Stability of free-floating ship

Part I

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ABSTRACT

Problem of calculation of righting arms of the free-floating ship, i.e. longitudinally balanced at any heel angle, was formulated. In such a case of particular interest for a ship in the damage condition, the righting arms are ambiguous as they depend on a way the heeling moment acts. Two cases were considered : when the heeling moment is parallel to the ship plane of symmetry, and the case when it performs the least work, i.e. when the moment is parallel to the main axis of ship waterplane. It was demonstrated that angular translations (heel and trim) are then the Euler angles associated with a relevant reference axis. Some cases of the incorrect defining and using of those angles in today design practice were indicated. The most important features of the curve of righting arms of free-floating ship were demonstrated.

Keywords : stability, free floating ship

INTRODUCTION

Rules of the classification societies require the intact ship stability to be investigated for the ship floating at even keel. The rules, however, do not clearly state how to calculate the damaged ship stability, which often leads to large discrepances in obtained results.

For the undamaged ship it is practically meaningless whether the stability calculations are performed for the ship having a fixed trim, constant in function of the heeling angle, or for the free-floating ship which changes its trim depending on its longitudinal equilibrium state. This is due to a small asymmetry of the ship relative to its midship section plane. However for the damaged ship the problem is important as it significantly influences the course of the righting arm curve for the heeeling angles greater than the entrance angle of the deck into water (Fig.1). The righting arm is understood here as the distance between the action line of buoyancy force and that of gravity force, occurring in still water, at a given heeling angle.

The influence of calculation assumptions is especially important in the case of flooding compartments far off the midship, which is understable due to the then occurring high asymmetry and a small entrance angle of the deck into water. Moreover the influence very strongly increases along with the ratio L/B decreasing. Hence it is greater for catamarans and SWATH units. The differences between righting arms may even reach a few hundred percent. For this reason the regulations should clearly define a way of carrying out calculations : whether at a fixed or free trim. It should be remembered that the final aim of stability calculations is to determine an expected final state of a considered ship under action of disturbing moments – and as a result – to correctly assess stability safety of the ship.

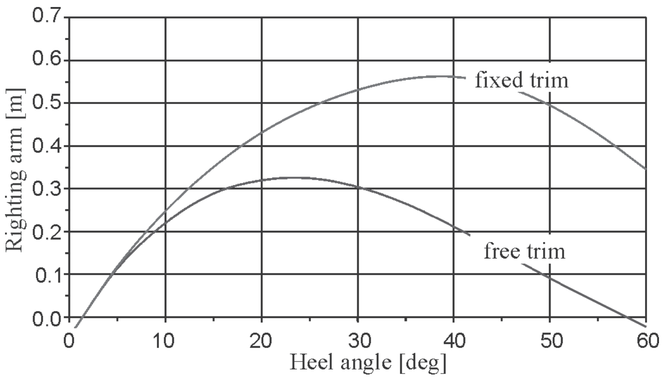

Fig. 1. *Righting arm curves for a free-floating damaged ship and that having a fixed trim [1]*

It is obvious that the routine stability calculations should be performed for a free-floating ship. However in such commonly accepted case the problem of correctly understood ship heel angle arises because then it is an ambiguous notion, which is manifested in existing various definitions of that angle. Most often the heel angle is assumed to be the slope angle of water-level trace line on the frame planes, further marked φ, or the slope angle of the baseplane (BP) relative to water-level , further marked α. The first of the angles is loosely associated with correctly defined ship heel angle, whereas the second is the heel angle of the ship having minimum righting arms.

HEEL ANGLE OF FREE-FLOATING SHIP

The problem of correct defining the heel angle of the free-floating ship has been recently solved [2÷4]. Namely, the ship heel angle, further marked ϕ, is understood as the angle rotation of the ship plane of symmetry (PS) around the intersection line of water surface and PS, i.e. the angle of rotation around the water-level trace line on PS, in other words – this is the inclination angle of PS from the vertical. At the same time the angle is equal to the slope angle of the axis *y* relative to water surface. The definition stems from the assumption that the ship heels under action of the heeling moment of fixed direction in space, and parallel to PS. If the ship floats freely it heels in such a way as to be logitudinally balanced all the time. It means that the direction of the righting moment is fixed in space[1)] and the same applies to the heeling moment. The conclusion immediately follows that the curve of buoyancy centres is then exactly flat and situated on the *rotation plane* perpendicular to the heeling moment vector, and containing the ship centre of gravity.

A necessity to define the heel angle of free-floating ship is usually not felt – many surveyors and designers are just surprised that any problem of this kind exists at all. Hence various definitions of the angle in question have been still assumed, which obviously results in ambiguity of calculation and makes it not possible to compare different computer softwares.

HISTORICAL OUTLINE

Why a body floats in a liquid has been known already in antiquity since the times of Archimedes. However in which way to assess and investigate stability of floating bodies has been recognized only after discovery of the Newtonian laws. In 1746 Bouguer introduced the notion of metacentrum and metacentric height considered as a measure of initial stability. In 1749 Euler introduced a formula for metacentric radius, and a theorem for equi-volume waterplanes. In 1796 Atwood published a method for calculation of the righting arm at a given heel angle [5]. Nonetheless for over a hundred years only the initial metacentric height GM was used to assess ship stability. Only the ship stability accidents at the end of 19^{th} century revealed importance of the freeboard and necessity of applying the curve of righting arms in assessing stability of ships.

