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SPHERE – TO – SPHEROID COMPARISON – NUMERICAL ANALYSIS

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ABSTRACT

Differences of orthodromic distances calculated on the sphere in comparison to the spheroid are numerically analyzed in the full range of departure point latitudes, courses over the ground and orthodromic distances for the global and limited range of latitudes. Optimum solutions for the radii of the sphere are provided.

Keywords: sphere in navigation; great circle distance; geodesics; direct and inverse geodesic solutions; numerical simulations

INTRODUCTION

In traditional navigation, to reduce the number of calculations, calculations are done with many simplifications, for example on a plane or a sphere instead of a spheroid. In modern navigation the spheroid such as WGS-84 ellipsoid is applied but in simpler navigation devices and in instructions for practical navigation a sphere is still used. The question arises what are differences between these two models of the earth.

In [1] comparison has been made between great circle distance on the navigation sphere and great ellipse on the spheroid and it has been concluded that this difference is near to 0.5%.

In [3] similar results have been obtained by simple comparison of extreme radiuses of curvature and an average substitute radius of the sphere.

In [4] the problem of differences for this radius have been partially numerically analysed for the global range of latitudes.

There are many global average radii defined such as mean, authalic, volumetric or rectifying. In [3] it has been proposed the radius of the sphere which gives minimum absolute percentage value of differences given by the equation (for the global range of latitudes)

$$R_S = \frac{R_{M\,max} + R_{M\,min}}{2} \quad (1)$$

where

$$R_{M\,min} = R_M(\varphi = 0°) = \frac{{b_0}^2}{a_0} \quad (2)$$

$$R_{M\,max} = R_M(\varphi = 90°) = R_N(\varphi = 90°) = \frac{{a_0}^2}{b_0} \quad (3)$$

and R_M, R_N are the radius of curvature in meridian and the radius of curvature in the prime vertical, respectively

$$R_M = \frac{a_0(1-e^2)}{\sqrt{(1-e^2\sin^2\varphi)^3}} \quad (4)$$

$$R_N = \frac{a_0}{\sqrt{1 - e^2 \sin^2 \varphi}} \quad (5)$$

For WGS-84

$$R_S = 6\ 367\ 516 \text{ m} \quad (6)$$

The set of procedures for calculating orthodromes (defined as the path of the shortest distance on any surface) by the application of solutions of the problems known in geodesy as the direct and the inverse geodetic problems, have been presented in [2,3].

In formal notations:

$$\varphi_2, \lambda_2 = \text{DGP}\ (\varphi_1, \lambda_1, S, C_{gs}) \quad (7)$$

$$S = \text{IGP}\ (\varphi_1, \lambda_1, \varphi_2, \lambda_2) \quad (8)$$

$$\varphi_{max} = \text{FIMAX}(\varphi_1, \lambda_1, \varphi_2, \lambda_2) \quad (9)$$

where φ_1, λ_1 and φ_2, λ_2 are the departure point and the destination point, respectively, S is the orthodromic distance, C_{gs} is the Course Over the Ground (COG) at the departure point of the orthodrome, DGP is the procedure of solving the direct geodetic problem, IGP is the procedure of solving the inverse geodetic problem, FIMAX is the procedure of calculating maximum latitude.

In this paper these procedures, with results based on full accuracy Sodano's solutions [5,6,7] on WGS-84 reference ellipsoid (as in [2,3]), will be analysed in comparison to the great circle distance

$$S_S = R_S \cos^{-1}((\sin\varphi_1 \sin\varphi_2 + \cos\varphi_1 \cos\varphi_2 \cos(\lambda_2 - \lambda_1))) \quad (10)$$

for the full range of φ_1, C_{gs} and S and for the global and limited range of latitudes.

GLOBAL RANGE OF LATITUDES

For the departure latitude φ_1 taken from 0° to 90° in steps of 10°, $\lambda_1 = 0$, the departure course C_{gs} -from 0° to 180° in steps of 10° and the spheroidal orthodromic distance S – from 100 NM to 10000 NM the destination points φ_2, λ_2 are calculated with the use of the procedure (7) and for these points the great circle distance S_S is calculated by using Eq. (10). Tables for the given orthodromic distances S are generated from the differences

$$\Delta S = S_S - S \quad (11)$$

or percentage differences

$$\Delta S_{\%} = \frac{S_S - S}{S} \cdot 100\% \quad (12)$$

Such exemplary tables are: Tab. 1 ($\Delta S_{\%}$ for S = 100 NM) and Tab. 2 (ΔS for S = 5400 NM).

