Numerical simulation of heat flow processes

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ABSTRACT

Numerical simulation method is based on mathematical models of physical processes, which describe run of a given process with a various accuracy. This paper presents a method for elaborating the computer simulation models based on differential equations. This makes it possible to obtain a high accuracy of representing dynamic features of a real process. The derived results are given in the form of numerical series. An example of modelling the dynamics of heat transfer through a flat wall is presented in the second part of the paper. Basing on the model one can simulate the process of operation of ship engine as well as such auxiliary devices as a cooler, evaporator, condenser, tank heating system etc.

Keywords : simulation, dynamic model, heat transfer, **Z** - transform

INTRODUCTION

Computer simulation has presently become a common tool. Simulators are used in didactic process, scientific research, and training courses, especially for operators. This way they substitute real technical devices, for economical and safety reasons. Today operational and maintenance conditions of machines are closer and closer to those represented in simulators. Educational process of to-be operators of machines starts from using the simulators.

The simulation makes it possible to represent difficult and dangerous operational situations and to learn to give an appropriate response to hazards. A boundary between a real situation and simulated one becomes obliterated. A simulation must be close to reality in order an operator always could consider a given situation as real one. By continuous improving the simulators a higher and higher level of education and safety can be ensured.

Developments in process automation and control have made it necessary to have at one's disposal a model of a given process [8, 9, 12].

By comparing a model of a given process with its real run it is possible to determine an instantaneous point of operation select appropriate control parameters and generate a correct message for operator, that rises safety level of the process. Simulation has also found wide application in training of sea-going ship's operators, both navigators and marine engineers. Hence simulators have become an indispensable instrumentation of maritime academies.

Simulation approach has also its important place in forming new scientific theories and widening the knowledge. In this way one looks for confirmation of laws in engineering, physics, economy and medicine. An expected chance for space missions is also checked by means of simulations.

In this paper has been presented the problem of modelling heat flow processes by using **Z -** transform applied to differential equation of a modeled process. This facilitates elaborating the digital models on which the work of any simulator is based.

SIMULATION MODELLING

During process simulation one tends to exactly represent statical and dynamical features of a given process. A simulation model should appropriately respond to external signals, and in the case of some kinds of simulators it should provide real-time responses. Real processes are complex and depending on many parameters. Only the simulation models based on differential equations are capable in providing a high conformance with real processes [10,14]. To find solutions of such equations is time-consuming. Modelling the objects of complex parameters leads to partial differential equations [1, 3, 4, 11]. For this reason one often introduces simplifications in expense of accuracy of process representation. For example, either simple algebraic equations are introduced, or process statical and dynamical features are separately considered . However the so seperated dynamical term does not account for all variables of the object in question.

Applying another approach one substitutes finite difference equations for differential equations, that impairs accuracy of solution. In solving partial differential equations by means of the finite difference method a condition for stability of solution is introduced [13]. For instance the heat transfer equation :

$$\frac{\partial U(x,t)}{\partial t} = a\frac{\partial^2 U(x,t)}{\partial x^2} \qquad (1)$$

obtains, after digitization respective to time and spatial variable, the following form of response in successive time instants :

$$U(x_i, t_{n+1}) = a\frac{T}{\Delta x^2}U(x_{i-1}, t_n) + \left(1 - 2a\frac{T}{\Delta x^2}\right)U(x_i, t_n) + a\frac{T}{\Delta x^2}U(x_{i+1}, t_n) \qquad (2)$$

To reach a higher calculation accuracy a digitizing step, both in the time and space domain, is made shorter and simultaneously the following stability condition has to be satisfied :

$$a\frac{T}{\Delta x^2} \le \frac{1}{2} \qquad (3)$$

The time step T is imposed by the assumed digitizing step Δx. The application of a variable digitizing step is necessary to accelerate a response being always of recurrent type. How to find a compromise between simulation accuracy and its duration time is an open question. During simulation process much time is usually devoted to graphical presentation of a model.

This author has proposed to combine the advantages of solving differential equations and finite difference ones in order to determine process models.

