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Fatigue strength assessment of beams under variable combined random stress

SUMMARY

The paper deals with high-cycle fatigue of the beams subjected to variable combined load consisting of bending moment, torsional moment and axial force. It is assumed that the envisaged service life can be considered as a set of states and that the stress components in each state are stationary random processes of known power spectral densities. An equivalent state of stress is determined which makes formulating the fatigue criteria possible.

INTRODUCTION

For design purposes fatigue analysis of ship hull structures is usually based on fatigue strength tests (S-N curves) and a suitable structure-load model. In the following some problems are considered which concern the beam elements of the hull structure loaded by variable wave forces. Since the wave loads fluctuate in a stochastic manner, it is assumed that the response of a structural member is given in terms of power spectral densities (PSDs) of the resulting stress and its derivative. To estimate the fatigue damage one must simulate the time history representative of the stress PSD. However, the simulation requires taking into account a finite number of sinusoidal frequency components, whereas the response PSD is made up of contributions of an infinite number of modes of vibration of the structure [1]. Therefore an alternative way of fatigue strength assessment by means of PSDs and Cempel's theory of energy transforming systems [4] was developed [2,3]. It consists in modelling the actual state of stationary random stress by the equivalent stress state with random-amplitude components and in fatigue analysis based on such a model. Similar approach is presented here for the case of a sequence of stationary states of random stress which are modelled by one equivalent stress state with random-amplitude components. It is assumed that the stress components are statistically independent zero-mean processes.

EQUIVALENT STATE OF STRESS UNDER VARIABLE CONDITIONS

To determine the equivalent stress state for a sequence of states of stationary random stress, the following equivalence conditions between one stress state A of the duration T and a set of the stress states $B_1, B_2, \dots$ of the durations $T_1, T_2, \dots$ can be used [5]:

$$\int_0^T \sigma_i^2 dt = \sum_r \int_0^{T_r} \sigma_{ri}^2 dt \qquad (1)$$

$$i = a, b, t \qquad r = 1, 2, \dots$$

$$\sum_i \frac{\eta_i}{E_i^2} \int_0^T \dot{\sigma}_i^2 dt = \sum_i \sum_r \frac{\eta_i}{E_i^2} \int_0^{T_r} \dot{\sigma}_{ri}^2 dt \qquad (2)$$

where:

- $\sigma_a, \sigma_b, \sigma_t$ - stress components due to axial force, bending moment and torsional moment, respectively
- σ_{ri} - i-th stress component in B_r-th state
- $E_a = E_b = E$ - Young modulus
- $E_t = G$ - shear modulus
- η_i - material internal viscous damping coefficient corresponding to i-th strain/ stress component

$$T = \sum_r T_r \qquad (3)$$

If the stress components in the individual states $B_1, B_2, \dots$ are stationary zero-mean processes with PSDs $G_{ri}(\omega)$, the equivalent stress components in the states can be defined as the stationary and periodic (in the sense of mean square) processes [6]:

$$\sigma_{ri}(t) = \sigma_{rio}\sin(\omega_r t + \varphi_{ri}) \tag{4}$$

where: σ_{rio} - random variable with the mean-square value V_{ri}

The PSD of the process σ_{ri} is given by (5):

$$S_{\sigma_{ri}}(\omega) = \frac{1}{4}V_{ri}\left[\delta(\omega-\omega_r)+\delta(\omega+\omega_r)\right] \tag{5}$$

The quantity V_{ri} is calculated as:

$$V_{ri} = 2\int_{-\infty}^{\infty} G_{ri}(\omega)d\omega \tag{6}$$

and the formula for the frequency ω_r is as follows:

$$\omega_r = \left[\frac{\sum_i \frac{\eta_i}{E_i^2}\int_{-\infty}^{\infty}\omega^2 G_{ri}(\omega)d\omega}{\sum_i \frac{\eta_i}{E_i^2}\int_{-\infty}^{\infty} G_{ri}(\omega)d\omega}\right]^{1/2} \tag{7}$$

The stress components in the equivalent state A can be determined in the form analogous to (4) as :

$$\sigma_i(t) = \sigma_{io}\sin(\omega_e t + \varphi_i) \tag{8}$$

where: σ_{io} - random variable with the mean-square value V_i

The PSD of the process σ_i is given by (9):

$$S_{\sigma_i}(\omega) = \frac{1}{4}V_i\left[\delta(\omega-\omega_e)+\delta(\omega+\omega_e)\right] \tag{9}$$

