Effective fatigue damage summation

Janusz Kolenda
Gdańsk University of Technology
Polish Naval University

ABSTRACT

The cumulative fatigue damage caused by uniaxial variable stress is considered. To facilitate the lifetime prediction at complex stress patterns, has been defined an effective stress of constant parameters, claimed to be equivalent to the sequence of original stresses in terms of fatigue life of material. The equivalence conditions are based on the theory of energy transformation systems. To the effective stress the Palmgren-Miner rule is applied. Thereby the cycle counting is not required.

Keywords : fatigue, design criteria, damage cumulation, stress modeling

INTRODUCTION

Most engineering components are subjected to variable loading. The fatigue damage in materials under cyclic loads of variable level, direction and/or mode, is accumulated at an unsteady rate. Though many cumulative damage theories have been developed, none of them enjoys universal acceptance. Each fatigue damage model can only account for one or several phenomenological factors, such as load dependence, multiple damage stages, non-linear damage evolution, load sequence and interaction effects, mean stress, overload effects and small amplitude cycles below fatigue limit. The most convenient is the Palmgren-Miner rule [1,2], and it has been adopted and used extensively, mainly because of its simplicity. According to the original concept :

- ✓ the fatigue process is cumulative,
- ✓ the fatigue effect is proportional to the work done by active loads,
- ✓ the increment of damage caused by n stress cycles of constant amplitude can be estimated as n/N, where N is the number of stress cycles which would cause failure of material in the same load state.

As to the loads of variable amplitude, the concept holds that the total damage can be estimated as the sum of the damage increments, each corresponding to a specified stress level. This can be symbolically expressed by the equation :

$$D = \sum_{j=1}^{w} \frac{n_j}{N_j} \qquad (1)$$

where :

D - total fatigue damage which equals unity at the failure
n_j - number of stress cycles at j-th stress level
N_j - number of cycles to failure at j-th stress level
w - number of stress levels.

The quantity N_j can be calculated as follows [3] :

$$N_j = \frac{K}{\sigma_j^m}\left(1 - \frac{\sigma_{mj}}{R}\right)^m \qquad (2)$$

where :

K - fatigue strength coefficient (in equation of relevant S-N curve)
m - fatigue strength exponent
R - yield strength - for ductile materials, ultimate strength - for brittle materials
σ_j - stress amplitude at j-th stress level
σ_{mj} - mean stress at j-th stress level.

Similar approach is applied to loads of variable level, direction and mode, namely :

$$D = \sum_i \sum_{j=1}^{w_i} \frac{n_{ij}}{N_{ij}} \quad ; \quad i = x, y, z, xy, yz, zx \qquad (3)$$

where the indices i = x,y,z are associated with normal stresses, and i = xy, yz, zx with shear stresses in Cartesian coordinate system.

The Palmgren-Miner rule implicitly assumes that the damage cumulation is independent of the order in which the stress cycles of different levels are applied and of the order of load modes. The assumptions have been found inadequate in certain cases [4]. Moreover, the cycle counting may be a source of uncertainties and D values may differ significantly from unity at the failure. Therefore several other cumulative damage hypotheses have been proposed [5] but their applicability varies from case to case. In this context a relatively simple approach based on the Palmgren-Miner rule and effective stress models, can be mentioned. Such approach for zero mean variable stresses has been presented in [6]; it will be used below for nonzero mean variable stresses.

EFFECTIVE STRESS

In accordance with the theory of energy transformation systems [7] the following can be stated.

An effective stress model and an original stress can be regarded equivalent in terms of fatigue life of material if during its service life the internally and externally dissipated energies under effective and original stresses are equal, respectively.

To illustrate the determination of the effective stress, let us consider a variable stress of i-th mode in the high-cycle fatigue regime :

$$\tilde{\sigma}_j(t) = \sigma_{mj} + \sigma_j(t) \;;\; j = 1, 2, \ldots, w \tag{4}$$

where $\sigma_j(t)$ is a zero mean time-varying stress.