The digitization is applied respective to the time variable **t** only. Equations in such a form can be solved with the use of the transform **Z**. The time variable **t** passes into **z** and becomes a parameter. The so obtained equation can be solved respective to spatial variables as continuous one by integrating it and applying the known methods for solving partial differential equations, e.g. the method of separation of variables or that of successive integral transformations [5]. In this phase dynamic relations of modeled process is not solved. Now the inverse transformation **Z** of the initial function is made to obtain its course respective to time.

The proposed approach makes it possible to obtain a solution equivalent to continuous one. This way the digitization in the domain of spatial variables as well as the necessity to satisfy the associated stability condition, is avoided. The digitization in the time domain only enables to choose indpendently a time step limited only by an assumed solution accuracy. The inverse transform **Z** is calculated by expanding the function into numerical series by means of the following formula [6]:

$$f_n = \lim_{p\to 0}\frac{1}{n!}\frac{d^n F\left(z = \frac{1}{p}\right)}{dp^n} \qquad (4)$$

If an expansion of a given function into Taylor series respective to the variable *p* exists then it is always possible to determine a solution. This is much easier to do than to use the Laplace transform [2, 7, 10]. The final solution contains all process variables and the time variable in a discrete form. The result is yielded in the form of numerical series, which leads to simple procedures of numerical programming; the calculation error depends on digitizing the function respective to time, only.

In each computational step the process parameters take constant values. In Eq.(1) the parameter **a** is constant and not dependent on time and position. The parameters can be changed in successive computational steps. The equation is reduced to a differential equation of constant coefficients, and a linear form of a given model is obtained.

EXAMPLE: HEAT TRANSFER THROUGH A WALL

Heat transfer through a flat wall occurs in many ship systems and technical devices. The phenomenon is described by Eq.(1) which can be solved by using the proposed method. After application of the transform **Z** the Eq. (1) takes the following form :

$$\frac{d^2U(x,z)}{dx^2} - \frac{1}{aT}\frac{z-1}{z}U(x,z) = 0 \qquad (5)$$

where :

$$a = \frac{\lambda}{\rho c}$$

After transformation one can obtain a parabolic 2nd order differential equation of constant coefficients. Its solution is as follows :

$$U(x,z) = Ae^{r_1 x} + Be^{r_2 x} \qquad (6)$$

where :

$$r_{1,2} = \mp\frac{1}{\sqrt{aT}}\sqrt{\frac{z-1}{z}} \qquad (7)$$

In order to solve Eq.(5) the following boundary conditions have been defined :

1. Temperature on the inner side of the wall varies in compliance with the assumed function :

$$U(0,z) = A + B = U_0(z) \qquad (8)$$

2. The outer side of the wall is insulated, hence :

$$-\lambda\frac{dU(x=w,z)}{dx} = -\lambda\left(Ar_1e^{r_1 x} + Br_2e^{r_2 x}\right) = 0 \qquad (9)$$

The solution of Eq.(5) is described by the following formula :

$$U(x,z) = U_0(z)\frac{e^{r_1(x-w)} + e^{r_2(x-w)}}{e^{r_1 w} + e^{r_2 w}} \qquad (10)$$

The wall's outer side temperature is described by the formula :

$$U(w,z) = U_0(z)\frac{2}{e^{\frac{w}{\sqrt{aT}}\sqrt{\frac{z-1}{z}}} + e^{-\frac{w}{\sqrt{aT}}\sqrt{\frac{z-1}{z}}}} \qquad (11)$$