The metacentric height, which is not an unimportant index of stability, does not make it possible to directly assess either a range of the curve of righting arms nor on a value of the maximum righting arm. Here it is worth mentioning the widely described case of sinking the HMS *Captain* in 1870, whose metacetric height GM = 0.79 m [6]. The ship capsized during a storm in the Bay of Biscay, whereas the accompanying battleship *Monarch* of a similar size and characteristics survived unharmed despite having its metacentric height GM = 0.73 m, i.e. smaller than that of the first ship. The fact was very surprising for the then naval architects. It is very easy to explain the accident if one observes that the freeboards of the two ships much differred to each other : the *Captain* had the freeboard F = 1.98 m and the *Monarch* - F = 4.27 m. As a result, despite the smaller metacentric height of the *Monarch*, its curve of righting arms was of much better parameters than that of the *Captain*, whose GZmax = 0.55 m instead of 0.25 m, $\phi_{max} = 40$ deg instead of 19 deg, and the stability range angle $\phi_v = 70$ deg instead of 54 deg.

The *Captain's* accident has finally proved that the metacentric height is an insufficient measure of safety against capsizing and it has made it necessary to examine ship's stability also at large heel angles. As a result, at the end of the 19^{th} century the curves of righting arms, called the *Reed's* curves in memory of their propagator, began to be used for the ship stability assessment. The first stability criteria, given by Rahola [7], appeared as late as in 1939. The well-documented recommendations dealing with a minimum size of the curve of righting arms have been elaborated on the basis of the analysis of the curves of righting arms of both for capsized ships and stable ones. At the end of the 1960s those criteria were adopted by IMCO (Intergovernmental Maritime Consultative Organization, presently IMO (International Maritime Organization), and they have been valid until now [8].

Though the curve of righting arms has been applied to assess stability of intact ships, stability of damaged ships has been further controlled by means of the metacentric height and freeboard. In the SOLAS conventions including the last one of 1974 the residual freeboard of as low as only 3 inches and the metacentric height of 2 inches have been assumed permissible. With such parameters the curves of righting arms usually show marginal values. A change took place as late as in 1990 when criteria for the curve of righting arms were introduced for damaged ships, in the form of the SOLAS 90 criteria [9]. However it is worth remembering that the criteria have not represented any important progress as they resulted from an administrative decision. Therefore they have only an alleged, but not real, relation to actual safety of a ship in damage condition. A breakthrough in that regard has happened during the last six years [10,11].

FORMULATION OF THE PROBLEM

Almost all of the commonly known calculation methods of the curve of righting arms concern the ship floating on even keel. This means indirectly that the buoyancy centre is assumed not to translate longitudinally during ship's heeling. It has been no necessity to consider earlier a different situation as the calculations dealt only with intact ships for which the assumption has been well satisfied. However the fact cannot be neglected any longer in the situations when the buoyancy centre translates longitudinally due to an asymmetrical distribution of buoyancy relative to the plane of rotation, as in the case of semi-submersible platforms arbitrarily orientated relative to wind direction, ships of low L/B ratio, or ships in damage conditions, and then the calculations should be carried out for a free-floating object. Determination of the curve of righting arms becomes in such cases ambiguous and the problem must be defined. Especially the way in which the righting moment acts should be defined.

It should be said that angular motions of a free-floating object are not considered in the basic ship theory as this is a 3-D motion. And, such motion is spatial and requiring good spatial imagination. For this reason as well as for making calculations easier the vectorial calculus is applied in this paper.

Calculations of the righting arms curve for free-floating ship is carried out under the following assumptions :

a) ***a pure heeling moment is statically exerted on the ship***. It means that ship inclinations are equi-volumetric and the horizontal position of the ship gravity centre remains constant (the moment cannot induce any translational motion hence change location of ship gravity centre on the sea level)

b) ***the heeling moment vector is strictly horizontal***. If this is not the case a vertical component of the moment, able to rotate the ship around its vertical axis, will exist

c) ***the heeling moment direction is fixed in space and parallel to PS***. As the moment is simultaneously parallel to the water-plane hence it is also parallel to the trace line of water on PS. Therefore the intersection edge of PS and the water-plane is also fixed in space. The edge determines orientation of the ship at the sea level as well it defines the direction of the heeling moment [2)]

d) ***the ship is in static equilibrium***, i.e. the sum of forces and moments acting on it equals zero. Hence ship's weight is equal to its buoyancy, and the statically applied heeling moment is balanced by the righting moment of the same direction and opposite sense

e) ***the righting moment is formed by the couple of forces*** : i.e. the gravity force applied in the ship's centre of gravity and the buoyancy force applied in the ship's centre of buoyancy – the forces are equal and of opposite sense to each other. The moment's vector is horizontally directed, perpendicular to the vertical plane determined by the gravity force and buoyancy force.