For each given distance S the maximum positive and negative differences $\Delta S_{max\%+}$, $\Delta S_{max\%-}$, ΔS_{max+}, ΔS_{max-} can be found. These differences are presented in Tab. 3, Fig. 1 and 2.

Maximum differences have been found with the use of extremes finding tool (Microsoft Excel Solver) where values φ_1 and C_{gs} are not limited to values in Tables and therefore maximum differences in Tab. 2 are not the same as in Tab. 3 for S = 5400 NM.

It has been proved that R_S from Eq.(1) is the only one which gives symmetrical (and therefore minimum) absolute and percentage differences reaching up to ±0.5% and ±17 NM, respectively. A small asymmetry is due to the fact that this equation gives exact symmetry for S near to zero distance.

Maximum positive differences appear for meridional orthodromes symmetrical to the equator and negative differences appear for meridional orthodromes symmetrical to the poles.

Maximum percentage differences appear for shorter orthodromes due to the bigger part of extreme difference between the radius of curvature in meridian and R_S at the equator and the poles in shorter orthodromes and the smaller part in longer orthodromes.

Maximum absolute differences of about ±17 NM appear for S = 5400 NM for meridional orthodromes from 45° N to 45° S (and reversed) and from 45° N to 45° N or from 45° S to 45° S through the poles.

From S = 7840 NM the maximum negative differences of orthodromes change position from the poles to the equator where the percentage difference is constant and the absolute difference grows proportionally to the orthodromic distance.

LIMITED RANGE OF LATITUDES

LIMITED HIGHER LATITUDES

In marine navigation higher latitudes are not accessible. For this reason and for navigation in limited range of latitudes, orthodromic differences can be analysed for orthodromes for which

$$\varphi_{max} \le \varphi_{lim} \quad (14)$$

where φ_{max} is from the procedure (9) and φ_{lim} is a limited latitude.

Differences in Tab. 1 and 2 are the functions

$$\Delta S = f(\varphi_1, C_{gs}, S, R_S) \quad (15)$$

which can be extended to

$$\Delta S_{lim} = f(\varphi_1, C_{gs}, S, R_S, \varphi_{lim}) \tag{16}$$

The tables can be generated for a given S and φ_{lim} with points

$$\Delta S_{lim} = f(\varphi_1, C_{gs}, S, R_S, \varphi_{max} > \varphi_{lim}) = 0 \tag{17}$$

and for each table

$$\Delta S_{lim\,max} = \max(|\Delta S_{lim}|) \tag{18}$$

is calculated and R_S is searched for which

$$\Delta S_{limmax} = \min \tag{19}$$

Therefore such R_S is optimum one to minimize maximum differences.

Since maximum percentage differences are for S = 100 NM (and are similar for all shorter orthodromes – Fig. 1) and maximum absolute differences are for S = 5400 NM therefore exemplary tables for φ_{lim} = 60° for the two distances are presented as Tab. 4 and 5.

Optimum R_S and ΔS_{limmax} in function of φ_{lim} are presented in Tab. 6, Fig. 3, Fig. 4 and Tab. 7, Fig. 5, Fig. 6.

Results for S = 5400 NM and lower φ_{lim} are rather of little importance because for these values tables of ΔS_{lim} have a few possible orthodromes only.

LIMITED LOWER LATITUDES

Nothing prevents from limiting also lower latitudes. In this case Eq. (17) can be extended to

$$\Delta S_{lim}=f(\varphi_1,C_{gs},S,R_S,\varphi_1 < \varphi_{min},\varphi_2<\varphi_{min},\varphi_{max}>\varphi_{lim}) =0 \tag{20}$$

where φ_{min} is limited lower latitude.

Exemplary table for φ taken from 20° to 60° and for S = 100 NM is presented as Tab. 8.

CONCLUSIONS

The performed numerical analysis has proved that assessments of the sphere – to - spheroid differences in distances reaching ±0.5%, which can be found in the referred to literature sources, are valid for short orthodromes only. For longer orthodromes the maximum of absolute differences of ±17 NM exists for 5400 NM orthodrome. For global and limited ranges of latitudes the ready- to- use optimum radii of the sphere, which minimize maximum absolute and percentage differences, are provided in this paper.