Finally, it is necessary to calculate the inverse transform **Z** by using the relation (4) :

$$F(z) = e^{\alpha\sqrt{\frac{z-1}{z}}} + e^{-\alpha\sqrt{\frac{z-1}{z}}} \qquad (12)$$

where :

$$\alpha = \frac{w}{\sqrt{aT}}$$

The inverse transform is obtained this way :

$$f(nT) = 0,-1,-\sum_{n=2}^{\infty}\frac{1}{2^{2n-3}}\frac{(2n-3)!}{n!(n-2)!}\alpha\sinh\alpha - \sum_{n=3}^{\infty}\sum_{k=1}^{\frac{n-1}{2},\frac{n-2}{2}}\frac{1}{2^{2n-2k-2}}\frac{(n-k-1)(2n-2k-3)!}{kn!(2k-1)!(n-2k-1)!}\alpha^{2k+1}\sinh\alpha +$$
$$+2\cosh\alpha,0,\sum_{n=2}^{\infty}\sum_{k=0}^{\frac{n-3}{2},\frac{n-2}{2}}\frac{1}{2^{2n-2k-3}}\frac{(2n-2k-3)!}{n!(2k+1)!(n-2k-2)!}\alpha^{2k+2}\cosh\alpha \tag{13}$$

A choice of **k** value is realized in accordance with the following principle :

k = 1 for odd values of n ; k = 2 for even values of n

The exciting function $\mathbf{U}_0(z)$ may be arbitrary, however this leads to the operation of convolution. If $\mathbf{U}_0(z)$ is assumed a step function then the response is a sum of terms of given series. This way the mentioned operation of convolution can be avoided. For realization of other functions the approximation by a staircase function is advised. Such approximation method is used for digital control algorithms.

In the time domain Eq.(11) takes the form :

$$U(w,nT) = U_0\frac{2}{a_0+a_1+a_2+a_3+\ldots+a_{n-1}+a_n} \tag{14}$$

where :

$$a_0 = 2\cosh\alpha \;;\; a_1 = -\alpha\sinh\alpha \;;\; a_2 = -\frac{1}{4}\alpha\sinh\alpha+\frac{1}{4}\alpha^2\cosh\alpha \;;\; a_3 = -\frac{3}{24}\alpha\sinh\alpha+\frac{3}{24}\alpha^2\cosh\alpha-\frac{1}{24}\alpha^3\sinh\alpha$$

. . . .

$$a_n = -\frac{1}{2^{2n-3}}\frac{(2n-3)!}{n!(n-2)!}\alpha\sinh\alpha-\sum_{k=1}^{\frac{n-1}{2},\frac{n-2}{2}}\frac{1}{2^{2n-2k-2}}\frac{(n-k-1)(2n-2k-3)!}{kn!(2k-1)!(n-2k-1)!}\alpha^{2k+1}\sinh\alpha+$$
$$+\sum_{k=0}^{\frac{n-3}{2},\frac{n-2}{2}}\frac{1}{2^{2n-2k-3}}\frac{(2n-2k-3)!}{n!(2k+1)!(n-2k-2)!}\alpha^{2k+2}\cosh\alpha \tag{15}$$

After dividing by the series $\mathbf{a}_n$ the response at discrete instants is obtained. For the exciting step signal it amounts to :

$$U(w,nT) = U_0\frac{1}{\cosh\alpha}\left(\begin{array}{c}\frac{1}{2^{2n-2}}\frac{(2n-3)!}{n!(n-2)!}\alpha\tanh\alpha+ \\ +\sum_{k=1}^{\frac{n-1}{2},\frac{n-2}{2}}\sum_{m=0}^{k}\frac{1}{2^{2n-2k-1}}\frac{(n-k-1)(2n-2k-3)!}{kn!(2k-1)!(n-2k-1)!}\alpha^{2k+1}D_{2k+1,m}\tanh^{2m+1}\alpha+ \\ +\sum_{k=0}^{\frac{n-3}{2},\frac{n-2}{2}}\sum_{m=0}^{k}\frac{1}{2^{2n-2k-2}}\frac{(2n-2k-3)!}{n!(2k+1)!(n-2k-2)!}\alpha^{2k+2}D_{2k,m}\tanh^{2m}\alpha\end{array}\right) \tag{16}$$

The coefficients $D_{k,m}$ are derived from the recurrence algorithm based on the Pascal triangle principle, namely from the formulae (17 and 18) :

k = 1	1					
k = 2	1	-2!				
k = 3	1	-3!				
k = 4	1	-10·2!	4!			
k = 5	1	-10·3!	5!			
k = 6	1	-91·2!	35·4!	-6!		
k = 7	1	-91·3!	35·5!	-7!		
k = 8	1	-820·2!	966·4!	-84·6!	8!	
k = 9	1	-820·3!	966·5!	-84·7!	9!	
k = 10	1	-7381·2!	24970·4!	-5082·6!	165·8!	-10!
k = 11	1	-7381·3!	24970·5!	-5082·7!	165·9!	-11!