From those assumptions some consequences follow :

- ⇨ As in the state of equilibrium directions of the moments are the same the centre of buoyancy must be situated on the plane perpendicular to the direction of action of the heeling moment (i.e. on the rotation plane), and on which the centre of gravity is located. As the direction of the heeling moment action is, under the assumption, fixed in space, hence the rotation plane is also fixed in space. The versor perpendicular to the rotation plane, further marked ***e***, is hence constant and stands for *the rotation axis*.
- ⇨ For ship heels at a fixed trim the centre of buoyancy need not to be located on the rotation plane, therefore the moment acting on the ship has not a constant direction at the horizontal plane.
- ⇨ The rotation plane is vertical, motionless in space, perpendicular to the direction of the moment action (trace of water on PS), and passing through the ship gravity centre G motionless at the horizontal plane.
- ⇨ It is the rotation plane about which the buoyancy centre of free-floating ship moves. Hence the curve of buoyancy centres is strictly flat in space. For ship heels at a fixed trim the curve of buoyancy centres is the projection of the spatial curve onto the rotation plane.
- ⇨ The trace of water on PS determines the direction of the heeling moment. The ship heel angle ϕ is the rotation of PS around the trace of water on PS, i.e. the deflection angle of PS from the vertical. As Oy-axis is perpendicular to PS the angle ϕ is simultaneously the slope angle of that axis against the water-level. The angle ϕ can be interpreted as the rotation angle of Oy-axis around the trace of water on PS, in the rotation plane.
- ⇨ In order the ship to be balanced at any heel angle ϕ it has also to rotate around the versor normal to PS, i.e. the Oy--axis, in such a way as to bring the centre of buoyancy on the rotation plane (Fig. 2). The ship rotation angle in PS, further marked θ, is called the trim angle – this is the definition commonly used in ship hydrostatics. A change of the trim angle θ does not influence the heel angle ϕ. The two angles are the *Euler angles* associated with Oy-axis. The third angle associated with the rotation of Oy-axis projection onto the horizontal plane (i.e. yaw) is not present as the rotation plane is fixed in space. Under the above given assumption b) any yaw-inducing moments are not considered in ship hydrostatics.
- ⇨ The righting arm GZ is the arm of the couple of forces forming the righting moment, i.e. the distance, measured in the rotation plane, between the action line of gravity force and that of buoyancy force.
- ⇨ As the righting moment is all the time parallel to the trace of water on PS, the righting moment work is the integral of the moment, respective to the heel angle ϕ. Simultaneously this is the least work which is to be performed in order to heel the ship up to a given angle ϕ . In other words, for a ship of a fixed trim or that not fully balanced the work of righting moment is greater.

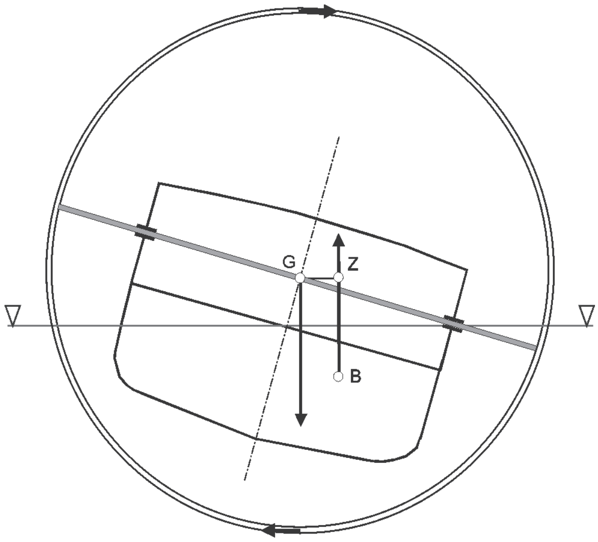

Fig. 2. *Rotation plane of free-floating ship which trims in PS*

It can be observed that the projection of Oy-axis onto the horizontal plane is perpendicular to the trace of water on PS. Hence the presented model of the heeling moment action strictly corresponds to the heeling moment due to a shift of cargo in ship's transverse plane. This deals also with the heeling moment of ro-ro ships in the damage condition, resulting from the accummulation of water on the car deck when a symmetrical midship compartment has been flooded. For the same reason the curve of righting arms measured by means of the Di Belli method[3)] is strictly consistent with the above given model of inclinations. Finally, the heeling moment of fixed direction in space, parallel to the trace of water on PS, is an accepted idealization of the wind-induced moment.

BASIC RELATIONSHIPS

A right hand side coordinate frame Oxyz, shown in Fig.3, fixed to the ship, is assumed. The point O is identical with the point K*,* Oy-axis points port, and Oz-axis – upwards.