BIBLIOGRAPHY

1. Earle M.A.: *Sphere to Spheroid Comparisons*. The Journal of Navigation, 59 (2006), pp. 491-496.
2. Lenart A.S.: *Solutions of Direct Geodetic Problem in Navigational Applications*. TransNav – The International Journal on Marine Navigation and Safety of Sea Transportation, 5(4) (2011), pp. 527-532.
3. Lenart A.S.: *Solutions of Inverse Geodetic Problem in Navigational Applications*. TransNav – The International Journal on Marine Navigation and Safety of Sea Transportation, 7(2) (2013), pp.253-257.
4. Nadrowski L.: *Analysis of errors of orthodromic distances on the sphere in comparison to the ellipsoid* (in Polish). Gdynia Maritime University, Faculty of Navigation – Engineer's Thesis 2016.
5. Sodano E.M.: *A rigorous non-iterative procedure for rapid inverse solution of very long geodesics*. Bulletin Géodésique, 47/48 (1958), pp. 13-25.
6. Sodano E.M: *General non-iterative solution of the inverse and direct geodetic problems*. Bulletin Géodésique, 75 (1965), pp. 69-89.
7. Sodano E.M.: *Supplement to inverse solution of long geodesics*. Bulletin Géodésique, 85 (1967), 233-236.

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Tab. 1. $\Delta S_{\%}$ [%] for S = 100 NM

φ_1 [°]	C_{gs} [°]																		
	0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180
90	**-0.501**	**-0.501**	**-0.501**	**-0.501**	**-0.501**	**-0.501**	**-0.501**	**-0.501**	**-0.501**	**-0.501**	**-0.501**	**-0.501**	**-0.501**	**-0.501**	**-0.501**	**-0.501**	**-0.501**	**-0.501**	**-0.501**
80	-0.476	-0.476	-0.478	-0.480	-0.483	-0.486	-0.488	-0.490	-0.491	-0.491	-0.489	-0.487	-0.483	-0.479	-0.475	-0.471	-0.468	-0.466	-0.466
70	-0.393	-0.395	-0.401	-0.411	-0.423	-0.435	-0.447	-0.456	-0.461	-0.462	-0.458	-0.449	-0.437	-0.423	-0.409	-0.395	-0.384	-0.376	-0.374
60	-0.262	-0.267	-0.281	-0.302	-0.328	-0.356	-0.382	-0.402	-0.414	-0.417	-0.410	-0.393	-0.369	-0.340	-0.309	-0.280	-0.257	-0.242	-0.237
50	-0.100	-0.108	-0.131	-0.167	-0.211	-0.257	-0.300	-0.335	-0.357	-0.363	-0.352	-0.325	-0.286	-0.239	-0.189	-0.142	-0.104	-0.079	-0.071
40	0.075	0.063	0.030	-0.021	-0.084	-0.151	-0.213	-0.263	-0.295	-0.305	-0.290	-0.253	-0.199	-0.132	-0.062	0.004	0.057	0.092	0.104
30	0.241	0.226	0.183	0.117	0.036	-0.050	-0.130	-0.195	-0.237	-0.250	-0.233	-0.187	-0.117	-0.033	0.056	0.139	0.207	0.251	0.267
20	0.379	0.361	0.310	0.232	0.136	0.034	-0.062	-0.139	-0.189	-0.206	-0.186	-0.133	-0.052	0.046	0.150	0.248	0.328	0.380	0.398
10	0.471	0.451	0.395	0.309	0.203	0.090	-0.016	-0.102	-0.158	-0.177	-0.156	-0.098	-0.011	0.097	0.211	0.317	0.404	0.461	0.481
0	**0.506**	0.486	0.428	0.338	0.229	0.112	0.002	-0.088	-0.146	-0.167	-0.146	-0.088	0.002	0.112	0.229	0.338	0.428	0.486	**0.506**

