MACIEJ TACZAŁA, D.Sc.,N.A.
Technical University of Szczecin
Faculty of Maritime Technology
Ocean and Ship Technology Institute

Bifurcation load determination by analyzing imperfect structures

SUMMARY

In the paper a method is presented of bifurcation load determination by analyzing imperfect structural systems with no limit points. Results of the calculations carried out by applying the method to square plates with different boundary conditions at midsurface edges and different initial deflection values are presented and discussed.

The paper presented at the Symposium on Ship Structure and Mechanics - Ultimate Capacity of Ship Structures , held ad memoriam of Prof. M. Kmiecik, 20 and 21 March 1996, Szczecin. The concept of the paper emerged from the discussions held by the author with Prof. Kmiecik and is based on his unpublished paper [5].

INTRODUCTION

In the stability analysis the term of „buckling" is usually related to the phenomenon of equilibrium state bifurcation of a structure. The notion of bifurcation is associated with ideal structural systems i.e. those having no imperfection of any kind: either due to material, workmanship, loading or boundary conditions. Bifurcation buckling is connected with the loss of deformation symmetry. Obviously, real structures do not manifest the symmetry due to inherent imperfections, therefore the bifurcation buckling does not exist for them. If an unstable point of bifurcation appears on the equilibrium path of an ideal structure, then a limit point appears on the path of an imperfect structure, which corresponds to it. In the case the buckling consists in the snap through appearing on the further part of equilibrium states curve. The bifurcation load can be taken in this case as a limit value of the critical load determined for imperfect structures with the gradually reduced imperfection [7]. The situation is different when there is a stable bifurcation point for an ideal structure. Then no limit point appears on the real structure equilibrium path. A few simple examples which illustrate the problem are presented in the further part of the article.

The bifurcation load seems to be a theoretical concept only as there are no ideal structures in the real world. However in the structural stability analysis the determination of a critical load understood as a bifurcation load is important. Firstly it is a good way of critical load assessment which makes the determination of post-buckling deformations by post-bifurcation analysis methods also possible. This is because the bifurcation load is determined by solving an eigenvalue problem after linearization of the problem being solved. Moreover, the determining of an imperfection form and value is not possible or too complicated in most cases.

A method of the bifurcation load calculation for the imperfect structures with the stable bifurcation point of the corresponding ideal structures is presented here. The method was proposed by M.Kmiecik [5].

CRITICAL POINTS

The model presented by Bergan [1] and Kleiber [3] is an illustrative example of instability of ideal and real structures. This is a space three-bar framework loaded by the conservative force P (Fig.1).

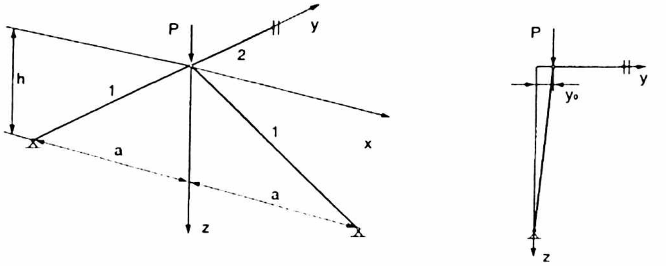

Fig. 1. Space framework of two degrees of freedom

Equilibrium states of the system were determined on the basis of the equilibrium conditions which emerge from vanishing variation of virtual displacement potential. The system potential is given by the following equation:

$$\Pi = \frac{E_1A_1}{l_1}(\Delta l_1)^2 + \frac{1}{2}\frac{E_2A_2}{l_2}(\Delta l_2)^2 - Pz \qquad (1)$$

where:

Δl_1 - contraction of the bar 1,

Δl_2 - contraction of the bar 2, given by (2) and (3):

$$\Delta l_1 = \sqrt{a^2 + y^2 + (h-z)^2} - \sqrt{a^2 + y_0^2 + h^2} \tag{2}$$

$$\Delta l_2 = y - y_0 \tag{3}$$

The equilibrium state of the system is determined from the condition (4):

$$\delta\Pi = 0 \tag{4}$$

Typical equilibrium curves of geometrically nonlinear systems can be exemplified on the basis of the framework in question (Fig.2).

