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# Time reversal symmetry in noncommutative phase space of the canonical type

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## Abstract

An algebra with noncommutativity of coordinates and noncommutativity of momenta which describes a rotationally- and time-reversal invariant quantum space is considered. A particle in a uniform field is studied in the space and the effect of space quantization on the energy levels of the particle is examined. We find a perihelion shift of a particle in a Coulomb potential in noncommutative phase space. Upper bounds for the parameters of noncommutativity are estimated.

## Keywords:

noncommutative space, time-reversal symmetry, rotational symmetry, gravitational field, minimal length

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# 1. Introduction

In ordinary space, commutation relations for coordinates and momenta

$$[X_1, X_2] = 0 \tag{1}$$

$$[X_1, P_1] = [X_2, P_2] = i\hbar, \tag{2}$$

$$[P_1, P_2] = 0, \tag{3}$$

are invariant upon time reversal [1].

If we consider transformations of coordinates and momenta upon time reversal as in the ordinary case

$$X_i \to X_i, \tag{4}$$

$$P_i \to -P_i, \tag{5}$$

taking into account that in the quantum case the time reversal operation also involves the operation of complex conjugation [1], in the case of a noncommutative algebra of the canonical type

$$[X_1, X_2] = i\hbar\theta, \tag{6}$$

$$[X_1, P_1] = [X_2, P_2] = i\hbar(1+\gamma), \tag{7}$$

$$[P_1, P_2] = i\hbar\eta, \tag{8}$$

with $\theta$, $\eta$, $\gamma$ being the parameters of noncommutativity, we find

$$[X_1, X_2] = -i\hbar\theta, \tag{9}$$

$$[X_1, P_1] = [X_2, P_2] = i\hbar(1+\gamma), \tag{10}$$

$$[P_1, P_2] = -i\hbar\eta. \tag{11}$$

In the present paper, we introduce an algebra with noncommutativity of coordinates and noncommutativity of momenta that does not lead to the violation of rotational and time-reversal symmetries and is equivalent to the noncommutative algebra of the canonical type. Within this algebraic framework, the motion of a system of free particles is studied, and the spectrum of particles in a uniform field is determined. Additionally, the motion in a gravitational field is analyzed. We obtain a stringent upper bound for the momentum scale based on studies of the perihelion shift of the planet Mercury.

The paper is organized as follows. In Section 2 transformation of noncommutative coordinated and noncommutative momenta upon time reversal is considered. In Section 3, circular motion is studied in a noncommutative phase space of the canonical type. In Section 4, we construct an algebra that is rotationally invariant and does not lead to the breaking of time-reversal symmetry. Section 5 studies the motion of a free particle system in a noncommutative phase space with preserved rotational and time-reversal symmetries. Section 6 is devoted to studying the energy of a particle in a uniform field in noncommutative phase space. In Section 7, the motion of a particle in a uniform gravitational field is analyzed, and the weak equivalence principle is examined. In Section 8, the equivalence principle is studied in the case of motion in a non-uniform gravitational field. Section 9 is devoted to the calculation of the perihelion shift of the planet Mercury in a rotationally invariant and time-reversal invariant noncommutative phase space. In Section 10, upper bounds for the parameters of coordinate noncommutativity and momentum noncommutativity are obtained. The conclusions are presented in Section 11

Results presented in this paper are published in [2–4].

# 2. Transformation of noncommutative coordinates and noncommutative momenta upon time-reversal

Because of (9)-(11) the transformation of coordinates and momenta $X_i$, $P_i$ after time reversal depends on representation. Noncommutative coordinates and momenta satisfying (6)-(8) can be represented by coordinates and momenta that satisfy the ordinary commutation relations as

$$X_1 = \varepsilon\left(x_1 - \theta_1' p_2\right), \tag{12}$$

$$X_2 = \varepsilon\left(x_2 + \theta_2' p_1\right), \tag{13}$$

$$P_1 = \varepsilon\left(p_1 + \eta_1' x_2\right), \tag{14}$$

$$P_2 = \varepsilon\left(p_2 - \eta_2' x_1\right). \tag{15}$$

Here $\varepsilon$, $\theta_1'$, $\theta_2'$, $\eta_2'$, $\eta_2'$ are constants.