The sequence of stresses (4) can be modeled by the effective stress :

$$\tilde{\sigma}_e(t) = \sigma_{me} + \sigma_e \sin\omega_e t \tag{5}$$

where σ_{me} , σ_e and ω_e are its mean value, amplitude and circular frequency, respectively. The equivalence conditions corresponding to the above given statement are [6] :

$$\int_0^T \tilde{\sigma}_e^2(t)\,dt = \sum_{j=1}^{w} \int_0^{t_j} \tilde{\sigma}_j^2(t)\,dt \tag{6}$$

$$\int_0^T \dot{\tilde{\sigma}}_e^2(t)\,dt = \sum_{j=1}^{w} \int_0^{t_j} \dot{\tilde{\sigma}}_j^2(t)\,dt \tag{7}$$

where :

t_j - the duration of j-th load state, dot denotes differentiation with respect to time, and T :

$$T = \sum_{j=1}^{w} t_j \tag{8}$$

Eqs (6) and (7) determine parameters of the effective stress. For example, when **:**

$$\tilde{\sigma}_j(t) = \sigma_{mj} + \sigma_j \sin\omega_j t \;\;;\;\; j = 1, 2, \ldots, w \tag{9}$$

these equations yield :

$$\sigma_{me} = \left[\frac{1}{T}\sum_{j=1}^{w} t_j \sigma_{mj}^2\right]^{1/2}$$

$$\sigma_e = \left[\frac{1}{T}\sum_{j=1}^{w} t_j \sigma_j^2\right]^{1/2} \tag{10}$$

$$\omega_e = \left[\frac{\sum_{j=1}^{w} t_j \omega_j^2 \sigma_j^2}{\sum_{j=1}^{w} t_j \sigma_j^2}\right]^{1/2}$$

In Eq (9) σ_j is the stress amplitude and ω_j - the stress circular frequency at j-th level.

FATIGUE DAMAGE

On determination of the effective stress parameters and the number of stress cycles during w load states of i-th mode :

$$n_e = \frac{\omega_e}{2\pi} T \tag{11}$$

the following is postulated :

- the fatigue process at the effective stress is cumulative,
- the fatigue effect is proportional to the dissipative energy,
- the increment of damage, corresponding to n_e cycles of the effective stress, can be estimated as follows :

$$D_e = \frac{n_e}{N_e} = D_{ei} \tag{12}$$

where :

$$N_e = \frac{K}{\sigma_e^m}\left(1 - \frac{\sigma_{me}}{R}\right)^m \tag{13}$$

Additionally, if the load is variable then the total fatigue damage based on the concept of the effective stress is assumed to be :

$$D_e = \sum_i D_{ei} \;;\; i = x, y, \ldots, zx \tag{14}$$

Of course, with the presented approach the aforementioned drawback of the Palmgren-Miner rule, associated with ignoring the load sequence effects, cannot be avoided, but the cycle counting is not required.

EXAMPLE

Task :

- Compare the cumulative fatigue damages D and D_e if the load is of i-th mode and the stress sequence consists of n_1 cycles of the stress :

$$\tilde{\sigma}_1(t) = \sigma_{m1} + \sigma_1 \sin\omega_1 t$$

and n_2 cycles of the stress :

$$\tilde{\sigma}_2(t) = \sigma_{m2} + \sigma_2 \sin\omega_2 t$$

in the high-cycle fatigue regime when :

$$\frac{\sigma_{mj}}{R} + \frac{\sigma_j}{F} > 1 \geq \frac{\sigma_{mj}}{R} + \frac{\sigma_j}{L} \;\;;\;\; j = 1, 2$$

where :

F - the fatigue limit under fully reversed stress of i-th mode

L - the maximum stress amplitude σ satisfying equation of i-th S-N curve

$$N\sigma^m = K$$

above which the low-cycle fatigue may occur.

