JANUSZ KOLENDA, Prof.,D.Sc.,M.E.
Technical University of Gdańsk
Polish Naval Academy, Gdynia

Fatigue assessment of structural elements under complex variable periodic loading

SUMMARY

The paper deals with the multiaxial states of stress produced by a series of short-term periodic loadings. Making use of the formulae presented in [5], each state is modelled by the equivalent state of stress with in-phase sinusoidal components. Such equivalent states can be treated by means of the classical strength theories.

In this paper, Huber-Mises theory is applied to determination of the reduced stress for each equivalent state. Then, using the constant-amplitude stress model [6], the set of the reduced stresses is modelled by the long-term effective stress which enables the fatigue lifetime to be estimated from the S-N curve. Hereby the application of the conventional cycle counting method and fatigue damage cumulation rule is not required.

The paper presented at the Seminar on „Design against fatigue" held on 28 April 1997 in the Faculty of Ocean Engineering and Ship Technology, Technical University of Gdańsk

INTRODUCTION

The fatigue design check of structural elements subjected to variable loading is frequently performed with the aid of S-N curves and the Miner rule [1]. The application of the latter requires the stress-time history to be represented by a stress histogram consisting of an equivalent set of constant-amplitude stress states. For this purpose various procedures commonly referred to as cycle counting methods have been devised. In the case of complex stress states, an appropriate multiaxial fatigue criterion must be also applied.

However the Miner rule may lead to large uncertainties in fatigue life calculations [1,2]. Moreover all multiaxial fatigue criteria can demonstrate large scatter [3]. The reasons of that a. o. may be that the linear criteria ignore the role of energy in the fatigue process and that the non-linear ones distort the stress spectra [4]. Therefore in this paper an alternative method of fatigue analysis is considered. It is restricted to high-cycle fatigue in multiaxial stress states produced by a series of periodic loadings and can be summarized as follows.

First, each stress component is expanded into Fourier series. Next, making use of the formulae presented in [5], each stress state is modelled by the equivalent state of stress with in-phase sinusoidal components. Then, the reduced stress based on the Huber-Mises distortion-energy theory is determined for each equivalent state of stress. Finally, taking advantage of the constant-amplitude stress model [6], the set of the reduced stresses is modelled by an effective stress. The corresponding algorithm is described in the following section.

Having determined the amplitude and the frequency of the effective stress, the fatigue lifetime can be calculated in the high-cycle fatigue range from the S-N curve modified to account for an actual condition (surface roughness, geometry, environment).

For the sake of simplicity the effects of mean stress are neglected and the stress components are assumed to be physically independent of each other.

CALCULATION OF THE EFFECTIVE STRESS

Consider a set of multiaxial stress states $S_1, S_2, ..., S_r, S_{r+1}, ...$with Cartesian stress components :

$$\sigma_{ri}(t) = \sigma_{ri}\left(t + \tau_{or}\right)$$

$$i = xx,\ yy,\ zz,\ xy,\ yz,\ zx \qquad (1)$$

$$r = 1, 2, ...$$

where τ_{or} is the common period of the components in r-th stress state.

After expansion of these components in the Fourier series one gets :

$$\sigma_{ri}(t) = \sum_{p=1}^{\infty} \sigma_{rip} \sin\left(p\omega_{or}t + \alpha_{rip}\right)$$

$$\omega_{or} = 2\pi / \tau_{or} \qquad (2)$$

where σ_{rip} and α_{rip} are the amplitude and the phase angle of p-th term.