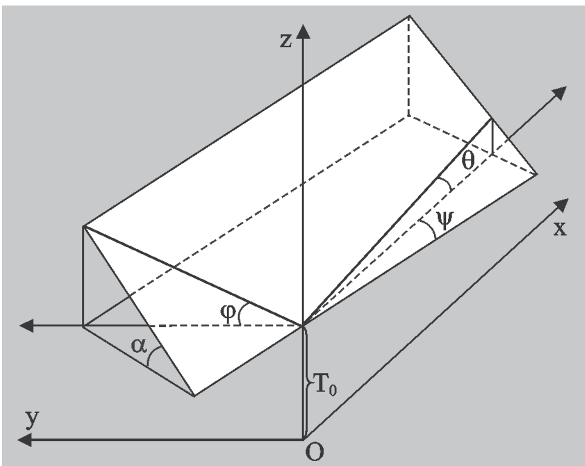

Fig. 3. *The right hand side coordinate frame Oxyz*

An arbitrary attitude of the waterplane can be described by the equation :

$$z = T_0 + x\mathrm{tg}\theta + y\mathrm{tg}\varphi \tag{1}$$

in which three independent parameters appear. These are :

- ✦ the angle θ called the trim angle which is the slope angle of the trace of water on PS relative to Ox- axis

- the angle φ which is the slope angle of the trace of water on the midship plane relative to Oy-axis, and
- T_o – draught of Oz-axis.

The above mentioned angles are positive when a positive increment of z corresponds to a positive increment of x (or y), as shown in Fig.3. Therefore the trim angle $\theta > 0$ is positive when the ship is trimmed by bow, and the angle $\varphi > 0$ is positive when it is heeled a-port. Both the angles are easy to be measured as :

$$\mathrm{tg}\theta = \Delta T_{DR}/L_{pp} \text{ and } \mathrm{tg}\varphi = \Delta T_{LP}/B$$

where :

ΔT_{DR} – difference of bow and stern draughts measured at the respective perpendiculars

ΔT_{LP} – difference of port and starboard draughts measured midships.

The angles φ and θ unambiguously describe position of the ship against water-level (or water-level against the ship) and are called the *analytical angles*. From the analytic geometry it results that the vector **R** normal to the waterplane given by (4) has the components :

$$\mathbf{R} = (\mathrm{tg}\theta\ ,\ \mathrm{tg}\varphi\ ,\ -1) \tag{2}$$

which points downwards and its absolute value equals :

$$R = \sqrt{1 + \mathrm{tg}^2\theta + \mathrm{tg}^2\varphi} \tag{3}$$

Hence the versor **n** normal to the waterplane and pointing upwards is equal to :

$$\mathbf{n} = -\mathbf{R}/R \tag{4}$$

The angle α contained between the water-level and the base plane BP (or initial waterplane) is one of the heel angles taken for calculation of the curve of righting arms. The angle between the planes is the same as that between the versors **n** and **k** normal to them. As a result, $\cos\alpha = \mathbf{k}\cdot\mathbf{n}$. Therefore :

$$\cos\alpha = 1/R = 1/\sqrt{1 + \mathrm{tg}^2\theta + \mathrm{tg}^2\varphi}$$

Hence :

$$\mathrm{tg}\alpha\sqrt{\mathrm{tg}^2\theta + \mathrm{tg}^2\varphi} \tag{5}$$

The sign of the angle α is the same as that of the angle φ. Taking into account that $1/R = \cos\alpha$ the formula (4) yields the following components of the versor **n** :

$$\mathbf{n} = (-\mathrm{tg}\theta\cos\alpha\ ,\ -\mathrm{tg}\varphi\cos\alpha\ ,\ \cos\alpha) \tag{6}$$

The heel angle ϕ is equal to the inclination angle of PS from the vertical. The angle is the same as that of Oy-axis relative to water-level. Hence $\cos(90° + \phi) = \mathbf{j}\cdot\mathbf{n} = n_y$. And, $\sin\phi = \mathrm{tg}\varphi\cos\alpha$, or simpler :

$$\mathrm{tg}\phi = \cos\theta\,\mathrm{tg}\varphi \tag{7}$$

The rotation plane rotates around the rotation axis parallel to the direction of the heeling moment action, defined by the versor **e**. The rotation angle is equal to the angle ϕ . The heeling moment direction is parallel to the water-level trace line on PS, whose versor $\mathbf{e}_1 = (\cos\theta, 0, \sin\theta)$, see Fig.3. Hence $\mathbf{e} = \mathbf{e}_1$.

Righting arm

In the course of heeling the ship its displacement remains constant and its buoyancy centre shifts in the rotation plane perpendicular to the rotation axis. Therefore the following is yielded :

$$\mathbf{e}\cdot\mathbf{r} = 0 \tag{8}$$

for any heel angle ϕ, where :

$\mathbf{r} = (x_B - x_G\ ,\ y_B - y_G\ ,\ z_B - z_G)$ – buoyancy centre radius vector relative to the ship gravity centre $\mathbf{r} = \mathbf{GB}$.

The righting moment is given by the formula :

$$\mathbf{M} = \mathbf{r}\cdot\mathbf{n}\Delta \tag{9}$$

where :

$\Delta = \rho g\nabla$ - ship buoyancy.

The vector is parallel to the rotation axis **e**, hence :

$$M = \mathbf{e}\cdot(\mathbf{r}\cdot\mathbf{n})\Delta \tag{10}$$

Therefore the righting arm $GZ = M/\Delta$ is given by the formula :

$$GZ = \mathbf{e}\cdot(\mathbf{r}\cdot\mathbf{n})$$

which is a function of the heel angle ϕ defined by (7). It is worth mentioning that the angle $\phi \le \alpha$, which can be observed directly from the formula $\cos\alpha = \cos\theta\cos\phi$ obtained by dividing $\sin\phi$ by $\mathrm{tg}\phi$. From (7) it also results that $\phi \le \varphi$. Therefore the actual rotation angle ϕ is never greater either than the angle α or the angle φ, which have been taken as the heel angles of free-floating ship, so far.

