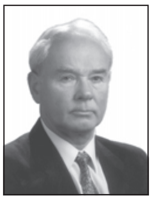
High-cycle fatigue criterion for anisotropic metals under multiaxial constant and periodic loads

Janusz Kolenda
Gdańsk University of Technology

ABSTRACT



Periodic stress with Cartesian components given in the form of Fourier series is considered. To account for the mean stress effect the modified Soderberg's formula is employed. An equivalent stress with synchronous components is defined. The design criterion for a finite fatigue life of metal elements is formulated. It covers the conditions of both static strength and fatigue safety in the high-cycle regime and includes material constants which have simple physical interpretation, can be determined by uniaxial tests, are related directly to the applied loads, and can reflect material anisotropy.

Key words : design criteria, multiaxial loading, periodic stress, mean stress effect

INTRODUCTION

In static problems, the load capacity of metal elements is referred to the yield strength and/or ultimate strength. In the case of multiaxial static loads, a reduced uniaxial stress, equivalent to the original one in terms of effort of the material, is taken into account. For ductile metals such stress model can be defined, for instance, with the aid of the Huber-von Mises-Hencky distortion-energy strength theory [1,2]. In the case of dynamic loading conditions, the design criteria of engineering elements made of ductile metals are frequently based on their fatigue limits and/or S-N curves (Wöhler curves) [3÷5] and, if the stress state is multiaxial and the stress components are proportional to each other, also on the distortion-energy theory [6,7]. In [8] the design criterion for an infinite fatigue life of metal elements under the stress with non-proportional components has been formulated with the aid of the average-distortion-energy strength hypothesis [9] and theory of energy transformation systems [10]. In this paper the design criteria for a finite fatigue life of metal elements under the stress with non-proportional components in the high-cycle regime are considered.

A CRITERION OF FINITE FATIGUE LIFE UNDER UNIAXIAL STRESS

If a metal element is subjected to an axial load producing the stress :

$$\sigma_a(t) = c + a \sin \omega t \tag{1}$$

where :

- a - stress amplitude
- c - mean value
- ω - circular frequency,

in design for an infinite fatigue life various „failure diagrams" or equations can be used [5,11]. For example, in [8] the Soderberg's equation has been utilized. Its necessary modification to indicate finite fatigue lives resulted in the following formula [11] :

$$a = \sigma\,(1 - c/R_e) \tag{2}$$

where :

- R_e - tensile yield strength
- σ - amplitude of the fully reversed stress at a given number, N, of cycles to failure
- a - amplitude of the stress (1) which leads to that fatigue life.

For the relation between σ and N in the high-cycle fatigue regime the following equation of the S - N curve is commonly accepted :

$$N\sigma^m = K \tag{3}$$

where :

- K - fatigue strength coefficient
- m - fatigue strength exponent.

Introducing the safety factor :

$$f = \frac{N}{N_d} = \frac{T}{T_d} \tag{4}$$

where :

- $N_d = \omega T_d/(2\pi)$ - required number of stress cycles to achieve a given design life T_d
- T - time to failure under the stress (1)

and partial safety factors :

$$f_d = \frac{N_a}{N_d} \qquad f_s = \frac{R_e}{c} \tag{5}$$

where :

$$N_a = \frac{K}{a^m} \quad (6)$$

one gets

$$f = f_d(1 - f_s^{-1})^m \quad (7)$$

The criterion in question reads :

$$f \geq 1 \quad (8)$$

that is :

$$f_d(1 - f_s^{-1})^m \geq 1 \quad (9)$$

Eq. (9) can be rewritten as :

$$f_d^{-1/m} + f_s^{-1} \leq 1 \quad (10)$$

i.e.,

$$\left(\frac{N_d}{K}\right)^{1/m} a + \frac{1}{R_e} c \leq 1 \quad (11)$$

It is seen that Eq.(11) covers both the condition of static strength and the fatigue safety requirement. This advantage will be maintained in what follows. To be on the safe side, in the case of c < 0 it is recommended to insert into Eq. (11) the compressive yield strength R_{ec}< 0 in place of R_e .