The PSD of the derivative processes $\dot{\sigma}_{ri}$ and $\dot{\sigma}_i$ are as follows:

$$S_{\dot{\sigma}_{ri}}(\omega) = \omega_r^2 S_{\sigma_{ri}}(\omega) \qquad S_{\dot{\sigma}_i}(\omega) = \omega_e^2 S_{\sigma_i}(\omega) \tag{10}$$

In the considered case, (1) and (2) must be replaced with equivalence conditions imposed on integrals of appropriate PSDs in the frequency domain. In view of energy balance, each of the integrals must be multiplied by the corresponding time period, i.e.:

$$T\int_{-\infty}^{\infty} S_{\sigma_i}(\omega)d\omega = \sum_r T_r \int_{-\infty}^{\infty} S_{\sigma_{ri}}(\omega)d\omega \tag{11}$$

$$T\sum_i \frac{\eta_i}{E_i^2}\int_{-\infty}^{\infty} S_{\dot{\sigma}_i}(\omega)d\omega = \sum_i\sum_r T_r\frac{\eta_i}{E_i^2}\int_{-\infty}^{\infty} S_{\dot{\sigma}_{ri}}(\omega)d\omega \tag{12}$$

After substituting (5), (9) and (10) into (11) and (12) one gets the following:

$$TV_i = \sum_r T_r V_{ri} \tag{13}$$

$$T\sum_i \frac{\eta_i}{E_i^2}\omega_e^2 V_i = \sum_i\sum_r T_r\frac{\eta_i}{E_i^2}\omega_r^2 V_{ri} \tag{14}$$

Hence:

$$\omega_e = \left(\frac{\sum_i\sum_r T_r \frac{\eta_i}{E_i^2}\omega_r^2 V_{ri}}{\sum_i\sum_r T_r\frac{\eta_i}{E_i^2}V_{ri}}\right)^{1/2} \tag{15}$$

$$V_i = \frac{1}{T}\sum_r T_r V_{ri} \tag{16}$$

FATIGUE CRITERIA

The above derived formulae may find application in various engineering problems. In particular they can be useful for comparing the effort of structural elements under variable conditions, in parametric studies of structures and for fatigue design criteria formulation. The criteria presented in [3] for beams under random loading can also be used in the latter problem, if the amplitudes of the equivalent stress components are zero-mean Gaussian variables. To avoid the limitation, the fatigue design criteria for beams under periodic loading should be considered [3] as follows:

$$f = \left[\left(\frac{\sigma_a^{(eq)}}{Z_a}+\frac{\sigma_b^{(eq)}}{Z_b}\right)^2+\left(\frac{\sigma_t^{(eq)}}{Z_t}\right)^2\right]^{-1/2} \geq 1 \tag{17}$$

in the safe region of the basic variable space, and:

$$\frac{\omega_e T}{2\pi}\left\{\left[\frac{\left(\sigma_a^{(eq)}\right)^{m_a}}{K_a}+\frac{\left(\sigma_b^{(eq)}\right)^{m_b}}{K_b}\right]^2+\left[\frac{\left(\sigma_t^{(eq)}\right)^{m_t}}{K_t}\right]^2\right\}^{1/2} \leq 1 \tag{18}$$

in the failure subregion, i.e. for:

$$f < 1 \leq l \tag{19}$$

where:

Z_i - faigue limit at i-th simple loading
$\sigma_i^{(eq)}$ - amplitude of i-th equivalent stress component
K_i, m_i - material dependent constants in the equations of the S-N curves (20):

$$N_i\sigma_i^{m_i} = K_i \tag{20}$$

$$l = \left[\left(\frac{\sigma_a^{(eq)}}{L_a}+\frac{\sigma_b^{(eq)}}{L_b}\right)^2+\left(\frac{\sigma_t^{(eq)}}{L_t}\right)^2\right]^{-1/2} \tag{21}$$

where: L_i - maximum stress amplitude satisfying (20).