Tab. 2. ΔS [NM] for S = 5400 NM

φ_1 [°]	C_{gs} [°]																		
	0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180
90	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
80	-5.90	-5.82	-5.57	-5.17	-4.62	-3.93	-3.13	-2.22	-1.24	-0.22	0.83	1.85	2.83	3.72	4.50	5.14	5.62	5.91	6.01
70	-11.12	-10.98	-10.57	-9.89	-8.94	-7.75	-6.32	-4.70	-2.91	-1.00	0.98	2.95	4.86	6.62	8.18	9.47	10.43	11.03	11.23
60	-14.98	-14.82	-14.34	-13.53	-12.40	-10.94	-9.16	-7.08	-4.74	-2.20	0.49	3.22	5.89	8.41	10.65	12.52	13.93	14.80	15.09
50	**-17.02**	-16.87	-16.42	-15.67	-14.57	-13.11	-11.28	-9.08	-6.53	-3.67	-0.58	2.62	5.81	8.86	11.61	13.92	15.67	16.76	**17.13**
40	-16.98	-16.89	-16.58	-16.03	-15.20	-14.02	-12.45	-10.46	-8.05	-5.24	-2.11	1.23	4.63	7.93	10.94	13.51	15.46	16.68	17.10
30	-14.90	-14.88	-14.80	-14.60	-14.21	-13.55	-12.52	-11.06	-9.12	-6.72	-3.91	-0.79	2.48	5.72	8.74	11.34	13.33	14.59	15.02
20	-11.03	-11.10	-11.30	-11.55	-11.74	-11.77	-11.49	-10.80	-9.62	-7.93	-5.75	-3.18	-0.36	2.53	5.28	7.68	9.55	10.74	11.15
10	-5.84	-6.01	-6.51	-7.24	-8.08	-8.88	-9.47	-9.71	-9.48	-8.72	-7.42	-5.66	-3.56	-1.28	0.97	2.98	4.58	5.60	5.96
0	0.06	-0.21	-0.99	-2.20	-3.67	-5.24	-6.72	-7.93	-8.72	-8.99	-8.72	-7.93	-6.72	-5.24	-3.67	-2.20	-0.99	-0.21	0.06

Tab. 3. ΔS_{max} for different S

S	100	500	1000	2000	4000	5400	6000	7840	10000	NM
$\Delta S_{max\%-}$	-0.501	-0.499	-0.494	-0.474	-0.396	-0.320	-0.283	-0.167	-0.167	%
$\Delta S_{max\%+}$	0.506	0.505	0.499	0.478	0.399	0.322	0.285	0.168	0.041	%
ΔS_{max-}	-0.501	-2.497	-4.942	-9.472	-15.842	-17.263	-17.003	-13.083	-16.651	NM
ΔS_{max+}	0.506	2.523	4.992	9.561	15.964	17.377	17.113	13.199	4.095	NM

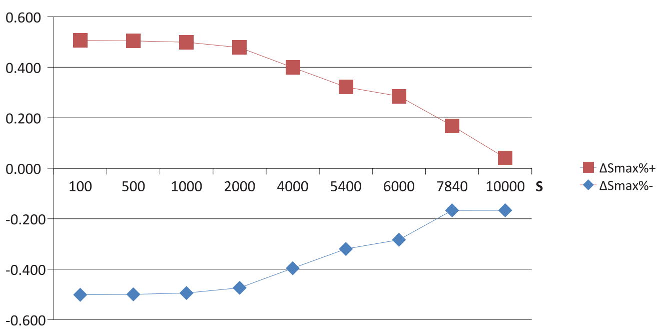

Fig. 1. ΔSmax% [%] for different S [NM]

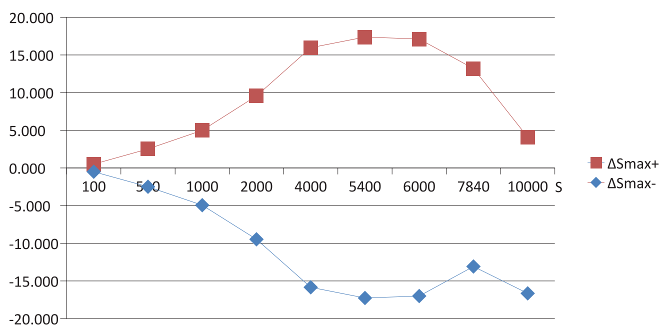

Fig. 2. ΔS_{max} [NM] for different S [NM]