A CRITERION OF FINITE FATIGUE LIFE UNDER OUT-OF-PHASE STRESS

On the basis of the average-distortion-energy strength hypothesis [9], the stress with Cartesian components :

$$\sigma_i(t) = c_i + a_i \sin(\omega t + \beta_i) \quad (12)$$

$$i = x, y, z, xy, yz, zx$$

can be modelled by the reduced stress [12] :

$$\sigma_{eq}(t) = c_{eq} + a_{eq} \sin \omega t \quad (13)$$

equivalent in terms of effort of the material under the stress with the components (12), where :

c_i , a_i , β_i - mean value, amplitude and phase angle of i-th stress component, respectively

c_{eq} , a_{eq} - mean value and amplitude of the reduced stress, given by (for the sake of brevity the stress components σ_z , σ_{yz} and σ_{zx} have been dropped)

$$c_{eq} = (c_x^2 + c_y^2 - c_x c_y + 3c_{xy}^2)^{1/2} \quad (14)$$

$$a_{eq} = [a_x^2 + a_y^2 - a_x a_y \cos(\beta_x - \beta_y) + 3a_{xy}^2]^{1/2} \quad (15)$$

Consequently, the criterion (11) becomes :

$$\left(\frac{N_d}{K}\right)^{1/m} a_{eq} + \frac{1}{R_e} c_{eq} \leq 1 \quad (16)$$

or, in explicit form

$$\left(\frac{N_d}{K}\right)^{1/m} [a_x^2 + a_y^2 - a_x a_y \cos(\beta_x - \beta_y) + 3a_{xy}^2]^{1/2} + \frac{1}{R_e}(c_x^2 + c_y^2 - c_x c_y + 3c_{xy}^2)^{1/2} \leq 1 \quad (17)$$

In particular, the criterion (17) may be useful for isotropic steels where [1] :

$$R_{es} = \frac{1}{\sqrt{3}} R_e \qquad F_t = \frac{1}{\sqrt{3}} F_b \quad (18)$$

where :

R_{es} - shear yield strength
F_b - fatigue limit under fully reversed bending
F_t - fatigue limit under fully reversed torsion.

With Eqs (18), the criterion of finite fatigue life for steel elements under out-of-phase stress can be also expressed as :

$$\left(\frac{N_d}{K}\right)^{1/m} [a_x^2 + a_y^2 - a_x a_y \cos(\beta_x - \beta_y) + \left(\frac{F_b}{F_t}\right)^2 a_{xy}^2]^{1/2} + \frac{1}{R_e}[c_x^2 + c_y^2 - c_x c_y + \left(\frac{R_e}{R_{es}}\right)^2 c_{xy}^2]^{1/2} \leq 1 \quad (19)$$

A CRITERION OF FINITE FATIGUE LIFE UNDER OUT-OF-PHASE STRESS FOR ANISOTROPIC METALS

Metals may be isotropic or anisotropic with respect to any type of behaviour, such as thermal conductivity, electrical resistivity, thermal expansion, plastic deformation, or fatigue strength. This paper is concerned with differences in yield strengths and fatigue limits in different directions as well as under loads of different modes. For instance, uniform stress field under axial load versus nonuniform one under bending contributes to significant difference in the relevant fatigue limits.

Anisotropy can be introduced by cold-working operations such as rolling, forging, drawing or stretching. For example, even small amounts of strain can cause a considerable difference in the longitudinal and transverse yield strengths. In other words, as a result of the longitudinal prestrain the tensile and compressive yield strengths are different in the transverse direction, as well as in longitudinal direction, though to a smaller degree [13]. Cold rolling of a wide plate and cold drawing of a tube over a mandrel both result in a lengthening and thinning, with no change in the transverse dimension (width or diameter), and so the resulting anisotropy is similar in the two products. In both cases, the anisotropy exhibits three mutually perpendicular planes of symmetry. In drawing of a rod (or tubing without a mandrel), the two transverse strains are equal, and so the anisotropy reveals an axis of symmetry.

For different metals and different fabrication methods the degree of anisotropy varies within wide limits; some metals are so nearly isotropic that it may be difficult to detect a difference in properties in different directions, such as in the case of the yield strengths in the normal and rolling directions of hot-rolled mild-steel plate; however other metals may be highly anisotropic.