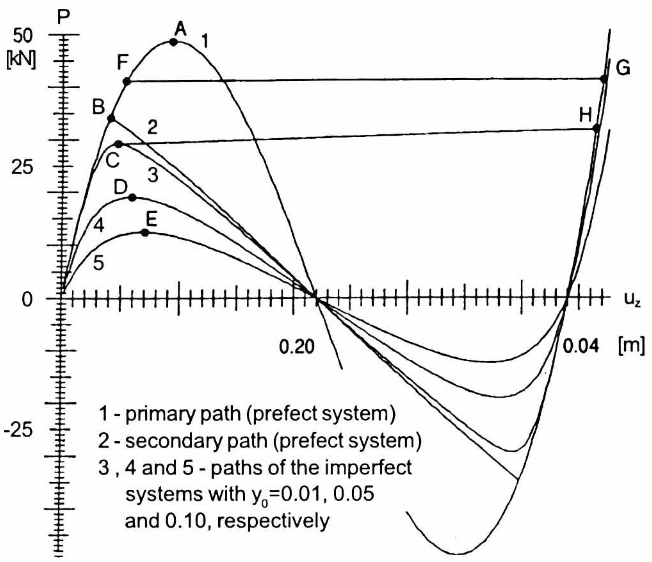

Fig. 2. Equilibrium paths of an ideal bar system and that with imperfections

It can be learned, when examining behaviour of the perfect system (y_0=0), that two courses of equilibrium states are possible. Bifurcation appears in the point B. The primary equilibrium path 1 could be theoretically continued, but any arbitrarily small disturbance, when the bifurcation point is passed, is able to cause the system to enter the secondary path 2. The system would be in the point G, having been disturbed in the point F, because the point of bifurcation B is unstable. The limit point A on the primary path corresponds to the case of the stability loss of two bars at zero displacement, y = 0.

If the imperfect systems ($y_0 \neq 0$) are examined (e.g. 3, 4, 5 in Fig.2) the limit points (C,D,E respectively) appear on their equilibrium paths. The real structure buckling consists in the snap-through to the farther part of the curve of equilibrium states (e.g. along the line CH).

The point B in Fig. 2 belongs to the unstable bifurcation points. The systems with such points are sensitive to imperfections as they lower the critical load of the system.

The next example illustrates stability of the systems with stable bifurcation points. Fig.3 presents the system which consists of a stiff undeformable bar and a spring attached to the upper end of the bar. The bar can rotate in its lower point. The system has the imperfections (initial deformation): the angle θ_0 and displacement y_0. The bar rotates under the load P about its lower end causing the spring to elongate.

The system potential can be expressed by (5):

$$\Pi = \frac{1}{2}k\left[L(\theta-\theta_0)\right]^2 - Pz = \frac{1}{2}k\left[L(\theta-\theta_0)\right]^2 - PL\left[\cos\theta_0 - \cos\theta\right] \tag{5}$$

Equilibrium of the system can be expressed as follows:

$$\delta\Pi = \left[kL^2(\theta-\theta_0) - PL\sin\theta\right]\delta\theta = 0 \tag{6}$$

The equilibrium condition at θ_0=0 has the following form:

$$P = kL\frac{\theta}{\sin\theta} \tag{7}$$

The equation (7) is represented by the curve 0 in Fig.4.

If the angle θ tends to 0, the following bifurcation load value is obtained:

$$P_B = kL \tag{8}$$

The curves 1 to 3 in Fig.4 illustrate the equilibrium path courses at different progressing imperfection values ($\theta_0 \neq 0$). If k = 100 N/m and L = 1 m is assumed the bifurcation load value reaches 100 N.

