L01°52.5'S, λ171°17.3'W	L01°52.9'S, λ171°16.6'W

*Resource： Bowditch, 1981, pp. 616-618 (Ageton method).

Fig. 4. Results of condition 2 by running the GCTPro_VA program in example 2
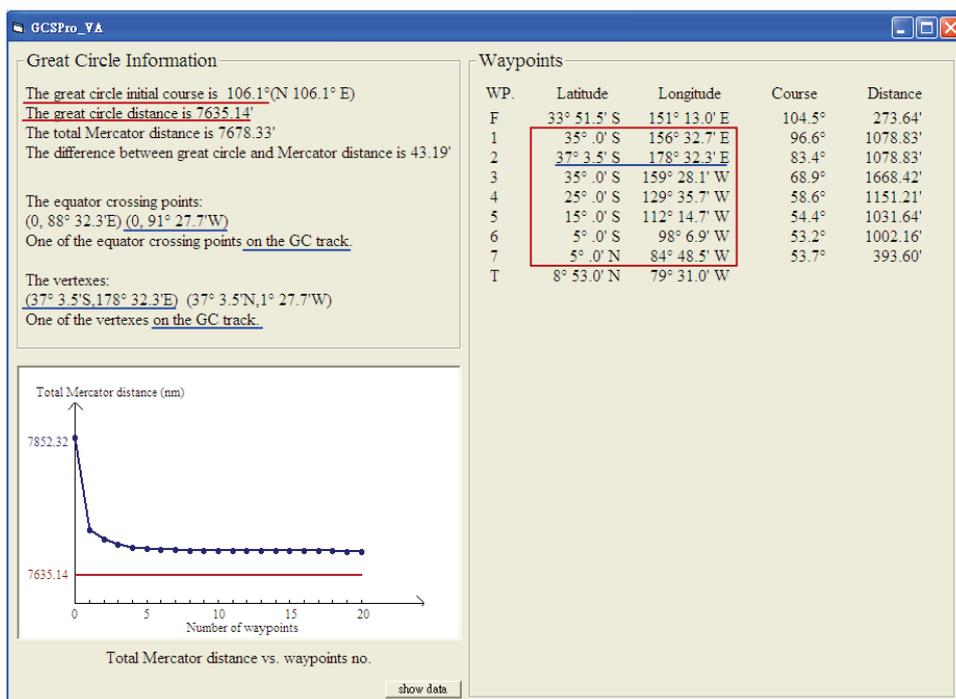
Fig. 5. Results of condition 3 by running the GCTPro_VA program in example 2

under all given initial conditions successfully. Because the mathematical properties of the geometry and algebra are both implied in the S-VA approach, the derivation process becomes more understandable and more straightforward as compared with those of conventional approaches. With the technique of transforming the Earth coordinates system into the Cartesian one and adopting the relative longitude concept, the derived governing equations by using the S-VA approach are simpler and clearer than those by the conventional approaches. Based on the S-VA approach, the GCTPro_VA program has been developed for friendly use of

the navigator. Several benchmark examples are conducted to validate the proposed S-VA approach effectively and to verify the developed GCTPro_VA program successfully.

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NOMENCLATURE

N	North pole
S	South pole
F	the departure
T	the destination
X	the waypoints on great circle track
E	the equator crossing point
V	the vertex
L	the latitude
L_F	the latitude of the departure
L_T	the latitude of the destination
L_X	the latitude of the waypoints
L_E	the latitude of the equator crossing point
L_V	the latitude of the vertex
λ	the longitude
λ_F	the longitude of the departure
λ_T	the longitude of the destination
λ_X	the longitude of the waypoints
λ_E	the longitude of the equator crossing point
λ_V	the longitude of the vertex
DLo	the difference of longitude from the departure to the destination
DLo_{FX}	the difference of longitude from the departure to the waypoints; a dihedral angle in Fig. 2
DLo_{FE}	the difference of longitude from the departure to the equator crossing point
DLo_{FV}	the difference of longitude from the departure to the vertex
D	the great circle distance from the departure to the destination
D_{FX}	the great circle distance from the departure to the waypoints
C	the great circle initial course angle from the departure to the destination; a dihedral angle in Fig. 2
M	the meridional parts between the given latitude and the equator
a_e	the equatorial radius of the Earth, a_e=3437.74677078 nm
e	the eccentricity of the Earth, e =0.081819190842622 from WGS-84
l	the difference of latitude between two adjacent waypoints
dlo	the difference of longitude between two adjacent waypoints
ΔM	the difference of meridional parts between two adjacent waypoints
cm	the course angle in spherical Mercator sailings
dm	the distance in spherical Mercator sailings

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