- Consider the following cases :

A :
$$n_1 = n_2 = n \;;\; \sigma_{m1}/R = 0.5 \;;\; \sigma_{m2}/R = 0.5\gamma$$
$$\sigma_1 = \sigma_2 = \sigma \;;\; \omega_1 = \omega_2 = \omega$$

B :
$$n_1 = n_2 = n \;;\; \sigma_{m1} = \sigma_{m2} = \sigma_m$$
$$\sigma_2 = \gamma\sigma_1 \;;\; \omega_1 = \omega_2 = \omega$$

C :
$$n_1 = n_2 = n \;;\; \sigma_{m1} = \sigma_{m2} = \sigma_m$$
$$\sigma_2 = \sigma_2 = \sigma \;;\; \omega_2 = \gamma\omega_1$$

Solution :

A. According to the Palmgren-Miner rule and Eq.(2), the fatigue damage D equals :

$$D = \frac{n_1}{N_1} + \frac{n_2}{N_2} = n\left(\frac{1}{N_1} + \frac{1}{N_2}\right)$$

where :

$$N_1 = \frac{K}{\sigma^m}(1-0.5)^m \ ; \ N_2 = \frac{K}{\sigma^m}(1-0.5\gamma)^m$$

The parameters of the effective stress are calculated from Eqs (10) for $t_1 = t_2 = 0.5T$, as follows :

$$\frac{\sigma_{me}}{R} = \left[\frac{1}{T}\left(t_1\frac{\sigma_{m1}^2}{R^2} + t_2\frac{\sigma_{m2}^2}{R^2}\right)\right]^{1/2} = \left[0.125(1+\gamma^2)\right]^{1/2}$$

$$\sigma_e = \left[\frac{1}{T}(t_1\sigma_1^2 + t_2\sigma_2^2)\right]^{1/2} = \sigma$$

$$\omega_e = \left[\frac{t_1\omega_1^2\sigma_1^2 + t_2\omega_2^2\sigma_2^2}{t_1\sigma_1^2 + t_2\sigma_2^2}\right]^{1/2} = \omega$$

Since the load duration is :

$$T = \frac{2\pi}{\omega_1}n_1 + \frac{2\pi}{\omega_2}n_2 = \frac{4\pi}{\omega}n$$

the number of cycles of the effective stress amounts to :

$$n_e = \frac{\omega_e}{2\pi}T = 2n$$

and the number of its cycles to failure is :

$$N_e = \frac{K}{\sigma_e^m}\left(1-\frac{\sigma_{me}}{R}\right)^m = \frac{K}{\sigma^m}\left\{1-\left[0.125\,(1+\gamma^2)\right]^{1/2}\right\}^m$$

Hence

$$D_e = \frac{2n\sigma^m}{K\left\{1-\left[0.125\,(1+\gamma^2)\right]^{1/2}\right\}^m}$$

and

$$\frac{D}{D_e} = 0.5\left[\frac{1}{0.5^m} + \frac{1}{(1-0.5\gamma)^m}\right]\cdot \cdot\left\{1-\left[0.125\,(1+\gamma^2)\right]^{1/2}\right\}^m \qquad (15)$$

The results of the exemplary calculation of the ratio D/D_e are given in Tab. 1.

Tab. 1. *Results of calculation with the use of Eq. (15) .*

m	3				
γ	0.6	0.8	1.0	1.2	1.4
D/D$_e$	1.11	1.03	1.00	1.06	1.35
m	6				
γ	0.6	0.8	1.0	1.2	1.4
D/D$_e$	1.49	1.15	1.00	1.24	2.58

B. In the case of variable amplitude stress one obtains :

$$D = n\left(\frac{1}{N_1} + \frac{1}{N_2}\right)$$

where :

$$N_1 = \frac{K}{\sigma_1^m}\left(1-\frac{\sigma_m}{R}\right)^m \ ; \ N_2 = \frac{K}{(\gamma\sigma_1)^m}\left(1-\frac{\sigma_m}{R}\right)^m$$