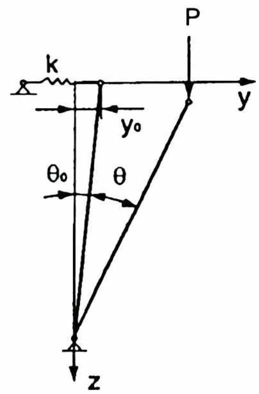

Fig. 3. Structural system composed of a stiff bar and spring

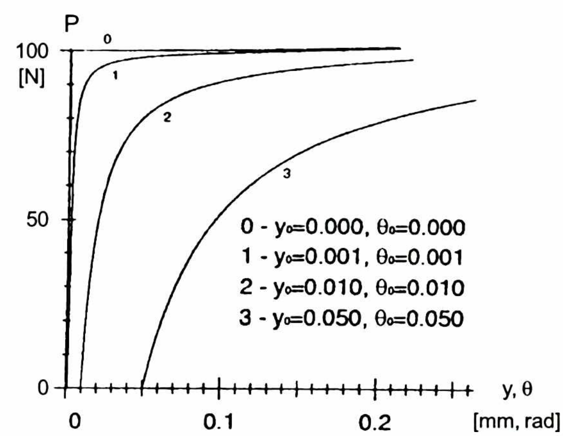

Fig. 4. Equilibrium paths of the prefect and imperfect stiff bar-spring systems

BIFURCATION POINT DETERMINATION

The bifurcation point of the systems which behave like that shown in Fig.3 is stable. It means that only a small displacement increment appears in the perfect system in response to a small disturbance, after the bifurcation load has been somewhat exceeded, opposite to the case of the system with unstable bifurcation point (see Fig.1), where a large displacement increment can be observed. It can be found, while considering behaviour of the imperfect systems, that when the load approaches the critical load value a substantial displacement increment (of the maximum value) happens, corresponding to a small load increase. Therefore the load increment ΔP lessens while approaching bifurcation load level.

Let s denote a system displacement measure (arc length):

$$s = \int ds, \quad ds^2 = dx_i dx_i \tag{9}$$

where:

dx_i - displacement increment at i-th degree of freedom.

It is assumed that the function ΔP reaches its minimum at the bifurcation load value:

$$\Delta P \to \min \tag{10}$$

which corresponds to the following condition:

$$d(\Delta P) = 0 \tag{11}$$

The expression (13) is obtained after expansion of the function P into Taylor series around a point with coordinate s, and with only the first term of the expansion left :

$$P(s+\Delta s) = P(s) + \frac{dP}{ds}\Delta s \tag{12}$$

$$\Delta P = P(s+\Delta s) - P(s) = \frac{dP}{ds}\Delta s \tag{13}$$

When assuming the length Δs constant, the differential $d(\Delta P)$ can be calculated, as follows:

$$d(\Delta P) = \frac{d}{ds}\left[\frac{dP}{ds}\Delta s\right]ds = \left[\frac{d^2P}{ds^2}\Delta s\right]ds \tag{14}$$

Therefore the condition of reaching the minimum of the function ΔP is the following:

$$\frac{d^2P}{ds^2} = 0 \tag{15}$$

or it is the inflexion point appearance on the curve P-s.

In Fig.5 courses of the curves of the load P and its first and second derivatives versus the coordinate s are shown.

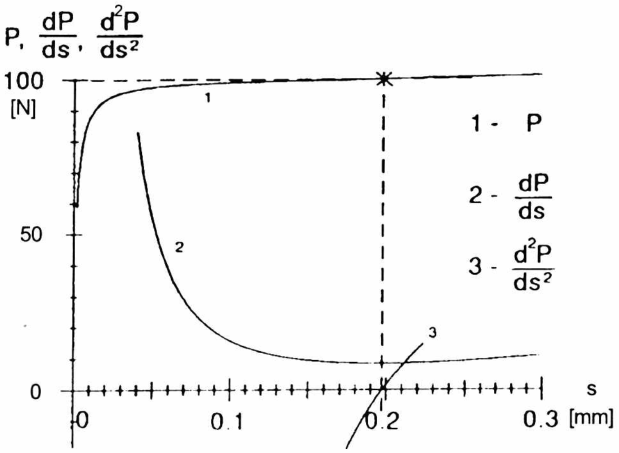

Fig. 5. Bifurcation load determination of the bar- spring system

Critical load values obtained with different imperfection values are given in Tab.1.