In order to take into account the material anisotropy and load modes, in this paper the following modification of Eq. (19) is suggested :

$$\left(\frac{N_d}{K}\right)^{1/m}\left[\left(\frac{F_b}{F_x}a_x\right)^2+\left(\frac{F_b}{F_y}a_y\right)^2-\left(\frac{F_b}{F_x}a_x\right)\left(\frac{F_b}{F_y}a_y\right)\cos(\beta_x-\beta_y)+\left(\frac{F_b}{F_t}\right)^2\left(\frac{F_t}{F_{xy}}a_{xy}\right)^2\right]^{1/2}+$$
$$+\frac{1}{R_e}\left[\left(\frac{R_e}{R_x}c_x\right)^2+\left(\frac{R_e}{R_y}c_y\right)^2-\left(\frac{R_e}{R_x}c_x\right)\left(\frac{R_e}{R_y}c_y\right)+\left(\frac{R_e}{R_{es}}\right)^2\left(\frac{R_{es}}{R_{xy}}c_{xy}\right)^2\right]^{1/2}\le 1 \tag{20}$$

where :

F_x - fatigue limit under fully reversed load relevant to the normal stress component of the amplitude a_x
F_{xy} - fatigue limit under fully reversed load associated with the shear stress component of the amplitude a_{xy}
R_x - absolute value of the yield strength relevant to the constant load inducing the normal stress component c_x
R_{xy} - yield strength associated with the constant load giving the shear stress component c_{xy} .

The remaining material constants R_i and F_i are defined analogously.

Eq. (20) yields the requested criterion :

$$\left(\frac{N_d}{K}\right)^{1/m}F_b\left[\sum_i\left(\frac{a_i}{F_i}\right)^2-\frac{a_x a_y}{F_x F_y}\cos(\beta_x-\beta_y)\right]^{1/2}+\left[\sum_i\left(\frac{c_i}{R_i}\right)^2-\frac{c_x c_y}{R_x R_y}\right]^{1/2}\le 1 \tag{21}$$

A CRITERION OF FINITE FATIGUE LIFE UNDER MULTIAXIAL CONSTANT AND PERIODIC STRESSES FOR ANISOTROPIC METALS

Let us consider a periodic stress with Cartesian components given by Fourier series :

$$\sigma_i(t)=c_i+\sum_{p=1}^{\infty}a_{ip}\sin(p\omega_o t+\beta_{ip}) \qquad i=x,y,xy \tag{22}$$

where :

c_i - mean value of i-th stress component
a_{ip} , β_{ip} - amplitude and phase angle of p-th term in Fourier expansion of i-th stress component
$\omega_0 = 2\pi/T_0$ - fundamental circular frequency
T_0 - common period of the stress components.

Following the results obtained in [8] and [12], the stress components (22) are modelled by the equivalent stress components :

$$\sigma_i^{(eq)}(t)=c_i^{(eq)}+a_i^{(eq)}\sin(\omega_{eq}t+\varphi_i) \qquad i=x,y,xy \tag{23}$$

where [8] :

$$c_i^{(eq)}=c_i \qquad a_i^{(eq)}=\left(\sum_{p=1}^{\infty}a_{ip}^2\right)^{1/2} \qquad a_x^{(eq)}a_y^{(eq)}\cos(\varphi_x-\varphi_y)=\sum_{p=1}^{\infty}a_{xp}a_{yp}\cos(\beta_{xp}-\beta_{yp}) \tag{24}$$

and [12] :

$$\omega_{eq}=k\omega_0 \qquad k=\mathrm{Round}(\kappa) \tag{25}$$

$$\kappa=\left[\frac{\sum_{p=1}^{\infty}(pa_{xp})^2+\sum_{p=1}^{\infty}(pa_{yp})^2-\sum_{p=1}^{\infty}p^2a_{xp}a_{yp}\cos(\beta_{xp}-\beta_{yp})+3\sum_{p=1}^{\infty}(pa_{xyp})^2}{\sum_{p=1}^{\infty}a_{xp}^2+\sum_{p=1}^{\infty}a_{yp}^2-\sum_{p=1}^{\infty}a_{xp}a_{yp}\cos(\beta_{xp}-\beta_{yp})+3\sum_{p=1}^{\infty}a_{xyp}^2}\right]^{1/2} \tag{26}$$

Here k is a natural number, φ_i, is the phase angle of i-th equivalent stress component and ω_{eq} is the equivalent circular frequency. Eq. (26) has been derived by means of the theory of energy transformation systems [10] under assumption that the material is ductile and isotropic. Its modification for ductile and isotropic materials, similar to that of Eq. (17), leads to :

$$\kappa=\left[\frac{\sum_i\sum_{p=1}^{\infty}\left(\frac{pa_{ip}}{F_i}\right)^2-\sum_{p=1}^{\infty}\frac{a_{xp}a_{yp}}{F_xF_y}p^2\cos(\beta_{xp}-\beta_{yp})}{\sum_i\sum_{p=1}^{\infty}\left(\frac{a_{ip}}{F_i}\right)^2-\sum_{p=1}^{\infty}\frac{a_{xp}a_{yp}}{F_xF_y}\cos(\beta_{xp}-\beta_{yp})}\right]^{1/2} \tag{27}$$