Tab. 4. |ΔS_{lim}|[%] for S = 100 NM, φlim = 60°, RS = 6 364 691 m

φ_1 [°]	C_{gs} [°]																		
	0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180
90	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
80	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
70	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
60	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	**0.461**	0.454	0.437	0.413	0.384	0.353	0.325	0.302	0.286	0.281
50	0.144	0.152	0.176	0.211	0.255	0.301	0.345	0.379	0.401	0.407	0.396	0.369	0.330	0.283	0.233	0.186	0.148	0.124	0.115
40	0.031	0.019	0.014	0.066	0.128	0.195	0.257	0.308	0.339	0.349	0.334	0.298	0.243	0.177	0.106	0.041	0.013	0.048	0.060
30	0.197	0.182	0.139	0.073	0.008	0.094	0.174	0.240	0.281	0.294	0.277	0.231	0.162	0.078	0.011	0.095	0.163	0.207	0.222
20	0.334	0.316	0.265	0.187	0.092	0.010	0.106	0.183	0.234	0.250	0.230	0.177	0.096	0.002	0.106	0.204	0.283	0.335	0.353
10	0.426	0.406	0.350	0.264	0.158	0.046	0.060	0.146	0.202	0.221	0.200	0.143	0.055	0.052	0.166	0.273	0.360	0.416	0.436
0	**0.461**	0.441	0.383	0.294	0.184	0.067	0.042	0.132	0.190	0.211	0.190	0.132	0.042	0.067	0.184	0.294	0.383	0.441	**0.461**

Tab. 5. |ΔS_{lim}|[NM] for S = 5400 NM, φ_{lim} = 60°, R_S = 6 365 799 m

φ_1 [°]	C_{gs} [°]																		
	0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180
90	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
80	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
70	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
60	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	3.65	0.97	1.76	4.44	6.95	9.19	11.06	12.47	13.34	13.63
50	0.00	0.00	0.00	0.00	0.00	0.00	12.74	10.53	7.98	5.12	2.04	1.16	4.36	7.40	10.15	12.46	14.21	15.30	**15.67**
40	0.00	0.00	0.00	0.00	0.00	15.47	13.90	11.91	9.50	6.69	3.57	0.23	3.17	6.47	9.48	12.04	14.00	15.22	15.64
30	0.00	0.00	0.00	0.00	**15.67**	15.00	13.97	12.51	10.57	8.17	5.36	2.25	1.02	4.27	7.28	9.88	11.87	13.13	0.00
20	0.00	0.00	0.00	0.00	13.20	13.22	12.94	12.25	11.07	9.38	7.21	4.64	1.82	1.07	3.82	6.22	0.00	0.00	0.00
10	0.00	0.00	0.00	0.00	9.54	10.34	10.93	11.17	10.93	10.17	8.88	7.12	5.01	2.74	0.49	1.53	0.00	0.00	0.00
0	0.00	0.00	0.00	0.00	5.13	6.70	8.18	9.38	10.17	10.45	10.17	9.38	8.18	6.70	5.13	0.00	0.00	0.00	0.00

Tab. 6. $\Delta S_{limmax\%}$ and R_S for different φ_{lim} and S = 100 NM

φ_{lim}	10	20	30	40	50	60	70	80	90	°
$\Delta S_{limmax\%}$	0.341	0.355	0.378	0.405	0.434	0.461	0.484	0.498	0.503	%
R_S	6 357 045	6 357 965	6 359 376	6 361 109	6 362 954	6 364 691	6 366 108	6 367 033	6 367 355	m

