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A modification of distortion-energy theory at static-dynamic random loading

SUMMARY

The paper deals with stress modelling at static-dynamic random loadings. The material compatible with the Kelvin-Voigt's model and the stationary loading is assumed. A two-step transformation of the original multiaxial stress into the uniaxial reduced stress is performed. For this purpose the average-distortion-energy strength hypothesis (a modification of the distortion-energy theory) is utilized.

INTRODUCTION

Details of machines and structural elements are usually subjected to loadings that vary in magnitude and/or direction in a random manner. The stress varying in magnitude between unequal extreme values can be described as a static-dynamic combination of a mean stress and an alternating stress equal to half the difference. As the mean stress increases the fatigue-safe range of the alternating stress diminishes. The relative values of the fatigue strength of a metal is an important factor for the choice of material and geometry of constructional parts at a given static-dynamic loading.

The problem of fatigue assessment becomes more complicated if the loading is multiaxial. Various procedures may be indicated as its possible solutions [1÷3]. Another analytical procedure capable of dealing with complex stress patterns in the high-cycle fatigue regime is presented in [4]. It is concerned with static-periodic loadings and materials compatible with the Kelvin-Voigt's model. Its main feature is the two-step transformation of the original stress into the reduced stress and application of the calculation methods commonly accepted at uniaxial stress. The reduced stress is determined with the aid of the average-distortion-energy strength hypothesis being a modification of the distortion-energy theory. In the following a similar approach to static-dynamic random loading is considered. It is assumed that the loading is stationary and the components of the resulting stress are physically and statistically independent of each other.

EQUIVALENT STRESS

A stationary random stress with Cartesian components :

$$\sigma_i(t) = \sigma_{mi} + \sigma_{vi}(t) \qquad (1)$$

$$i = x, y, z, xy, yz, zx$$

is considered, where :

σ_{mi} - time-independent part of i-th stress component
$\sigma_{vi}(t)$ - zero mean time-varying part of i-th stress component.

In view of the difficulty of cycle counting and mean stress analysing in multiaxial random fatigue [3], the stress (1) is modelled by the equivalent stress with cyclic stationary components :

$$\sigma_i^{(eq)}(t) = \sigma_{mi} + a_i^{(eq)} \sin\left(\omega_{eq} t + \varphi_i\right) \qquad (2)$$

$$i = x, y, z, \ldots, zx$$

where the random variables $a_i^{(eq)}$ and φ_i fulfil the following conditions [5] :

$$E\{\sigma_{1i}\} = E\{\sigma_{-1i}\} = 0$$
$$E\{\sigma_{-1i}\sigma_{1i}^*\} = E\{\sigma_{-1i}^*\sigma_{1i}\} = 0 \qquad (3)$$

Here

$$\sigma_{1i} = \frac{1}{2j} a_i^{(eq)} \exp\left(j\varphi_i\right) \qquad \sigma_{-1i} = \sigma_{1i}^*$$

$E\{\ \}$ - expected value
$(\ )^*$ - complex conjugate

$$j = \sqrt{-1}$$

The equivalent conditions refer to dissipative energy and are presented in [6]. The mean-square values of the amplitudes of the equivalent stress components and the equivalent frequency are given as follows [6] :

$$E\left\{\left(a_i^{(eq)}\right)^2\right\} = 2\int_{-\infty}^{\infty} S_i(\omega)\,d\omega \qquad (4)$$

$$\omega_{eq} = \left[\frac{\sum_i \frac{\eta_i}{E_i^2}\int_{-\infty}^{\infty}\omega^2 S_i(\omega)\,d\omega}{\sum_i \frac{\eta_i}{E_i^2}\int_{-\infty}^{\infty} S_i(\omega)\,d\omega}\right]^{1/2} \qquad (5)$$

where :

- E_i - Young or shear modulus, associated with i-th stress component, respectively
- η_i - coefficient of internal viscous damping in the Kelvin-Voigt's model of the material, associated with i-th stress component
- $S_i(\omega)$ - power spectral density of the process $\sigma_{vi}(t)$.

REDUCED STRESS

As the stress components (2) are synchronous, it is necessary to adopt the distortion-energy theory and determine the reduced stress in the form of the cyclic stationary process :

$$\sigma_{red}(t) = \sigma_m + a_{red}\sin(\omega_{eq}t + \varphi) \qquad (6)$$

where :

- σ_m - time-independent part of the reduced stress
- a_{red}, φ - amplitude and phase angle of the reduced stress.