Adaptation of Eq. (21) to the equivalent stress with the components (23) gives :

$$\left(\frac{N_d}{K}\right)^{1/m} F_b\left[\sum_i\left(\frac{a_i^{(eq)}}{F_i}\right)^2-\frac{a_x^{(eq)}a_y^{(eq)}}{F_xF_y}\cos(\varphi_x-\varphi_y)\right]^{1/2}+\left[\sum_i\left(\frac{c_i^{(eq)}}{R_i}\right)^2-\frac{c_x^{(eq)}c_y^{(eq)}}{R_xR_y}\right]^{1/2}\le 1 \quad (28)$$

where :

$$N_d=\frac{\omega_{eq}T_d}{2\pi} \quad (29)$$

Hence the criterion of finite fatigue life for engineering details made of anisotropic ductile materials and subjected to multiaxial constant and periodic loads becomes :

$$\left(\frac{\omega_{eq}T_d}{2\pi K}\right)^{1/m} F_b\left[\sum_i\sum_{p=1}^{\infty}\left(\frac{a_{ip}}{F_i}\right)^2-\frac{1}{F_xF_y}\sum_{p=1}^{\infty}a_{xp}a_{yp}\cos(\beta_x-\beta_y)\right]^{1/2}+\left[\sum_i\left(\frac{c_i}{R_i}\right)^2-\frac{c_xc_y}{R_xR_y}\right]^{1/2}\le 1 \quad (30)$$

EXAMPLE

Assumptions

Suppose the stress components and material data (in MPa) are :

$\sigma_x(t)=c_x+a_x\sin\omega t$ – normal stress component

$\sigma_{xy}(t)=c_{xy}+a_{xy}\sin\omega t$ – shear stress component

$c_x=50 \quad a_x=90 \quad c_{xy}=40 \quad a_{xy}=60$

$R_e=260$(steel 25) $\quad R_{es}=160$

$R_{eg}=310$ – yield strength in bending

$F_b=Z_{go}=180 \quad F_t=Z_{so}=110$

$F_x=Z_{rc}=150$ – fatigue limit under fully reversed tension - compression.

Task

Compare the criteria (17) and (21) in the following cases :

A. the normal stress component is caused by bending moment
B. the normal stress component is caused by axial force.

Solution

A common and conservative strategy in classical fatigue design for applying the distortion-energy strength theory is to separately compute the equivalent stress corresponding to the mean stress and the equivalent stress corresponding to the amplitude [14]. Similar results have been obtained with the aid of the average - distortion - energy strength hypothesis, Eqs. (14) and (15), which will be used in the considered Example.

A. Eqs (14) and (15) yield :

$$c_{eq}=(c_x^2+3c_{xy}^2)^{1/2}=(50^2+3\cdot 40^2)^{1/2}=85.44\,\text{MPa}$$

$$a_{eq}=(a_x^2+3a_{xy}^2)^{1/2}=(90^2+3\cdot 60^2)^{1/2}=137.48\,\text{MPa}$$

So, Eq. (17) becomes :

$$137.48\left(\frac{N_d}{K}\right)^{1/m}+\frac{85.44}{260}\le 1$$

Hence :

$$\left(\frac{N_d}{K}\right)^{1/m}\le 48.83\cdot 10^{-4}$$

With Eq. (21) one obtains :

$$\left(\frac{N_d}{K}\right)^{1/m}180\left(\frac{90^2}{180^2}+\frac{60^2}{110^2}\right)^{1/2}+\left(\frac{50^2}{310^2}+\frac{40^2}{160^2}\right)^{1/2}\le 1$$

i.e.,

$$\left(\frac{N_d}{K}\right)^{1/m}\le 52.74\cdot 10^{-4}$$

B. In this case Eq. (17) leads to the same results as with the bending stress, that is :

$$\left(\frac{N_d}{K}\right)^{1/m}\le 48.83\cdot 10^{-4}$$

whereas Eq. (21) gives :

$$\left(\frac{N_d}{K}\right)^{1/m}180\left(\frac{90^2}{150^2}+\frac{60^2}{110^2}\right)^{1/2}+\left(\frac{50^2}{260^2}+\frac{40^2}{160^2}\right)^{1/2}\le 1$$

$$\left(\frac{N_d}{K}\right)^{1/m}\le 46.90\cdot 10^{-4}$$

The differences between these results are self-explanatory.