Metacentric radii

The buoyancy centre of the free-floating ship moves along a curve in the rotation plane which rotates as a disc around the rotation axis motionless in space. As the action lines of the buoyancy force are always vertical they are perpendicular to the actual waterplane. When changing the ship heel by $d\phi$ the buoyancy force lines rotate also by the angle $d\phi$ in the rotation plane, and their respective waterplanes – by the angle $d\alpha_1$ around the instantaneous axis of waterplane rotation, called the ship **floatation axis f**. The relationship between the differentials is given by the following formula, [4] :

$$d\alpha_1\cos\chi = d\phi \tag{11}$$

where :

χ – the angle defining situation of the floatation axis relative to the rotation axis $\mathbf{e} = \mathbf{e_1}$, being the PS trace line on waterplane.

The equation (11) reflects the fact that small angles have vectorial features. Hence the angle $d\phi$ is nothing else but the projection of the waterplane rotation angle $d\alpha_1$ onto the rotation axis **e**. In general case the angle $d\alpha_1$ is not equal to the change of the slope angle $d\alpha$ of waterplane relative to BP; the equality occurs only when the floatation axis **f** is parallel to the intersection edge of BP and water-level (see Fig.3).

The metacentric radius is understood as the curvature radius of the curve of buoyancy centres in the rotation plane; it depends on the angle ϕ. Translation of the buoyancy centre along the arc of the curve of buoyancy centres amounts to $ds = r_f d\alpha_1 = BMd\phi$. On accounting for the formula (11), the metacentric radius can be determined by the formula :

$$BM = \frac{r_f}{\cos\chi} \tag{12}$$

where :

$$r_f = (1/\nabla)\sqrt{I_f^2 + D_f^2}$$

f – instantaneous ship's floatation axis passing through the floatation centre F (the waterplane centre of gravity)

∇ – ship's hull volumetric displacement

I_f and D_f – transverse and deviation (cross-product) inertia moment of the instantaneous waterplane, respectively, associated with the floatation axis and floatation centre F.

The quantity r_f is a proportional coefficient between the buoyancy centre translation ds and the angle $d\alpha_1$.

The centre of buoyancy translates in the rotation plane, in parallel to the instantaneous waterplane (water-level). Therefore the buoyancy centre translation vector $d\mathbf{r} = (\mathbf{n} \cdot \mathbf{e})ds$.

Floatation axis

As it can be observed the metacentric radius of the free-floating ship , at a given heel angle, depends on location and orientation of the instantaneous floatation axis **f**. If the waterplane is rotated by the angle $d\alpha_1$ the transverse component of the buoyancy centre translation, BC, relative to the floatation axis (Fig. 4) is proportional to I_f, and the longitudinal component of this translation, AB,– to D_f, which results from the Euler's theorem of equi-volume waterplanes; this is the reason that the expression for the resultant translation AC, equal to $ds = r_f d\alpha_1$, appears in (12).

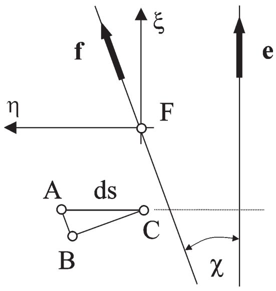

***Fig. 4.** Waterplane top view*

It is required the resultant translation to be normal to the direction of heeling moment action (the rotation axis **e**). In order to obtain this the angle C in Fig.4 must be equal to χ, which results from the properties of the angles having correspondingly perpendicular arms. Therefore the floatation axis slope angle relative to the rotation axis has to satisfy the following equation :

$$\mathrm{tg}\chi = D_f / I_f \tag{13}$$

The angle χ has the same sign as that of the waterplane deviation moment (in Fig.4 - positive). It should be remembered that the moments D_f and I_f also depend on χ. By applying the relationships resulting from the Mohr circle [4,12], known from the theory of strength of materials, they can be represented as follows: $D_f = D'$, and $I_f = I + a'$. The equation (13) obtains then the form : $D' - (I + a')\mathrm{tg}\chi = 0$, where the mark „ ' " stands for the quantities associated with the rotated coordinate system (ξ', η') whose axis ξ' coincides with the floatation axis **f**, and its origin is located in the floatation centre F; (the system is not shown in Fig. 4). The quantities marked „ ' " are expressed by means of the quantities taken from the system (ξ, η) :

$$D' = D\cos 2\chi + a\sin 2\chi \equiv r\sin(2\gamma + 2\chi)$$

$$a' = a\cos 2\chi - D\sin 2\chi \equiv r\cos(2\gamma + 2\chi)$$

where : $a = ½(I_{\xi\xi} - I_{\eta\eta})$ $I = ½(I_{\xi\xi} + I_{\eta\eta})$ and

$D = I_{\xi\eta}$, $I_{\xi\xi}$, $I_{\eta\eta}$ - waterplane deviation moment, transverse and longitudinal inertia moment, respectively, in the system ξη (Fig.4) whose origin coincides with the floatation centre F, and the axis ξ is parallel to the rotation axis $\mathbf{e} = \mathbf{e}_1$ (a trace of PS on the waterplane).