After time reversal, if we consider transformations for coordinates and momenta as in ordinary space $x_i \to x_i$, $p_i \to -p_i$, we obtain

$$X_1 \to X_1' = \varepsilon\left(x_1 + \theta_1' p_2\right), \tag{16}$$

$$X_2 \to X_2' = \varepsilon\left(x_2 - \theta_2' p_1\right), \tag{17}$$

$$P_1 \to -P_1' = \varepsilon\left(-p_1 + \eta_1' x_2\right), \tag{18}$$

$$P_2 \to -P_2' = \varepsilon\left(-p_2 - \eta_2' x_1\right). \tag{19}$$

The results (16)-(19) depend on the parameters $\varepsilon$, $\theta_1'$, $\theta_2'$,

$\eta_2'$, $\eta_2'$. So, the transformation of the noncommutative coordinates depends on the representation.

One can choose parameters $\varepsilon$, $\theta_1'$, $\theta_2'$, $\eta_2'$, $\eta_2'$ in different ways. On the basis of (12)-(15) we can write

$$[X_1, X_2] = i\hbar\varepsilon^2(\theta_1' + \theta_2'), \tag{20}$$

$$[X_1, P_1] = i\hbar\varepsilon^2(1 + \theta_1'\eta_1') \tag{21}$$

$$[X_2, P_2] = i\hbar\varepsilon^2(1 + \theta_2'\eta_2'), \tag{22}$$

$$[P_1, P_2] = i\hbar\varepsilon^2(\eta_1' + \eta_2'). \tag{23}$$

Comparing (6)-(8) and (20)-(23) we obtain

$$\varepsilon^2 = 1, \ \ \theta_1'\eta_1' = \theta_2'\eta_2' = \gamma, \tag{24}$$

$$\theta_1' + \theta_2' = \theta, \tag{25}$$

$$\eta_1' + \eta_2' = \eta. \tag{26}$$

Based on the equations, we find

$$\theta_1' = \frac{1}{2}\left(\theta \pm \sqrt{\theta^2 - 4\frac{\theta\gamma}{\eta}}\right), \tag{27}$$

$$\theta_2' = \frac{1}{2}\left(\theta \mp \sqrt{\theta^2 - 4\frac{\theta\gamma}{\eta}}\right), \tag{28}$$

$$\eta_1' = \frac{1}{2}\left(\eta \mp \sqrt{\eta^2 - 4\frac{\eta\gamma}{\theta}}\right), \tag{29}$$

$$\eta_2' = \frac{1}{2}\left(\eta \pm \sqrt{\eta^2 - 4\frac{\eta\gamma}{\theta}}\right), \tag{30}$$

and $\gamma \leq \theta\eta/4$. So, we have two different representations for noncommutative coordinates and noncommutative momenta. These representations determine two different transformations after time reversal (16)-(19).

Well-known is the symmetric representation

$$\varepsilon = 1, \tag{31}$$

$$\theta_1' = \theta_2' = \frac{\theta}{2}, \tag{32}$$

$$\eta_1' = \eta_2' = \frac{\eta}{2}. \tag{33}$$

In this case

$$\gamma = \frac{\theta\eta}{4}, \tag{34}$$

see [5]. If $\gamma = 0$, one has the ordinary commutation relation for coordinates and momenta. The commutator for coordinates and momenta is equal to $i\hbar$. Taking into account (6)-(8), (20)-(23), $\gamma = 0$ we have

$$\varepsilon^2 = \frac{1}{1 + \theta_1'\eta_1'}, \tag{35}$$

$$\theta_1'\eta_1' = \theta_2'\eta_2', \tag{36}$$

$$\varepsilon^2(\theta_1' + \theta_2') = \theta, \tag{37}$$

$$\varepsilon^2(\eta_1' + \eta_2') = \eta, \tag{38}$$

One has one free parameter. Namely, five parameters $\varepsilon$, $\theta_1'$, $\theta_2'$, $\eta_1'$, $\eta_2'$ are related with four equations (35)-(38). So, by choosing one of the parameters one can obtain different representations for noncommutative coordinates and momenta which satisfy (6)-(8) with $\gamma = 0$. So, one can write different transformations after time reversal (16)-(19).