Tab. 1. Critical load values of the bar-spring system with different imperfection values

y_0 imperfection [mm]	critical load [N]
0.050	95.9224
0.010	98.7465
0.001	99.8168
0.000	100.0000

Limit load values converge to the bifurcation load, i.e. 100 N, as the imperfection diminishes.

PLATE CALCULATIONS

A course of the equilibrium curves analogical to that of the case shown in the previous example can be obtained for plates, e.g. the square ones, simply supported and axially compressed, with the initial deflection form relevant to their buckling form (single deflection half-wave both length- and widthwise, see Fig.6).

The bifurcation load (critical stress) of the simply-supported square plate is given by the following known formula :

$$\sigma_{buck} = 4\frac{\pi^2 E}{12(1-\nu^2)}\left(\frac{t}{b}\right)^2 \tag{16}$$

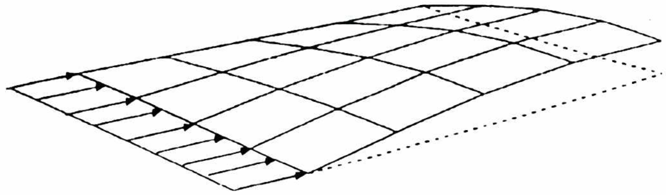

Fig.6. The analyzed square plate (a quarter of panel)

When assuming:

- the plate dimensions: c=b=1026 mm, t=10 mm
- the material constants: E= 210 000 N/mm², ν=0.3 the value of σ_{buck} = 72.12 N/mm² is achieved which, if related to the yield point σ_y = 240 N/mm², amounts to $\sigma_{buck}/\sigma_y = 0.3$.

FEM calculations were carried out of the plate divided into 4x4 elements, with different initial deflection values. Their results are shown in Fig.7 and 8 where σ_{ym} is the average stress value in the load direction at the compressed edge.

The calculations were also performed of the plate with the initial deflection value w_0/t = 0.010, divided into 2x2 and 8x8 elements. The bifurcation load of the plates was calculated by using the presented method. The load derivative $d^2\sigma_{ym}/ds^2$ was calculated with the application of the finite difference method and higher-order approximation [4]:

$$\left(\frac{d^2\sigma_{ym}}{ds^2}\right)_i = \frac{1}{12\Delta s^2}\left[-(\sigma_{ym})_{i-2} + 16(\sigma_{ym})_{i-1} + \right.$$
$$\left. -30(\sigma_{ym})_i + 16(\sigma_{ym})_{i+1} - (\sigma_{ym})_{i+2}\right] \tag{17}$$

where, on the basis of (9), Δs is the finite displacement increment given by (18):

$$\Delta s^2 = \Delta x_i \Delta x_i \tag{18}$$

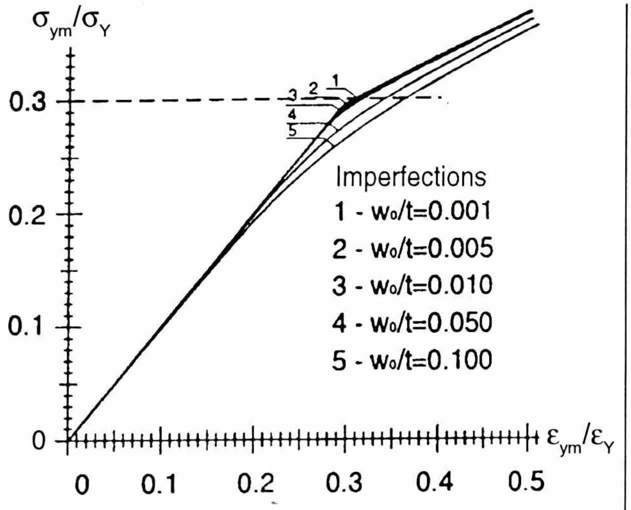

Fig. 7. Equilibrium paths of the compressed square plates in the dimensionless stress-strain coordinate system