According to Cempel's theory of energy transforming systems [7], the r-th stress state, S_r, can be modelled by the equivalent stress state, $S_r^{(eq)}$, with in-phase components :

$$\sigma_{ri}^{(eq)} \sin \omega_r t \tag{3}$$

Here $\sigma_{ri}^{(eq)}$ is the amplitude of i-th component in the r-th equivalent stress state calculated in compliance with [5] :

$$\sigma_{ri}^{(eq)} = \left\{ \frac{8}{k_r^2 \tau_{or}} \int_0^{\tau_{or}} \left[\sum_{p=1}^{\infty} \sigma_{rip} \sin\left(p\omega_{or} t + \alpha_{rip}\right) \right]^2 \cdot \right.$$

$$\left. \cdot \left[\sum_{p=1}^{\infty} p\sigma_{rip} \cos\left(p\omega_{or} t + \alpha_{rip}\right) \right]^2 dt \right\}^{1/4} \tag{4}$$

and ω_r is the multiple of the fundamental circular frequency in the r-th state, ω_{or}, given by :

$$\omega_r = k_r \omega_{or} \qquad k_r = Round\ (\kappa_r) \tag{5}$$

$$\kappa_r = \left[\frac{\sum_i \sum_{p=1}^{\infty} \eta_i \left(\frac{p\sigma_{rip}}{E_i} \right)^2}{\sum_i \sum_{p=1}^{\infty} \eta_i \left(\frac{\sigma_{rip}}{E_i} \right)^2} \right]^{1/2} \tag{6}$$

In (6) :

E_i - Young or shear modulus

η_i - the coefficient of internal viscous damping in Kelvin-Voigt's model of a given material [8], both corresponding to i-th Cartesian coordinate.

For the equivalent stress states $S_1^{(eq)}$, $S_2^{(eq)}$, ..., the reduced stresses :

$$\sigma_r \sin \omega_r t \qquad r = 1, 2, \ldots \tag{7}$$

can be determined by means of the classical strength theories (Huber-Mises or Tresca). This opinion follows from the literature on fatigue under in-phase loading [9,10]. If Huber-Mises strength theory is chosen, the amplitudes of the reduced stresses are calculated in compliance with [8] :

$$\sigma_r = \left\{ \left(\sigma_{rxx}^{(eq)}\right)^2 + \left(\sigma_{ryy}^{(eq)}\right)^2 + \left(\sigma_{rzz}^{(eq)}\right)^2 - \sigma_{rxx}^{(eq)}\sigma_{ryy}^{(eq)} - \sigma_{ryy}^{(eq)}\sigma_{rzz}^{(eq)} + \right.$$

$$\left. - \sigma_{rzz}^{(eq)}\sigma_{rxx}^{(eq)} + 3\left[\left(\sigma_{rxy}^{(eq)}\right)^2 + \left(\sigma_{ryz}^{(eq)}\right)^2 + \left(\sigma_{rzx}^{(eq)}\right)^2 \right] \right\}^{1/2} \tag{8}$$

The amplitudes $\sigma_{ri}^{(eq)}$ are defined by (4) as positive quantities in comparison with the analogous formula for the reduced stress in the general state of static stress where positive or negative values of stress components can be inserted. Therefore the signs „-" in (8) must be replaced with „+", if necessary, in order to account for the bending stress in outer fibres on both sides of the cross-section with respect to the neutral axis of structural elements.

The constant-amplitude stress model developed in [6] with the aid of Cempel's theory of energy transforming systems [7] makes it possible to model the set of reduced stresses by the effective stress :

$$\sigma_{eff} \sin \omega_{eff} t \tag{9}$$

if the durations τ_1, τ_2, ... of the stress states S_1, S_2, ... are given.

The calculation formulae for the amplitude and circular frequency of the effective stress are acc. to [6] as follows :

$$\sigma_{eff} = \left(\frac{\sum_r \omega_r^2 \sigma_r^4 \tau_r \sum_r \sigma_r^2 \tau_r}{\sum_r \omega_r^2 \sigma_r^2 \tau_r \sum_r \tau_r} \right)^{1/4} \tag{10}$$

$$\omega_{eff} = \left(\frac{\sum_r \omega_r^2 \sigma_r^2 \tau_r}{\sum_r \sigma_r^2 \tau_r} \right)^{1/2} \tag{11}$$