The quantities D' and a' determine the parametric equation of the Mohr circle shown in Fig.5. When applying the introduced notation the equation (13) gets the following form :

$$r\sin(2\gamma + 2\chi) - [I + r\cos(2\gamma + 2\chi)]\mathrm{tg}\chi = 0 \tag{14}$$

where :

$$r = \sqrt{a^2 + D^2} \ , \ 2\gamma_o = \mathrm{arctg}(D/a)$$

$2\gamma = 2\gamma_o$ if $a > 0$, in the opposite case : $2\gamma = 2\gamma_o + 180°$

The equation (13) having the quantities marked „ ' " is easier to be solved, and the equation (14) more simple for the geometrical interpretation shown in Fig.5; where *a* and γ_o are assumed negative.

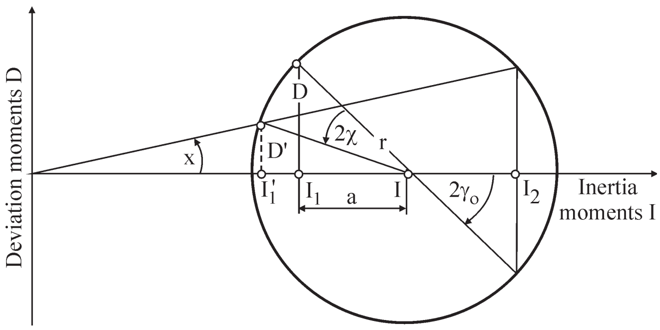

***Fig. 5.** Mohr circle and geometric characteristics of waterplane*

When cos2χ and sin2χ appearing in (13) is expressed by tgχ the equation (13) can be reduced to the simple 1st order equation :

$$D = (I - a)\mathrm{tg}\chi \tag{15}$$

Therefore : $\mathrm{tg}\chi = D/I_{\eta\eta}$

where : $|\chi| \le \arcsin(r/I)$ – see Fig.5.

As the angle χ is known, the quantities $D_f = D'$ and $I_f = I + a'$, necessary to express the radius on the basis of (12), as well as direction of the floatation axis, are defined. The floatation axis versor is given by the formula :

$$\mathbf{f} = \mathbf{e}\cos\chi + (\mathbf{n} \cdot \mathbf{e})\sin\chi \tag{16}$$

The equation (13) results from the asssumption that : $\mathbf{e} \cdot d\mathbf{r} = 0$, i.e. that the buoyancy centre translation in the ship-fixed coordinate system is normal to the rotation axis. It would be this way if the rotation axis were fixed to ship; however the axis is fixed in space but this does not mean it is fixed to the ship. It can be observed that when changing the heel also, in general, the trim has to be changed to balance the ship, which results in changing the orientation of the axis $\mathbf{e}_1 = (\cos\theta , 0 , \sin\theta)$ relative to ship, determined by a trace of water on PS.

After differentiation of the equation (8) the following is obtained : $\mathbf{e} \cdot d\mathbf{r} = - d\mathbf{e} \cdot \mathbf{r}$, i.e. that the buoyancy centre translation in the hull-fixed coordinate frame is not strictly normal to the rotation axis. It should be intuitively obvious : the buoyancy centre translation in the ship-fixed frame has to be oblique to it as the buoyancy centre is to be permanently located in the rotation plane which changes its orientation relative to ship during inclinations. When this is accounted for, the following relationship for the angle χ between the floatation axis and rotation axis is obtained [4] :

$$\mathrm{tg}\chi = \frac{D}{I_{\eta\eta} - BZ\nabla} = \frac{D}{\nabla\left(BM_L - BZ\right)} \tag{17}$$

where :

$BZ = -\mathbf{r} \cdot \mathbf{n}$ – is the height of the gravity centre over the buoyancy centre (Fig.2)

BM_L – is the longitudinal metacentric radius represented by the bracketed term in (17).

As the term BZ∇ is negligibly small in comparison with the longitudinal inertia moment of waterplane, $I_{\eta\eta}$, the equation (17) practically yields the same solution as the equation (15).

Hence it can be observed that to determine the longitudinal radius of free-floating ship at a given heel angle it is necessary to know three geometrical characteristics of waterplane, viz. its deviation moment and transverse and longitudinal inertia moments in the $\xi\eta$ coordinate system associated with the rotation axis $\mathbf{e} = \mathbf{e}_1$ (a trace of PS on the waterplane) and the buoyancy centre. In calculating these characteristics it should be accounted for that the water trace lines on frame planes are not normal to the water trace line on PS. Denoting this angle by $90° + \beta$ the following is obtained : $\cos(90° + \phi) = \mathbf{e}_1 \cdot \mathbf{e}_2$, where : $\mathbf{e}_2 = (0, \cos\varphi, \sin\varphi)$ is unit versor of the trace of water on frame planes (Fig.3).

Hence :

$$\sin\beta = -\sin\theta\sin\varphi \qquad (18)$$

The way of calculation of the geometrical characteristics of the waterplane arbitrarily heeled relative to ship is discussed in [4] and [13].