If we choose $\theta_2' = 0$ we find $\varepsilon = 1$, $\eta_1' = 0$, $\eta_2' = \eta$, $\theta_1' = \theta$. The representation is the following

$$X_1 = x_1 - \theta p_2, \tag{39}$$

$$X_2 = x_2, \tag{40}$$

$$P_1 = p_1, \tag{41}$$

$$P_2 = p_2 - \eta x_1. \tag{42}$$

So, upon time reversal the coordinate $X_2$, and momentum $P_1$ transform as in ordinary space $X_2 \to X_2$, $P_1 \to -P_1$. For coordinates and momenta $X_1$, $P_2$ we obtain

$$X_1 \to X_1' = x_1 + \theta p_2, \tag{43}$$

$$P_2 \to -P_2' = -p_2 - \eta x_1. \tag{44}$$

If we choose

$$\varepsilon = (1 + \theta'\eta')^{-\frac{1}{2}}, \tag{45}$$

$$\theta_1' = \theta_2' = \frac{1 \pm \sqrt{1 - \theta\eta}}{\eta}, \tag{46}$$

$$\eta_1' = \eta_2' = \frac{1 \pm \sqrt{1 - \theta\eta}}{\theta}, \tag{47}$$

we can write two symmetric representations (12)-(15) [5, 6]. These representations also lead to different transformations under the time reversal.

# 3. Circular motion in noncommutative phase space of the canonical type

Considering the Hamiltonian

$$H=\frac{P_1^2}{2m}+\frac{P_2^2}{2m}-\frac{k}{X}, \tag{48}$$

(here $X=\sqrt{X_1^2+X_2^2}$) and taking into account that coordinates and momenta $X_i$, $P_i$ satisfy relations of noncommutative algebra of the canonical type, we find

$$\dot{X}_1=\frac{P_1}{m}(1+\gamma)+\frac{k\theta X_2}{X^3}, \tag{49}$$

$$\dot{X}_2=\frac{P_2}{m}(1+\gamma)-\frac{k\theta X_1}{X^3}, \tag{50}$$

$$\dot{P}_1=\frac{\eta P_2}{m}-\frac{kX_1}{X^3}(1+\gamma), \tag{51}$$

$$\dot{P}_2=-\frac{\eta P_1}{m}-\frac{kX_2}{X^3}(1+\gamma). \tag{52}$$

Solutions of the equations that correspond to the circular motion read

$$X_1(t)=R_0\cos(\omega t), \tag{53}$$

$$X_2(t)=R_0\sin(\omega t), \tag{54}$$

$$P_1(t)=-P_0\sin(\omega t), \tag{55}$$

$$P_2(t)=P_0\cos(\omega t). \tag{56}$$

Here $R_0$ is the radius of the circle. The momentum reads

$$P_0=\frac{m\omega R_0^3+km\theta}{R_0^2(1+\gamma)}, \tag{57}$$

and frequency is defined as

$$\begin{aligned}\omega &= \frac{1}{2}\left(\sqrt{\frac{4k}{mR_0^3}((1+\gamma)^2-\theta\eta)+\left(\frac{k\theta}{R_0^3}+\frac{\eta}{m}\right)^2}+\right.\\ &\left.-\quad\frac{\eta}{m}-\frac{k\theta}{R_0^3}\right).\end{aligned} \tag{58}$$