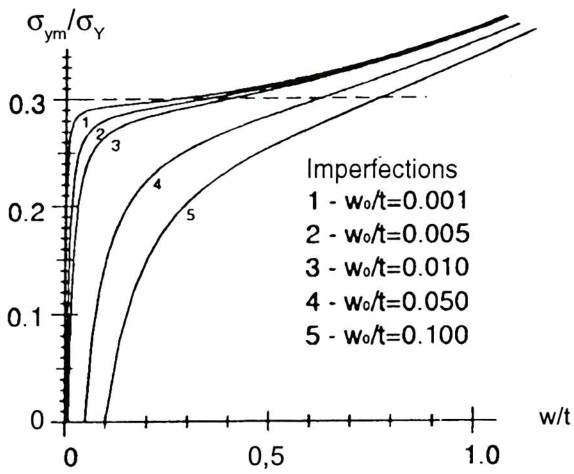

Fig. 8. Equilibrium paths of the compressed square plates in the dimensionless stress - max. deflection coordinate system

The plates were analyzed with the use of the *constant arc-length* algorithm which ensures the Δs value kept constant in consecutive load steps [2]; therefore the equation (14) is valid here. The load value at which the second derivative vanishes, was calculated by applying linear interpolation. Results of the critical load calculations (related to the material yield point σ_y) are collected in Tab.2. Initial deflection influence is weak; even the result of $\sigma_{buck} / \sigma_y = 0.3024$ obtained at the large initial deflection value of $w_0/t = 0.1$ is close to the perfect system bifurcation load value $\sigma_{buck} / \sigma_y = 0.3$.

Tab. 2. Critical load values of the square plate at different imperfection values

w_0/t imperfection	σ_{ym}/σ_y critical load
0.001	0.2957
0.005	0.2955
0.010	0.2955
0.050	0.2976
0.100	0.3024

Element mesh density influence on calculation exactness is illustrated in Fig. 9. The calculation result is very close to the exact value already in the case of application of 16 elements (1.5% difference), and the values are practically the same ($\sigma_{buck} / \sigma_y = 0.2993$) when 8x8 mesh is used.

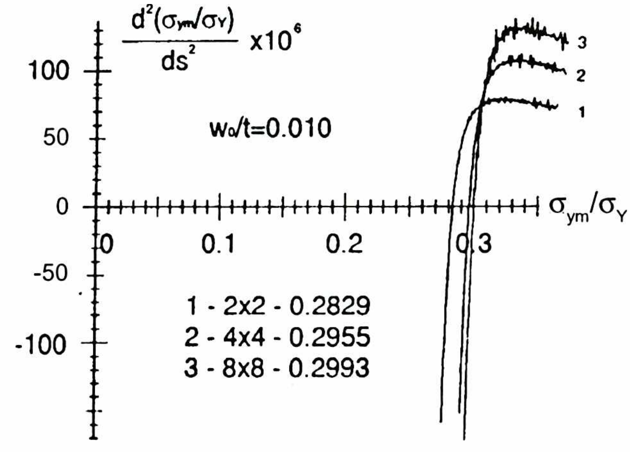

Fig. 9. $d^2(\sigma_{ym} / \sigma_y) / ds^2$ versus σ_{ym} / σ_y at different initial deflection values

The bifurcation load determination of the plate similar to that earlier considered, but with changed conditions of the mid-surface edge fixation, was used as an application example of the presented method. Entire displacement freedom of the plate edge (*unrestrained*) was assumed in the previous case. In the present example the calculations were carried out of the plate with the fixed (*restrained*) edges of the mid-surface, and of the plate with forced parallel displacement of the side edges whose shape was kept linear *(constrained)*, edges. The boundary conditions better correspond to the real fixation conditions of a plate panel in the ship hull plating structure. Based on the experience from the results of the previous example the calculations were performed with the use of 8 x 8 element mesh and the initial deflection value $w_0/t = 0.01$. Courses of the equilibrium curves of the *unrestrained, restrained and constrained* plates are shown in Fig.10.