When the floatation axis **f** is known it is easy to find the analytic angles φ and θ describing ship's location relative to water at a new heel angle. Namely, by changing the slope of the waterplanes relative to each other, by the angle $\Delta\alpha_1$, the rotation of the versor **n** around the floatation axis by the angle $\Delta\alpha_1$ is induced. Hence the new versor $\mathbf{n}_1$ is given by the formula :

$$\mathbf{n}_1 = \mathbf{n}\cos\Delta\alpha_1 + (\mathbf{f} \cdot \mathbf{n})\sin\Delta\alpha_1 \qquad (19)$$

As the new versor **n** is known, the new analytical angles corresponding to the versor can be easily found by using the formula (6). Namely, $tg\theta = -n_x/n_z$, and $tg\varphi = -n_y/n_z$. Because the translation d**r** and the new waterplane slope angles are known, it is very fast to find a balanced location of buoyancy centre at the new heel angle enlarged by the angle $\Delta\phi$.

The versor **n** can be directly expressed by the angles θ and ϕ serving as degrees of freedom. To this end, in the formula (6) the expression $\cos\alpha = \cos\theta\cos\phi$ as well as that for $tg\phi$ given by (7) should be accounted for. The following is immediately obtained :

$$\mathbf{n} = (-\sin\theta\cos\phi, -\sin\phi, \cos\theta\cos\phi) \qquad (20)$$

Knowing the new versor **n**, one has as before : $tg\theta = -n_x/n_z$, and $\sin\phi = -n_y$. It is not necessary to find $tg\varphi = -n_y/n_z$, as in the first approximation the equation of new waterplane (in the ship-fixed frame) can be represented by the formula :

$$n_x(x - x_F) + n_y(y - y_F) + n_z(z - z_F) = 0 \qquad (21)$$

where :

x_F, y_F, z_F - the coordinates of the previous floatation centre.

The equation (21) is more convenient than (1), as $tg\varphi$ and T_o tend to infinitely large values along with the heel angle increasing. The equation (1) is necessary to start calculations. The waterplane given by (21) is neither equi-volume nor balanced one. Correct values of draught and trim can be found by means of the method of successive approximations so as to maintain ship displacement constant and equal to an assumed value, and to make the equation (8) satisfied. The calculations become significantly shorter in case of making use of the properties of equi-volume waterplanes of free-floating ship. Within a finite interval of the heel angle $\Delta\phi$, such waterplanes roll on the surface of a cone whose parameters can be determined in advance [4]. The rolling waterplanes adhere to the cone along an instantaneous floatation axis.

Mechanism of equi-volume heels

An infinitesimal rotation of waterplane around the floatation axis **f** can be considered as resulting from two rotations : ship's rotation by the angle $d\phi$ around the rotation axis $\mathbf{e} = \mathbf{e}_1$ (trace of water on PS), and ship's rotation by the angle $d\theta$ around the normal to PS. The directed angle $\mathbf{j}d\theta$ is inclined by the angle ϕ relative to water-level. As small rotations have vectorial properties the directed angle $\mathbf{f}d\alpha_1$ is the resultant of two components : parallel and normal to the rotation axis **e** :

$$\mathbf{f}d\alpha_1 = (d\phi, -d\theta\cos\phi) \qquad (22)$$

The components are defined in the $\xi\eta$ coordinate frame (see Fig.4). The normal component to the rotation axis, equal to $-d\theta\cos\phi$ is further denoted $d\alpha_2$. The directed angle $\mathbf{f}d\alpha_1$ is inclined by the angle χ to the rotation axis (ξ-axis). It can be seen in Fig.4 that the positive normal component $d\alpha_2$ corresponds to the positive angle χ, and the trim change is negative (by stern), therefore the normal component of the opposite sign must be applied. The projection of $d\alpha_1$ onto the rotation axis yields from the relationship (11). Taking into account the relationships inherent to rectangular triangles, one can determine the normal component $d\alpha_2$ in three different ways :

$$d\alpha_2 = d\alpha_1\sin\chi = d\phi tg\chi = -d\theta\cos\phi \qquad (23)$$

From this formula it results that :

- the more inclined the floatation axis from the rotation axis, the greater changes of ship trim during heeling, which is consistent with intuition
- when $\chi = 0$, i.e. $\mathbf{f} = \mathbf{e}$, ship trim does not change due to the waterplane rotation itself as in the case of fixed-trim ship
- for $\phi = 90°$ (PS is then horizontal), $\chi = 0$, i.e. the rotation axis (which does not then exist in the sense of trace of water on PS) determines the floatation axis **f**.

The rotation angle of PS around the normal, equal to $\mathbf{j}d\theta$, has also the vertical component : $-d\theta\sin\phi$. It makes the angle χ changing as a result of a change of orientation of the rotation axis (relative to ship hull), which occurs in the course of trimming. The change of the angle χ, which results from the change of orientation of the rotation axis only, is further denoted by $d\chi_o$. Hence the angle $d\chi_o = -d\theta\sin\phi$. Accounting for the equation (23), one obtains :

$$d\chi_0 = d\alpha_1\sin\chi tg\phi = d\phi tg\chi tg\phi \qquad (24)$$

If for a new waterplane the floatation angle χ changes by $d\chi$ the new floatation axis rotates against the previous one by the angle $d\chi_f = d\chi - d\chi_o$, equal to the difference of both the changes. When the angle $d\chi_f > 0$ is positive then the new floatation axis **f** shifts towards the heel, i.e. it departs from the rotation axis.