These random variables cannot be determined from the single criterion provided by the distortion-energy theory. Similar problem at static-periodic stresses was solved in [4] by means of the average-distortion-energy strength hypothesis which resulted in the following relationships between the parameters of the reduced stress and those of the equivalent stress :

$$\sigma_m^2 = \sigma_{mx}^2 + \sigma_{my}^2 + \sigma_{mz}^2 - \sigma_{mx}\sigma_{my} - \sigma_{my}\sigma_{mz} - \sigma_{mz}\sigma_{mx} + \\ +3\left(\sigma_{mxy}^2 + \sigma_{myz}^2 + \sigma_{mzx}^2\right) \qquad (7)$$

$$a_{red}^2 = \left(a_x^{(eq)}\right)^2 + \left(a_y^{(eq)}\right)^2 + \left(a_z^{(eq)}\right)^2 + \\ \mp a_x^{(eq)}a_y^{(eq)} \mp a_y^{(eq)}a_z^{(eq)} \mp a_z^{(eq)}a_x^{(eq)} + \\ +3\left[\left(a_{xy}^{(eq)}\right)^2 + \left(a_{yz}^{(eq)}\right)^2 + \left(a_{zx}^{(eq)}\right)^2\right] \qquad (8)$$

This hypothesis is formulated within probabilistic approach as follows :

The reduced random stress is equivalent in terms of static-dynamic effort of a material to a given multiaxial random stress if :

- ❖ the time-independent parts of the strain energy of distortion per unit volume at both the stresses are equal in terms of expected values
- ❖ the integral averages of the instantaneous values of the strain energy of distortion per unit volume at both the stresses are equal in terms of expected values
- ❖ the reduced stress and the components of the multiaxial stress have the same frequency.

Hence (7) and (8) can be rewritten in the following form :

$$E\left\{\sigma_m^2\right\} = E\left\{\sigma_{mx}^2 + \sigma_{my}^2 + \sigma_{mz}^2 - \sigma_{mx}\sigma_{my} - \sigma_{my}\sigma_{mz} + \\ -\sigma_{mz}\sigma_{mx} + 3\left(\sigma_{mxy}^2 + \sigma_{myz}^2 + \sigma_{mzx}^2\right)\right\} \qquad (9)$$

$$E\left\{a_{red}^2\right\} = E\left\{\left(a_x^{(eq)}\right)^2 + \left(a_y^{(eq)}\right)^2 + \left(a_z^{(eq)}\right)^2 + \\ \mp a_x^{(eq)}a_y^{(eq)} \mp a_y^{(eq)}a_z^{(eq)} \mp a_z^{(eq)}a_x^{(eq)} + \\ +3\left[\left(a_{xy}^{(eq)}\right)^2 + \left(a_{yz}^{(eq)}\right)^2 + \left(a_{zx}^{(eq)}\right)^2\right]\right\} \qquad (10)$$

In order to account for the randomness of the phase angles φ_i in (2), equation (10) is replaced by :

$$E\left\{a_{red}^2\right\} = E\left\{\left(a_x^{(eq)}\right)^2 + \left(a_y^{(eq)}\right)^2 + \left(a_z^{(eq)}\right)^2 + \\ + a_x^{(eq)}a_y^{(eq)} + a_y^{(eq)}a_z^{(eq)} + a_z^{(eq)}a_x^{(eq)} + \\ +3\left[\left(a_{xy}^{(eq)}\right)^2 + \left(a_{yz}^{(eq)}\right)^2 + \left(a_{zx}^{(eq)}\right)^2\right]\right\} \qquad (11)$$

Hence the formulae for the mean-square values of the time-independent part of the reduced stress and of its amplitude are as follows :

$$E\left\{\sigma_m^2\right\} = E\left\{\sigma_{mx}^2\right\} + E\left\{\sigma_{my}^2\right\} + E\left\{\sigma_{mz}^2\right\} + \\ - E\left\{\sigma_{mx}\right\}E\left\{\sigma_{my}\right\} - E\left\{\sigma_{my}\right\}E\left\{\sigma_{mz}\right\} + \\ - E\left\{\sigma_{mz}\right\}E\left\{\sigma_{mx}\right\} + 3\left(E\left\{\sigma_{mxy}^2\right\} + E\left\{\sigma_{myz}^2\right\} + E\left\{\sigma_{mzx}^2\right\}\right) \qquad (12)$$

$$E\left\{a_{red}^2\right\} = E\left\{\left(a_x^{(eq)}\right)^2\right\} + E\left\{\left(a_y^{(eq)}\right)^2\right\} + \\ + E\left\{\left(a_z^{(eq)}\right)^2\right\} + E\left\{a_x^{(eq)}\right\}E\left\{a_y^{(eq)}\right\} + \\ + E\left\{a_y^{(eq)}\right\}E\left\{a_z^{(eq)}\right\} + E\left\{a_z^{(eq)}\right\}E\left\{a_x^{(eq)}\right\} + \\ +3\left[E\left\{\left(a_{xy}^{(eq)}\right)^2\right\} + E\left\{\left(a_{yz}^{(eq)}\right)^2\right\} + E\left\{\left(a_{zx}^{(eq)}\right)^2\right\}\right] \qquad (13)$$

where :

- $E\{\sigma_{mi}\}, E\{\sigma_{mi}^2\}$ - expected values and mean-square values of the time-independent parts of the original stress components (known from the analysis of the original stress)
- $E\{a_i^{(eq)}\}$ - expected values of the equivalent stress amplitudes (unknown)
- $E\{(a_i^{(eq)})^2\}$ - mean-square values of the equivalent stress amplitudes {given by (4)}.