It can be found from Fig.11 that the critical load values of the *unrestrained* ($\sigma_{buck} / \sigma_y = 0.2993$) and *constrained* ($\sigma_{buck} / \sigma_y = 0.2989$) plates are close to each other, but the value of the *restrained* plate is much smaller ($\sigma_{buck} / \sigma_y = 0.2300$). The knuckles observed on the equilibrium curves shown in Fig. 11 correspond to those values.

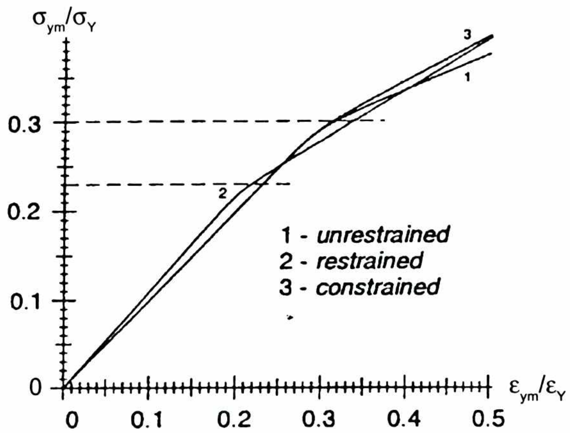

Fig. 10. Equilibrium paths of the compressed square plates with different boundary conditions

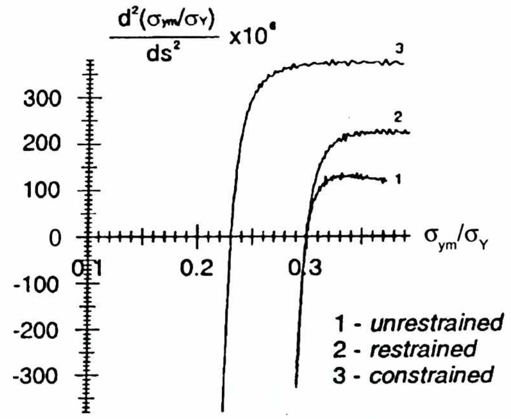

Fig. 11. $d^2(\sigma_{ym} / \sigma_y) / ds^2$ versus σ_{ym} / σ_y at different boundary conditions

CONCLUSIONS

The presented method allows to determine the bifurcation load as the critical load limit value calculated by analyzing a nonlinear structure, taking into account imperfections or laboratory experiment results. The method can be also used to verify calculation correctness of the programs for the nonlinear analysis of structures.

NOMENCLATURE

Lower indices 1, 2 - of the bar 1 or 2 respectively

DENOTATIONS

a, h	- bar system dimensions
A	- bar cross-section area
b	- plate breadth
c	- plate length
E	- Young modulus
k	- spring stiffness factor
l	- bar length
Δl	- bar length change
L	- bar length
P	- force (load)
ΔP	- increment of the load P
P_B	- bifurcation load
s	- curvilinear coordinate (measure of displacement)
Δs	- increment of the curvilinear coordinate s
t	- plate thickness
w	- plate deflection
w_0	- initial plate deflection
y	- displacement along y-axis
y_0	- initial value of the displacement y (imperfection)
z	- displacement along z- axis
ε_y	- yield point strain
ε_{ym}	- mean strain value in the load direction, at the compressed edge
θ	- slope angle
θ_0	- initial value of the slope angle q (imperfection)
$\delta\theta$	- virtual angular displacement
ν	- Poisson ratio
π	- constant (= 3.1415926...)
Π	- system energy potential
$\delta\Pi$	- potential variation
σ_{buck}	- critical stress (Euler stress)
σ_Y	- yield stress
σ_{ym}	- mean stress value in the load direction, at the compressed edge