APPLICATION OF THE EFFECTIVE STRESS TO HIGH-CYCLE FATIGUE ASSESSMENT OF STRUCTURAL ELEMENTS

According to Huber-Mises strength theory the reduced stress is referred to as the equivalent tensile stress. It means that the high-cycle fatigue analysis based on the effective stress expressed by (9) can be performed with the aid of the design S-N curve for symmetrical tension-compression. Its equation is usually given in the form :

$$N\sigma^m = K \tag{12}$$

valid for :

$$Z_{rc} < \sigma \le L \tag{13}$$

where :

- N - number of stress cycles to failure
- σ - stress amplitude
- K, m - selected constants
- Z_{rc} - fatigue limit
- L - maximum stress amplitude satisfying (12).

From (10) it follows that if the inequalities :

$$Z_{rc} < \sigma_r \le L \qquad r = 1,2, \ldots \tag{14}$$

are fulfilled, the condition :

$$Z_{rc} < \sigma_{eff} \le L \tag{15}$$

is also met.

In this case the fatigue lifetime, τ, can be calculated as follows :

$$\tau = \frac{2\pi N}{\omega_{eff}} = \frac{2\pi K}{\omega_{eff}} \sigma_{eff}^{-m} \tag{16}$$

The cumulative fatigue damage ratio [1] :

$$D = \sum_r \frac{n_r}{N_r} = \frac{1}{2\pi K} \sum_r \omega_r \sigma_r^m \tau_r \tag{17}$$

can be also evaluated with the use of (12) for the set of the reduced stresses expressed by (7) where :

n_r - the number of stress cycles applied at stress amplitude σ_r

N_r - the number of cycles to failure at stress amplitude σ_r.

However (16) makes it possible to define the alternative damage ratio :

$$D_{eff} = \frac{\sum_r \tau_r}{\tau} = \frac{\omega_{eff}\sigma_{eff}^m}{2\pi K}\sum_r \tau_r \qquad (18)$$

The quantity D_{eff} can be called the effective fatigue damage ratio. The results of comparative calculations of the ratios D and D_{eff} at variable uniaxial stresses are discussed in [6].

Example

Problem

There are two load states S_1 and S_2 of durations $\tau_1 = \tau_2 = 5\cdot10^4$ s. Consider the effective fatigue damage ratio at the point of the structural element where the Fourier analysis of stress components in both these states gives (stress amplitudes are given in MPa) :

$$S_1 \left| \begin{array}{l} \sigma_{1xx} = 195.6\sin\omega_{01}t + 35\sin3\omega_{01}t + 35\sin5\omega_{01}t \\ \sigma_{1xy} = 118.8\sin4\omega_{01}t + 59.6\sin8\omega_{01}t \qquad \omega_{01}=1.5\ s^{-1} \\ \sigma_{1yy} = \sigma_{1zz} = \sigma_{1yz} = \sigma_{1zx} = 0 \end{array} \right.$$

$$S_2 \left| \begin{array}{l} \sigma_{2xx} = 254.65\sin\omega_{02}t + 84.88\sin3\omega_{02}t + 50.93\sin5\omega_{02}t \\ \sigma_{2yy} = \sigma_{2zz} = \ldots = \sigma_{2zx} = 0 \qquad \omega_{02}=2.0\ s^{-1} \end{array} \right.$$

The material constants are:

$E_{xx} = E = 2.1\cdot10^5$ MPa $\quad E_{xy} = G = 8.077\cdot10^4$ MPa

$Z_{rc} = 175$ MPa $\quad K = 5.36\cdot10^{12}$

$L = 450$ MPa $\quad m=3 \quad \eta_{xx} = \eta_{xy}$

Solution

From (4)÷(6) the parameters of the equivalent stress states are obtained:

$S_1^{(eq)}$: $\sigma_{1xx}^{(eq)} = 113.2$ MPa $\qquad \sigma_{1xy}^{(eq)} = 165.5$ MPa