For the period of motion, we have

$$\begin{aligned}T &= 4\pi\left(\sqrt{\frac{4k}{mR_0^3}((1+\gamma)^2-\theta\eta)+\left(\frac{k\theta}{R_0^3}+\frac{\eta}{m}\right)^2}+\right.\\ &\left.-\quad\frac{\eta}{m}-\frac{k\theta}{R_0^3}\right)^{-1}.\end{aligned} \tag{59}$$

If we study the motion in the opposite direction with the same radius $R_0$, we find

$$X_1(t)=R_0\cos(\omega t), \tag{60}$$

$$X_2(t)=-R_0\sin(\omega t), \tag{61}$$

$$P_1(t)=P_0'\sin(\omega t), \tag{62}$$

$$P_2(t)=P_0'\cos(\omega t). \tag{63}$$

Here we use notions $P_0'$ to distinguish momentum in the case of motion in the opposite direction. Using (60)-(63), (49)-(52) we find

$$\begin{aligned}\omega' &= \frac{1}{2}\left(\sqrt{\frac{4k}{mR_0^3}((1+\gamma)^2-\theta\eta)+\left(\frac{k\theta}{R_0^3}+\frac{\eta}{m}\right)^2}+\right.\\ &\left.+\quad\frac{\eta}{m}+\frac{k\theta}{R_0^3}\right),\end{aligned} \tag{64}$$

$$\begin{aligned}T' &= 4\pi\left(\sqrt{\frac{4k}{mR_0^3}((1+\gamma)^2-\theta\eta)+\left(\frac{k\theta}{R_0^3}+\frac{\eta}{m}\right)^2}+\right.\\ &\left.+\quad\frac{\eta}{m}+\frac{k\theta}{R_0^3}\right)^{-1},\end{aligned} \tag{65}$$

and the momentum reads

$$P_0'=-\frac{m\omega' R_0^3-km\theta}{R_0^2(1+\gamma)}. \tag{66}$$

It is important to stress that the expressions (58), (59), (64), (65) are different. We have

$$\Delta\omega=\omega'-\omega=\frac{\eta}{m}+\frac{k\theta}{R_0^3}. \tag{67}$$

Expressions for $\omega'$, $T'$ contain terms with parameters of noncommutativity with opposite signs in comparison to (58), (59). It is also important to stress that $P_0'\neq -P_0$. All these conclusions are caused by the breaking of the time-reversal symmetry in the noncommutative phase space of the canonical type.

# 4. Noncommutative phase space with preserved time reversal and rotational symmetries

To preserve rotational and time-reversal symmetries in noncommutative space, we introduce tensors of noncommutativity $\theta_{ij}$, $\eta_{ij}$ that transform under the time reversal as follows

$$\theta_{ij} \to -\theta_{ij}, \tag{68}$$

$$\eta_{ij} \to -\eta_{ij}. \tag{69}$$

and read

$$\theta_{ij} = \frac{c_\theta}{\hbar}\sum_k \varepsilon_{ijk} p_k^a, \tag{70}$$

$$\eta_{ij} = \frac{c_\eta}{\hbar}\sum_k \varepsilon_{ijk} p_k^b. \tag{71}$$

Here $c_\theta$, $c_\eta$ are constants, and $p_i^a$, $p_i^b$ are additional momenta that correspond to harmonic oscillators

$$H_{osc}^a = \frac{(p^a)^2}{2m_{osc}} + \frac{m_{osc}\omega^2 a^2}{2}, \tag{72}$$

$$H_{osc}^b = \frac{(p^b)^2}{2m_{osc}} + \frac{m_{osc}\omega^2 b^2}{2}. \tag{73}$$

with very large frequency $\omega$ and $\sqrt{\hbar}/\sqrt{m_{osc}\omega} = l_P$, $l_P$ is the Planck's length. So, rotationally-invariant and time-reversal invariant algebra reads

$$[X_i, X_j] = ic_\theta \sum_k \varepsilon_{ijk} p_k^a, \tag{74}$$

$$[X_i, P_j] = i\hbar\left(\delta_{ij} + \frac{c_\theta c_\eta}{4\hbar^2}(\mathbf{p}^a \cdot \mathbf{p}^b)\delta_{ij} - \frac{c_\theta c_\eta}{4\hbar^2} p_j^a p_i^b\right), \tag{75}$$

$$[P_i, P_j] = ic_\eta \sum_k \varepsilon_{ijk} p_k^b. \tag{76}$$

Additional coordinates and additional momenta satisfy the ordinary commutation relations.