CONCLUSIONS

- The finite fatigue life design criterion covering the conditions of both static strength and fatigue safety of metal elements under multiaxial constant and periodic loads, has been formulated.
- The presented criterion includes material constants which
  - ⇨ have simple physical interpretation
  - ⇨ can be determined by uniaxial tests
  - ⇨ are directly related to the applied load
  - ⇨ can reflect material anisotropy.
- Similarly as in vibration problems, it is not possible to state a definite upper limit to *p* in (22), (24), (26), (27) and (30), since this depends upon the nature and origin of the considered stress variations, as well as upon the influence of minor terms on fatigue endurance of various materials, which may be the source of uncertainties.

NOMENCLATURE

a	-	stress amplitude
a_i	-	amplitude of i-th stress component (i = x, y, z, xy, yz, zx)
a_{eq}	-	amplitude of the reduced stress
a_{ip}	-	amplitude of p-th term in Fourier expansion of i-th stress component
$a_i^{(eq)}$	-	amplitude of i-th component of the equivalent stress
c	-	mean stress value
c_i	-	mean value of i-th stress component
c_{eq}	-	mean value of the reduced stress
$c_i^{(eq)}$	-	mean value of i-th component of the equivalent stress
f	-	safety factor
f_d , f_s	-	partial safety factors
F_b , F_t	-	fatigue limits under fully reversed bending and torsion, respectively
F_i	-	fatigue limit under fully reversed load associated with the stress component of the amplitude a_i
k	-	natural number obtained by rounding the number κ
K	-	fatigue strength coefficient in equation of the S-N curve for tension-compression
m	-	fatigue strength exponent in equation of the S-N curve for tension- compression
N , N_a	-	numbers of zero mean stress cycles to cause failure at the stress amplitudes σ and a, respectively
N_d	-	required number of stress cycles to achieve a given design life
R_e , R_{es}	-	tensile and shear yield strengths, respectively
R_i	-	yield strength associated with the constant load inducing the stress component c_i
t	-	time
T	-	time to fatigue failure
T_d	-	design life
T_0	-	stress period
β_i	-	phase angle of i-th stress component
β_{ip}	-	phase angle of p-th term in Fourier expansion of i-th stress component
κ	-	quantity given by Eq. (27)
σ	-	stress amplitude satisfying Eq. (3)
σ_a	-	stress produced by axial load
σ_i	-	i-th stress component
σ_{eq}	-	reduced stress
$\sigma_i^{(eq)}$	-	i-th component of the equivalent stress
φ_i	-	phase angle of i-th component of the equivalent stress
ω	-	circular frequency
ω_0	-	fundamental circular frequency of the periodic stress
ω_{eq}	-	equivalent circular frequency

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CONTACT WITH THE AUTHOR

Prof. Janusz Kolenda
Faculty of Ocean Engineering
and Ship Technology,
Gdańsk University of Technology
Narutowicza 11/12
80-952 Gdańsk, POLAND
e-mail : sek7oce@pg.gda.pl

Conference

Safety at sea and marine environment protection

These were the topics of 9th Technical Scientific Conference held in Kołobrzeg on Baltic Sea coast on 2 ÷ 3 June 2005. The conference program contained 17 papers presented during two sessions :

Safety of navigation and rescue of lives at sea
(13 papers)
Marine environment protection (4 papers).

Out of the presented topics, 8 papers dealt with local problems, and general ones were given in the following papers :

- *Safety of maritime turism in the polar regions, with Antarctica as an example* – by J. Frydecki and A. Wolski (Martime University of Szczecin)
- *Development trends in modelling the areas of search at sea* – by Z. Burciu (Gdynia Maritime University)
- *Contemporary methods for classification of free- drift phenomenon* – by M. Drogosiewicz and A. Wójcik (Polish Naval University)
- *Problems associated with dynamical determining the search area* – by T. Budny (Gdynia Maritime University)
- *Introduction of manoeuvrability standards in the low--speed range as an element of improvement of port operation safety* – by T. Abramowicz-Gerigk (Gdynia Maritime University)
- *Modelling the life- raft sinkage probability* by L. Smolarek (Gdynia Maritime University)
- *Probability of radar detection of rigid inflatable boats (RIB) fitted with active radar reflectors* – by A . Szklarski (Gdynia Maritime University)
- *Application of ultrafiltration technique to bilge water purification* – by Z. Jóźwiak and A. Kozłowski (Maritime University of Szczecin)
- *Research on deoiling process of oil/water emulsion by means of ceramic bulkheads* – by J. M. Gutteter – Grudziński (Maritime University of Szczecin).