BIBLIOGRAPHY

1. Bergan P.G.: „ Solution Algorithms for Nonlinear Structural Problems ". Computers and Structures, vol.12, 1980
2. Crisfield M.A.: „ A Fast Incremental/Iterative Solution Procedure that Handles « Snap - Through « ". Computers and Structures, vol.13, 1981
3. Kleiber M., Woźniak C.: „ Nonlinear Mechanics of Structures ". PWN, Warszawa, Kluwer Academic Publishers, Dordrecht/Boston/London, 1991
4. Kmiecik M.: „ Metody rozwiązywania równań nieliniowej teorii płyt ". Prace Naukowe Politechniki Szczecińskiej, no. 162, 1981
5. Kmiecik M.: „ Bifurcation Load Evaluation of Imperfect Structures ". 1993, (unpublished)
6. Riks E.: „ An Incremental Approach to the Solution of Snapping and Buckling Problems ". International Journal of Solids and Structures, vol.15, 1979
7. Waszczyszyn Z.: „ Numerical Problems of Nonlinear Stability Analysis of Elastic Structures". Computers and Structures, vol.17, 1983

Appraised by Tadeusz Jastrzębski, Assoc.Prof.,D.Sc.

Conferences

III EAST - WEST CONGRESS ON ENGINEERING EDUCATION

The Congress under the theme:
„ *Re-vitalising Academia/Industry links* "
was held at the Gdynia Maritime Academy,
Gdynia, between 15 and 20 September, 1996.

The paramount aim of the Congress has been to prolong interest in engineering education among academics and industry leaders worldwide. Another important objective of it has been to promote and continue the international co-operation between developed and developing countries and Central and Eastern Europe, which was so well initiated at the first East - West Congress held at the Jagiellonian University of Kraków in 1991 and was continued at the second Congress held at the Technical University of Łódz in 1993.

III Congress was organized by joint efforts of the UNESCO Supported International Centre for Engineering Education (USICEE) at Monash University, Melbourne, Austalia, Gdynia Maritime Academy, International Liaison Group for Engineering Education, and Australasian Association for Engineering Education.

The Congress enjoyed the patronage and support of the Australian Ambassador to the Czech Republic, Poland and Slovakia, His Excellency M. Jonathan Thwaites who took part in the Congress. Also UNESCO decided to sponsor the Congress and Dr Adnan Badran, Deputy Director of UNESCO, represented this organization at the Congress and gave the opening address.

Prof. Zenon Pudlowski from Monash University was the Chairman of the Congress, and Prof. Janusz Mindykowski and Prof. Romuald Cwilewicz from Gdynia Maritime Academy were co-organizers of the meeting.

III Congress presented and discussed research and development activities on engineering education carried out throughout the world. Particular emphasis was placed on the chosen theme *Re-vitalising Academia/Industry Links* to stress the importance and revelance of collaboration between university and industry in engineering education and industrial training in an area of advancing technology and modern production processes. The Congress program was structured to incorporate a number of plenary sessions, paper sessions with over 100 papers, and panel discussions. Several distinguished persons who represented academia, professional associations, industry and government presented keynote addresses and led useful discussions. Also, the Congress hosted the 9th Meeting of the International Liason Group for Engineering Education.

All the papers submitted for III Congress have been contained in the Congress Proceedings edited by the USICEE, Faculty of Engineering, Monash University, Clayton, Melbourne, VIC 3168, Australia.

IV East-West Congress on Engineering Education is to be held again in Kraków in 1998 together with V International Conference on Engineering Education and International Congress of Deans of Technical Universities and Industrial Managers.