The parameters of the effective stress are :

$$\sigma_{me} = \left[\frac{1}{T}\left(0.5T\sigma_m^2 + 0.5T\sigma_m^2\right)\right]^{1/2} = \sigma_m$$

$$\sigma_e = \left[\frac{1}{T}\left(0.5T\sigma_1^2 + 0.5T\gamma^2\sigma_1^2\right)\right]^{1/2} = \left[0.5\left(1+\gamma^2\right)\right]^{1/2}\sigma_1$$

$$\omega_e = \left[\frac{0.5T\omega^2\sigma_1^2 + 0.5T\omega^2\gamma^2\sigma_1^2}{0.5T\sigma_1^2 + 0.5T\gamma^2\sigma_1^2}\right]^{1/2} = \omega$$

so that :

$$N_e = \frac{K}{\left[0.5(1+\gamma^2)\right]^{m/2}\sigma_1^m}\left(1-\frac{\sigma_m}{R}\right)^m$$

and

$$\frac{D}{D_e} = \frac{n\left(\frac{1}{N_1}+\frac{1}{N_2}\right)}{\frac{2n}{N_e}} = \frac{1+\gamma^m}{2\left[0.5\,(1+\gamma^2)\right]^{m/2}} \qquad (16)$$

Exemplary values of D/D_e are presented in Tab. 2.

Tab. 2. *Results of calculation with the use of Eq. (16)* **.**

m	3				
γ	0.6	0.8	1.0	1.2	1.4
D/D$_e$	1.08	1.02	1.00	1.01	1.04
m	6				
γ	0.6	0.8	1.0	1.2	1.4
D/D$_e$	1.66	1.26	1.00	1.10	1.32

C. In Tab. 3 the values of the ratio D/D_e are presented as calculated for the stress of variable frequency at different relative frequency ranges :

$$\frac{\Delta\omega}{\omega_m} = \frac{|1-\gamma|}{\sqrt{\gamma}}$$

Here

$$\Delta\omega = |\omega_1 - \omega_2|$$

is the difference between higher and lower circular frequency of the stresses $\tilde{\sigma}_1(t)$ and $\tilde{\sigma}_2(t)$, and :

$$\omega_m = \sqrt{\omega_1\omega_2}$$

is the central circular frequency of the stress cycles. For the period :

$$T = t_1 + t_2 = \frac{2\pi}{\omega_1}n_1 + \frac{2\pi}{\omega_2}n_2 = \frac{2\pi n}{\omega}\left(1+\frac{1}{\gamma}\right)$$

the parameters of the effective stress are :

$$\sigma_{me} = \sigma_m \ ; \ \sigma_e = \sigma \ ; \ \omega_e = \sqrt{\gamma}\,\omega$$

and the number of its cycles is :

$$n_e = \frac{\omega_e}{2\pi}T = \frac{1+\gamma}{\sqrt{\gamma}}n$$

By applying Eqs (2) and (13), the number of cycles to failure under considered stresses is the same, namely :

$$N_1 = N_2 = N_e = \frac{K}{\sigma^m}\left(1-\frac{\sigma_m}{R}\right)^m$$

Consequently :

$$\frac{D}{D_e} = \frac{2n}{n_e} = \frac{2\sqrt{\gamma}}{1+\gamma} \qquad (17)$$

***Tab. 3.** Results of calculation with the use of Eq. (17) .*

γ	0.2	0.4	0.6	0.8	1.0	1.2
$\Delta\omega/\omega_m$	1.79	0.95	0.52	0.22	0.00	0.18
D/D_e	0.75	0.90	0.97	0.99	1.00	1.00
γ	1.4	1.6	1.8	2.0	4.0	10.0
$\Delta\omega/\omega_m$	0.34	0.47	0.60	0.71	1.50	2.85
D/D_e	0.99	0.97	0.96	0.94	0.80	0.57

From Tab. 1, 2 and 3 it can be observed that the linear summation of the fatigue damage corresponding to the effective stress, based on the theory of energy transformation systems, leads to less conservative results for the stresses of variable level (in particular at higher values of the fatigue strength exponent), and to even more conservative results for the stresses of variable frequency (especially at the relative frequency ranges greater than unity). The observed discrepancies between the values of the fatigue damages D and D_e result from different assumptions concerning the work of the active loads in the Palmgren-Miner rule and the dissipative energy in the equivalence conditions (6) and (7).