$\kappa_1 = 4.43 \quad k_1 = 4 \quad \omega_1 = 6\ s^{-1}$

$S_2^{(eq)}$: $\sigma_{2xx}^{(eq)} = 240$ MPa $\quad \kappa_2 = 1.6 \quad k_2 = 2 \quad \omega_2 = 4\ s^{-1}$

Eq.(8) yields the following amplitudes of the reduced stresses :

$\sigma_1 = 308$ MPa $\qquad \sigma_2 = 240$ MPa

so that the amplitude and the circular frequency of the effective stress amount to :

$$\sigma_{eff} = \left[\frac{\left(6^2\cdot308^4\cdot5\cdot10^4 + 4^2\cdot240^4\cdot5\cdot10^4\right)}{\left(6^2\cdot308^2\cdot5\cdot10^4 + 4^2\cdot240^2\cdot5\cdot10^4\right)}\cdot \frac{\left(308^2\cdot5\cdot10^4 + 240^2\cdot5\cdot10^4\right)}{\left(5\cdot10^4 + 5\cdot10^4\right)}\right]^{1/4} = 285\ \text{MPa}$$

$$\omega_{eff} = \left(\frac{6^2\cdot308^2\cdot5\cdot10^4 + 4^2\cdot240^2\cdot5\cdot10^4}{308^2\cdot5\cdot10^4 + 240^2\cdot5\cdot10^4}\right)^{1/2} = 5.33\ s^{-1}$$

Hence the effective damage ratio, in accordance with (18), equals to :

$$D_{eff} = \frac{5.33\cdot285^3}{2\pi\cdot5.36\cdot10^{12}}\left(5\cdot10^4 + 5\cdot10^4\right) = 0.349$$

It may be interesting to note that the cumulative damage ratio calculated from (17) for the above reduced stresses is D=0.326.

CONCLUDING REMARKS

The described algorithm of high-cycle fatigue assessment consists of :

- modelling multiaxial states of periodic stress by the equivalent states with in-phase components
- evaluating the reduced stresses
- employing the constant-amplitude stress model for calculation of the effective stress
- estimating the fatigue life.

It involves Fourier analysis of stress components and includes additional material constants (Young and shear moduli, coefficients of internal damping). The calculation formulae are based on the S-N curve equation , Cempel's theory of energy transforming systems and Huber-Mises strength theory. Hereby the application of the conventional cycle counting method and fatigue damage cumulation rule is not required.

NOMENCLATURE

D - cumulative fatigue damage ratio

D_{eff} - fatigue damage ratio at stress amplitude σ_{eff}

E_i - modulus of elasticity associated with i-th stress component (i=xx, yy, zz, xy, yz, zx)

k_r - natural number obtained by rounding the number κ_r (r = 1, 2,...)

K, m - constants in Eq. (12)

L - maximum stress amplitude satysfying Eq. (12)

n_r - number of cycles at stress amplitude σ_r

N - number of stress cycles to failure

N_r - number of cycles to failure at stress amplitude σ_r

t - time

Z_{rc} - fatigue limit at tension-compression

α_{rip}, σ_{rip} - phase angle and amplitude of p-th term in Fourier expansion of σ_{ri} (p=1,2,...)

κ_r - number given by Eq. (6)

η_i - coefficient of internal damping of the material, associated with i-th stress component

σ - stress amplitude in Eq. (12)

σ_{eff}, ω_{eff} - amplitude and circular frequency of the effective stress

σ_r, ω_r - amplitude and circular frequency of the reduced stress in r-th load state

σ_{ri} - i-th stress component in r-th load state

$\sigma_{ri}^{(eq)}$ - amplitude of i-th component in r-th equivalent stress state

τ - fatigue lifetime

τ_{or}, ω_{or} - period and fundamental circular frequency of the stress in r-th load state

τ_r - duration of r-th load state

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