Passing now to the processes (4) and (8), the following can be written:

$$E\left\{\left(\frac{\sigma_{rao}}{Z_a}+\frac{\sigma_{rbo}}{Z_b}\right)^2+\left(\frac{\sigma_{rto}}{Z_t}\right)^2\right\}\le 1 \qquad (22)$$

as the criterion in design for the infinite fatigue life, and:

$$\left(\frac{\omega_e T}{2\pi}\right)^2 E\left\{\left(\frac{\sigma_{ao}^{m_a}}{K_a}+\frac{\sigma_{bo}^{m_b}}{K_b}\right)^2+\left(\frac{\sigma_{to}^{m_t}}{K_t}\right)^2\right\}\le 1 \qquad (23)$$

as the criterion in design for a finite fatigue life. The latter is valid if:

$$E\left\{\left(\frac{\sigma_{rao}}{Z_a}+\frac{\sigma_{rbo}}{Z_b}\right)^2+\left(\frac{\sigma_{rto}}{Z_t}\right)^2\right\}>1\ge$$

$$\ge E\left\{\left(\frac{\sigma_{rao}}{L_a}+\frac{\sigma_{rbo}}{L_b}\right)^2+\left(\frac{\sigma_{rto}}{L_t}\right)^2\right\} \qquad (24)$$

where: E - expected value.

Hence, (25), (26) and (27) is obtained under assumption that the stress components are statistically independent of each other:

$$\sum_i \frac{V_{ri}}{Z_i^2}+\frac{2}{Z_a Z_b}E\{\sigma_{rao}\}E\{\sigma_{rbo}\}\le 1 \qquad (25)$$

$$\left(\frac{\omega_e T}{2\pi}\right)^2\left(\sum_i \frac{1}{K_i^2}E\{\sigma_{io}^{2m_i}\}+\frac{2}{K_a K_b}E\{\sigma_{rao}^{m_a}\}E\{\sigma_{rbo}^{m_b}\}\right)\le 1 \qquad (26)$$

$$\sum_i \frac{V_{ri}}{Z_i^2}+\frac{2}{Z_a Z_b}E\{\sigma_{rao}\}E\{\sigma_{rbo}\}>1\ge$$

$$\ge \sum_i \frac{V_{ri}}{L_i^2}+\frac{2}{L_a L_b}E\{\sigma_{rao}\}E\{\sigma_{rbo}\} \qquad (27)$$

The moments in (25) through (27) depend on the type of probability density functions of the amplitudes of the original stress components.

If a stress is a narrow-band normal process, the distribution of its amplitude follows Rayleigh's distribution. Assuming that this distribution can be applied to the amplitudes σ_{rio} and σ_{io}, the following is obtained:

$$E\{\sigma_{rio}\}=2^{1/2}\Gamma(1+0.5)s_{ri} \qquad (28)$$

$$E\{\sigma_{io}^{m_i}\}=2^{m_i/2}\Gamma(1+0.5m_i)s_i^{m_i} \qquad (29)$$

where: s_{ri}, s_i - standard deviations of the amplitudes s_{rio} and s_{io}, respectively

The quantity s_i can be determined from the condition that the mean-square values given by (16) and (29) are equal which yields (30):

$$s_i=\left[\frac{V_i}{2\Gamma(1+1)}\right]^{1/2}=(0.5V_i)^{1/2} \qquad (30)$$

Similarly:

$$s_{ri}=(0.5V_{ri})^{1/2} \qquad (31)$$

CONCLUDING REMARKS

Errors may be involved by approximating the stress components in different service conditions by the equivalent stress components (8) and the applied assumptions. Moreover, the calculation of the moments in (25) through (27) may be the source of uncertainties. Therefore the left-hand side of (26) should be made substantially less than one.

NOMENCLATURE

E	-	expected value, Young modulus
$G_{ri}(\omega)$	-	power spectral density (PSD) of i-th stress component in B_r-th state
i = a, b, t	-	lower index corresponding to stress components resulting from axial force, bending moment and torsional moment, respectively
r	-	state number
s_{σ_i}, $s_{\dot\sigma_i}$	-	PSD of the process σ_i and of its derivative
$s_{\sigma_{ri}}$, $s_{\dot\sigma_{ri}}$	-	PSD of the process σ_{ri} and of its derivative
t	-	time
T	-	summary duration of the considered stress states
T_r	-	duration of B_r-th stress state
V_i, s_i	-	mean-square value and standard deviation of σ_{io}
V_{ri}, s_{ri}	-	mean-square value and standard deviation of σ_{rio}
Γ	-	gamma function
φ_i, φ_{ri}	-	random variables
σ_i, σ_{io}	-	i-th equivalent stress component and its amplitude for the set of stationary random stress states
σ_{ri}, σ_{rio}	-	i-th equivalent stress component and its amplitude for the B_r-th state of stationary random stress
ω	-	circular frequency
ω_e	-	circular frequency of the equivalent stress for the set of stationary random stress states
ω_r	-	circular frequency of the equivalent stress for the B_r-th state of stationary random stress

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