. . . .

$$\begin{cases} D(2k+1,m) = (-1)^m \left[D_1(2k-1,m) + (2m+1)^2 D_1(2k-1,m)\right](2m+1)! \\ D(2k,m) = (-1)^m \left[D_1(2k-2,m) + (2m+1)^2 D_1(2k-2,m)\right](2m)! \end{cases} \tag{17}$$

$$D_{k,m} = \sum_{i=m}^{k} (-1)^m \frac{i!}{k!(i-k)!} D(k,i) \tag{18}$$

For m=0 the Euler numbers are yielded [15] :

$$D_{k,0} = \sum_{i=0}^{k} D(k,i) = E_k \tag{19}$$

The successive values of the function are as follows :

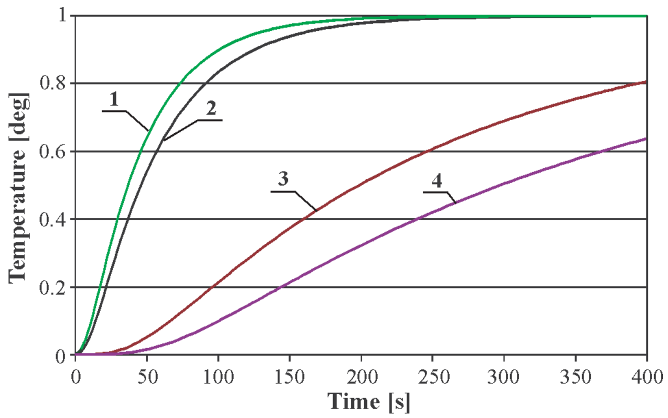
$$U(w,0) = 0$$

$$U(w,T) = U_0 \frac{1}{\cosh\alpha}$$

$$U(w,2T) = U_0 \frac{1}{\cosh\alpha}\left(1 + \frac{1}{2}\alpha\tanh\alpha\right)$$

$$U(w,3T) = U_0 \frac{1}{\cosh\alpha}\left(\begin{array}{c} 1 + \frac{1}{2}\alpha\tanh\alpha + \frac{1}{8}\alpha\tanh\alpha + \\ + \left[-\frac{1}{8} + \frac{1}{4}\tanh^2\alpha\right]\alpha^2 \end{array}\right)$$

Below in the figure results of the example calculations performed by using the described method are presented.



Dynamic characteristics of heat transfer through the wall of 0.1 m in thickness, the excitation $U_0(t)$ = 1[deg], for the following materials : 1 – copper; 2 – aluminium; 3 – brass; 4 – steel.

RECAPITULATION

The presented method has many advantages, namely :

- Simulation time is very short since any particular term of the series represents a solution.
- Calculations can be extended by means of non-zero initial conditions and arbitrary boundary conditions. This does not impair accuracy of the model.
- A simulation step may be chosen from a wide variability range. It makes calculation time shortening possible that is demanded during simulation to disregard its already known fragments.
- In some selected cases the model in question enables to determine a response value at n-th instant with disregarding the preceding ones. The so obtained solution is equivalent to continuous one.

NOMENCLATURE

- a – coefficient of temperature equalization [m²/s]
- a_n – coefficients of numerical series
- A, B – integration constants
- c – specific heat of a material [J/kgK]
- D – coefficients of expanded form of a initial function
- n – natural number
- r_1, r_2 – zero loci of a partial differential equation
- t – time[s]
- T – digitizing period [s]
- U – temperature function
- U_o – initial values
- w – wall thickness [m]
- x – spatial variable
- z, p – arguments of a function in the complex variable domain
- λ – thermal conductance [W/mK]
- ρ – density [kg/m³]

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