The equi-volume waterplanes roll on a cone whose axis is inclined relative to them by the angle ε determined by the following formula :

$$tg\varepsilon = d\chi_f/d\alpha_1$$

To find the formula is very simple : it is enough to observe that $tg\varepsilon$ is the ratio of the radius and the generatrix of the cone. When the angle $d\chi_f > 0$ is positive the cone is located above the waterplanes, otherwise – below them. The vertex of the cone is located in the distance *l*, of the generatrix from the floatation centre F, given by $l = -d\eta'_F/d\chi_f$, where $d\eta'_F$ is the translation of the floatation centre perpendicular to the floatation axis (when $l > 0$, the vertex is located fore). Taking into account that $d\eta'_F = r_F d\alpha_1$ one obtains :

$$l = -r_F d\alpha_1/d\chi_f = -r_F/tg\varepsilon$$

where :

$r_F = dI_f/dV$ – differential metacentric radius (curvature radius of the curve of floatation centres).

From the formula it results that the cone base radius at the level of the floatation centre equals the differential metacentric radius.

The normal waterplane rotation component $d\alpha_2$ represents the ship trim angle measured in the vertical plane passing through the trace of water on PS. If ship heel is enlarged by $d\phi$, the buoyancy centre translation perpendicular to the rotation angle is proportional to $Dd\phi$, where : D - waterplane deviation moment in the $\xi\eta$ coordinate frame (Fig.4) .

The translation must be compensated by the trim $I_{\eta\eta}d\alpha_2$. Equating them to each other one gets $d\alpha_2 = (D/I_{\eta\eta})d\phi$. Accounting for that $tg\chi = d\alpha_2/d\phi$, one obtains the formula (15). A more exact solution can be obtained by using the metacentric formula for $d\alpha_2 = (D/\nabla GM_L)d\phi$, where : $GM_L = BM_L - BZ$ is the longitudinal metacentric height. As $tg\chi = d\alpha_2/d\phi$, the above gives the formula (17) provided before without derivation. The formula accounts for the rotation plane slope change defined in the hull-fixed coordinate frame, which results from the ship trim change.

(***to be continued***)

NOMENCLATURE

a	-	height of gravity centre over buoyancy centre in upright position of ship
B	-	buoyancy centre
BM	-	transverse metacentric radius
BM_L	-	longitudinal metacentric radius
BP	-	baseplane
BZ	-	height of gravity centre over buoyancy centre
e	-	direction of rotation axis (versor normal to rotation plane)
$\mathbf{e}_1$, $\mathbf{e}_2$	-	versors of trace of water on PS and on midship plane, respectively
f	-	versor of floatation axis
F	-	freeboard
g	-	gravity acceleration
G	-	ship gravity centre
GM	-	metacentric height
GZ	-	righting arm
i, **j**, **k**	-	versors of the ship–fixed coordinate frame whose origin is in the point K (intersection point of the plane of symmetry, PS, midship plane and base plane BP)
l, l_d,	-	righting arm and dynamic arm, respectively
L, B, T	-	length, breadth and mean draught of ship, respectively
n,	-	upward pointing versor nomal to waterplane
PS	-	ship plane of symmetry
SOLAS	-	Safety of Life at Sea
SWATH	-	Small Waterplane Hull
T_o	-	draught of z-axis
WEGEMT	-	European Association of Universities in Marine Technology
∇	-	volumetric displacement of ship
α	-	angle between initial waterplane and water-level
β	-	angle between water trace lines on PS and midship plane
Δ	-	ship buoyancy (weight of the displaced water)
Δl	-	correction of righting arm obtained by means of cross-curves of stability, accounting for oblique translation of gravity centre relative to rotation plane at changing the height of ship gravity centre over BP
η	-	rotation angle of plane of rotation
θ	-	slope angle of trace of water on PS relative to x-axis of ship
ρ	-	water density
φ	-	slope angle of trace of water on the stations relative to y-axis of ship
ϕ	-	angle of PS inclination from the vertical
ϕ_V	-	angle of vanishing stability
χ	-	angle between floatation axis and rotation axis
ψ	-	angle between water-level trace line and PS trace line on initial waterplane

1) „Fix in space" does not mean : fixed relative to the ship hull coordinate system.

2) In case of the objects arbitrarily situated relative to the heeling moment vector or wind direction (e.g. semi-submersible floating units) the comments concerning PS should be applied to the reference plane initially situated perpendicularly to the wind direction and rotating together with the object in question. For the ship unsymmetricaly flooded this is the plane parallel to the main inertia axis of the waterplane in ship's upright position.

3) In the Di Belli method, a heel angle of ship model, induced by shifting a weight along an arm perpendicular to PS, is measured. The heel angle is the inclination angle of the arm against water--level.

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