Calculation of the expected values of the amplitudes requires their probability density functions to be known. When the up-crossing frequency of an uniaxial stress at an adequate level is equal to the frequency of the equivalent stress their amplitudes follow the same distribution [6]. In the opposite case this assumption may be erroneous but it can serve for a rough estimation of the afore-mentioned values also if the stress is multiaxial.

The quantity σ_m is a nonlinear function of the time-independent parts of the original stress. Its expected value can be approximately evaluated after expansion of σ_m in Taylor series about $\mu_{mi}=E\{\sigma_{mi}\}$ [7]. Retaining only the linear terms of the series one gets :

$$E\{\sigma_m\}=\left[\mu_{mx}^2+\mu_{my}^2+\mu_{mz}^2-\mu_{mx}\mu_{my}-\mu_{my}\mu_{mz}-\mu_{mz}\mu_{mx}+ +3\left(\mu_{mxy}^2+\mu_{myz}^2+\mu_{mzx}^2\right)\right]^{1/2} \qquad (14)$$

Similarly :

$$E\{a_{red}\}=\left[\mu_x^2+\mu_y^2+\mu_z^2+\mu_x\mu_y+\mu_y\mu_z+\mu_z\mu_x+ +3\left(\mu_{xy}^2+\mu_{yz}^2+\mu_{zx}^2\right)\right]^{1/2} \qquad (15)$$

where :

$$\mu_i=E\left\{a_i^{(eq)}\right\}$$

CONCLUDING REMARKS

The above given formulae may be applied to various technical problems. In particular they can be useful for :

- comparative calculations of a static-dynamic effort of engineering details
- parametric studies of structures, and
- fatigue analysis of structural elements and machinery parts.

For instance, an infinite fatigue life may be expected for a constructional member under multiaxial stationary random loading if :

$$E\{a_{red}\}+\lambda E\{\sigma_m\}\le Z_{rc} \qquad (16)$$

where :

Z_{rc} - fatigue limit at symmetrical tension-compression
λ - asymmetry sensitivity index at tension-compression calculated acc. [8]

$$\lambda=\frac{2Z_{rc}-Z_{rj}}{Z_{rj}} \qquad (17)$$

Z_{rj} - fatigue limit at pulsating tension.

NOMENCLATURE

$a_i^{(eq)}$ - amplitude of i-th component of the equivalent stress (i = x,y,z,xy,yz,zx)
a_{red} - amplitude of the reduced stress
$E\{\ \}$ - expected value
E_i - modulus of elasticity associated with i-th stress component
S_i - power spectral density of the time-varying part of i-th stress component
t - time
Z_{rc} - fatigue limit at symmetrical tension-compression
Z_{rj} - fatigue limit at pulsating tension
η_i - coefficient of the material damping associated with i-th stress component
λ - asymmetry sensitivity index at tension-compression
μ_i - expected value of the amplitude of i-th component of the equivalent stress
μ_{mi} - expected value of the time-independent part of i-th stress component
σ_i - i-th stress component
$\sigma_i^{(eq)}$ - i-th component of the equivalent stress
σ_m - time-independent part of the reduced stress
σ_{mi} - time-independent part of i-th stress component
σ_{red} - reduced stress
σ_{vi} - time-varying part of i-th stress component
φ - phase angle of the reduced stress
φ_i - phase angle of i-th component of the equivalent stress
ω - circular frequency
ω_{eq} - equivalent circular frequency

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Miscellanea

TOP KORAB Club meetings in 1998

In 1998 members of the Society of Polish Naval Architects and Marine Engineers, KORAB, took part in 7 so-called „club meetings" devoted to presentation and discussion of the following topics :

- „New refrigerating media in shipbuilding" by Z. Bonca, Technical University of Gdańsk
- „New ship designs of Gdynia Shipyard" by W. Żychski, Gdynia Shipyard
- „IMO and ship-related conventions" by J.Pacześniak, Polish Register of Shipping
- „Determination of ship manoeuvrability characteristics by model testing" by J.Kaliciński, Ship Design & Research Centre, Gdańsk
- „Model testing in solving problems of ship launching from domestic slipways" by T.Zdybek, Ship Design & Research Centre, Gdańsk
- „Problems of construction of a car carrier" by E.Piór, Gdynia Shipyard
- „On the history of engineering developments in Gdańsk" by A.Januszajtis, Gdańsk

The meetings are always a good occasion to be acquainted with different interesting problems of shipbuilding and marine economy, as well as to discuss them and exchange views.

Also, KORAB members visited Naval Port in Gdynia and several Polish naval vessels.