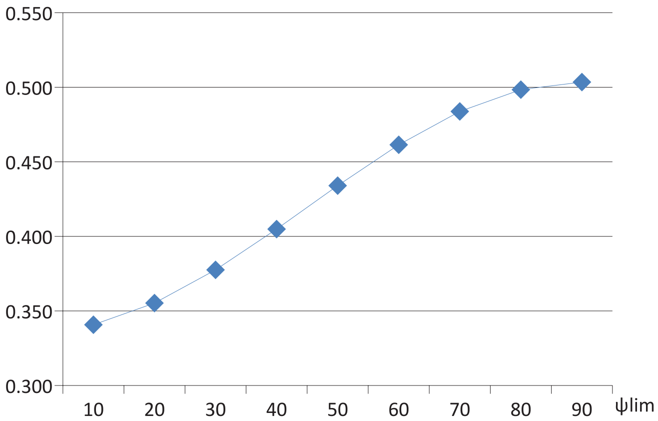

Fig. 3. $\Delta S_{limmax\%}$ for different φ_{lim} and S = 100 NM

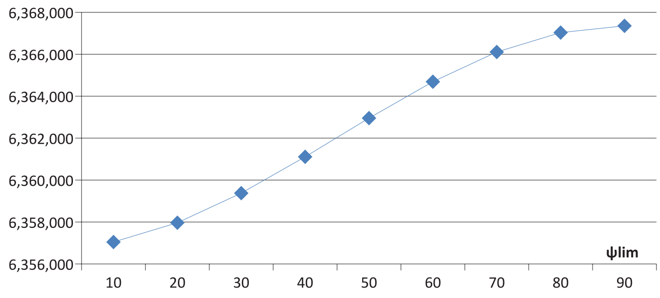

Fig. 4. R_S [m] for different φ_{lim} and S = 100 NM

Tab. 7. ΔS_{limmax} and R_S for different φ_{lim} and S = 5400 NM

φ_{lim}	10	20	30	40	50	60	70	80	90	°
ΔS_{limmax}	0.786	3.152	6.644	11.216	15.335	15.668	16.579	16.853	17.072	NM
R_S	6 377 209	6 374 991	6 372 423	6 367 838	6 365 408	6 365 799	6 366 871	6 367 193	6 367 450	m

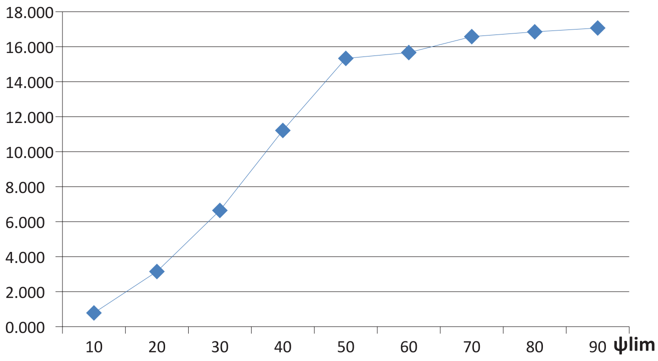

Fig. 5. ΔS_{limmax} for different φ_{lim} and S = 5400 NM

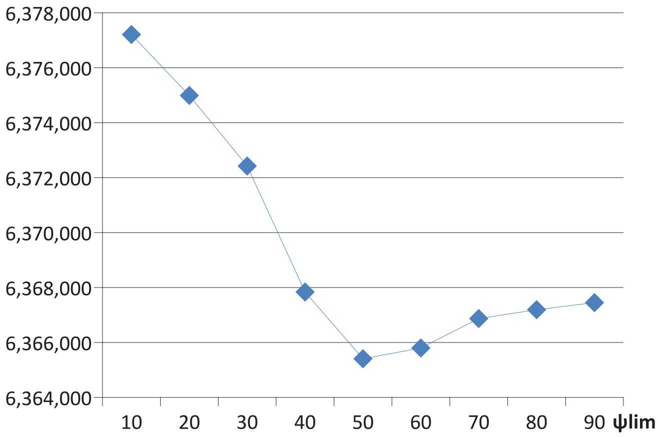

Fig. 6. R_S for different φ_{lim} and S = 5400 NM

Tab. 8. $|\Delta S_{lim}|$[%] for S = 100 NM, φ_{min} = 20°, φ_{lim} = 60°, R_S = 6 368 747 m

φ_1 [°]	C_{gs} [°]																		
	0	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180
90	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
80	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
70	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
60	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.398	0.391	0.374	0.350	0.321	0.290	0.261	0.238	0.223	0.218
50	0.080	0.089	0.112	0.148	0.192	0.238	0.281	0.316	0.338	0.343	0.333	0.306	0.267	0.219	0.169	0.123	0.085	0.060	0.052
40	0.094	0.083	0.049	0.002	0.065	0.132	0.194	0.244	0.276	0.285	0.271	0.234	0.179	0.113	0.043	0.023	0.076	0.111	0.123
30	0.261	0.246	0.203	0.137	0.056	0.030	0.111	0.176	0.218	0.231	0.213	0.167	0.098	0.014	0.075	0.159	0.226	0.271	0.286
20	0.398	0.380	0.329	0.251	0.155	0.053	0.042	0.120	0.170	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
10	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000