CONCLUSIONS

- For majority of engineering components, fatigue process under service conditions involves variable loading history. The cycle counting technique and the Palmgren-Miner rule are then frequently used for fatigue lifetime prediction. However, at complex stress patterns the cycle counting procedure may be complicated.
- The effective stress defined in the foregoing text, makes it possible to avoid this difficulty if the stress process is integrable in respect of time over the service period.
- From the exemplary calculations it follows that the linear summation of fatigue damages corresponding to the effective stress leads to less conservative results than that corresponding to the original stress of variable level , and to even more conservative results when the frequency of the original stress is variable.

NOMENCLATURE

D - fatigue damage
D_e - fatigue damage corresponding to the effective stress
D_{ei} - fatigue damage corresponding to the effective stress of i-th mode (i=x,y,z, xy, yz, zx)
F - fatigue limit under fully reversed stress
K - fatigue strength coefficient
L - maximum stress amplitude satisfying equation of the S-N curve above which the low-cycle fatigue may occur
m - fatigue strength exponent
n - number of stress cycles
n_e - number of the effective stress cycles
n_j - number of stress cycles at j-th level
n_{ij} - number of stress cycles of i-th mode at j-th level
N - number of zero mean stress cycles
N_e - number of the effective stress cycles to failure
N_j - number of stress cycles to failure at j-th level
N_{ij} - number of stress cycles to failure at i-th mode and j-th level
R - yield strength - for ductile materials, ultimate strength - for brittle materials
t - time
t_j - duration of j-th load state
T - duration of w load states
w - number of stress levels
w_i - number of levels of i-th mode stress
γ - multiplying factor
σ - stress amplitude
σ_e - amplitude of the effective stress
$\tilde{\sigma}_e(t)$ - effective stress
σ_j - amplitude of the stress $\tilde{\sigma}_j(t)$ in Eq. (9)
$\sigma_j(t)$ - zero mean time-varying stress of j-th level
$\tilde{\sigma}_j(t)$ - variable stress of j-th level
σ_m - mean value of the stress
σ_{me} - mean value of the effective stress
σ_{mj} - mean value of the stress $\tilde{\sigma}_j(t)$
ω - circular frequency
ω_e - circular frequency of the effective stress
ω_j - circular frequency of the stress $\tilde{\sigma}_j(t)$ in Eq. (9)

BIBLIOGRAPHY

1. Palmgren A.: *Die Lebensdauer von Kugellagern*. Verfahrenstechnik, 68, 1924
2. Miner M.A.: *Cumulative damage in fatigue. Journal of Applied Mechanics*, 67, 1945
3. Kolenda J.: *On fatigue safety of metallic elements under static and dynamic loads*. Gdańsk University of Technology Publishers, Gdańsk, 2004
4. Lin H., Nayeb-Hashemi H., Berg C.A.: *Cumulative damage behavior of anisotropic Al-6061-T6 as a function of axial-torsional loading mode sequence*. J. Engng Materials and Technology, 1, 1999
5. Fatemi A., Yang L.: *Cumulative fatigue damage and life prediction theories: a survey of the state of the art for homogeneous materials*. Int. J. Fatigue, 1, 1998
6. Kolenda J.: *Analytical procedures of high-cycle fatigue assessment of structural steel elements*. Gdańsk University of Technology Publishers, Gdańsk, 1997
7. Cempel C.: *Theory of energy transformation systems and their application in diagnostic of operating systems*. Applied Math. and Computer Sciences, 3, 1993.

CONTACT WITH THE AUTHOR
Prof. Janusz Kolenda
Faculty of Ocean Engineering
and Ship Technology,
Gdańsk University of Technology
Narutowicza 11/12
80-952 Gdańsk, POLAND
e-mail : sek7oce@pg.gda.pl

Main building of Gdańsk University of Technology