It is important to stress that independently of representation coordinates and momenta upon time reversal transforms as $X_i \to X_i$, $P_i \to -P_i$. Coordinates and momenta which satisfy relations of noncommutative algebra (74)-(76) can be represented as

$$X_i = x_i + \frac{c_\theta}{2\hbar}[\mathbf{p}^a \times \mathbf{p}]_i, \tag{77}$$

$$P_i = p_i - \frac{c_\eta}{2\hbar}[\mathbf{x} \times \mathbf{p}^b]_i, \tag{78}$$

where operators $x_i$, $p_i$ satisfy the ordinary relations

$$[x_i, x_j] = [p_i, p_j] = 0, \tag{79}$$

$$[x_i, p_j] = i\hbar\delta_{ij}. \tag{80}$$

Upon time reversal, we have

$$x_i \to x_i, \tag{81}$$

$$p_i \to -p_i, \tag{82}$$

$$p_i^a \to -p_i^a, \tag{83}$$

$$p_i^b \to -p_i^b. \tag{84}$$

So, from (77), (78) we obtain that upon time reversal noncommutative coordinates and noncommutative momenta transform as

$$X_i \to X_i, \tag{85}$$

$$P_i \to -P_i. \tag{86}$$

Also, it is important that algebra (74)-(76) is rotationally invariant. After transformations

$$X_i' = U(\varphi)X_i U^+(\varphi), \tag{87}$$

$$P_i' = U(\varphi)P_i U^+(\varphi), \tag{88}$$

$$p_i^{a\prime} = U(\varphi)p_i^a U^+(\varphi), \tag{89}$$

$$p_i^{b\prime} = U(\varphi)p_i^b U^+(\varphi), \tag{90}$$

we have

$$[X_i', X_j'] = ic_\theta \sum_k \varepsilon_{ijk} p_k^{a\prime}, \tag{91}$$

$$[X_i', P_j'] = i\hbar\left(\delta_{ij} + \frac{c_\theta c_\eta}{4\hbar}(\mathbf{p}^{a\prime} \cdot \mathbf{p}^{b\prime})\delta_{ij} - \frac{c_\theta c_\eta}{4\hbar} p_j^{a\prime} p_i^{b\prime}\right), \tag{92}$$

$$[P_i', P_j'] = ic_\eta \sum_k \varepsilon_{ijk} p_k^{b\prime}, \tag{93}$$

where $U(\varphi) = \exp(i\varphi(\mathbf{n}\cdot\mathbf{L}^{\mathbf{t}})/\hbar)$, with $\mathbf{L}^{\mathbf{t}} = [\mathbf{x}\times\mathbf{p}] + [\mathbf{a}\times\mathbf{p}^a] + [\mathbf{b}\times\mathbf{p}^b]$.

The algebra is consistent. This follows from the explicit representation (77), (78).

## 5. Effect of noncommutativity of momentum on the motion of a system of free particles in time reversal and rotationally invariant noncommutative space

Let us consider a system of $N$ particles in time-reversal and rotationally invariant noncommutative phase space. The Hamiltonian reads

$$H=\sum_n\frac{(\mathbf{P}^{(n)})^2}{2m_n}+H^a_{osc}+H^b_{osc}. \tag{94}$$

Here index $n$ labels the particles. Using representation for noncommutative momenta with coordinates and momenta satisfying the ordinary commutation relation, we can write

$$\begin{aligned} H &= \sum_n\left(\frac{(\mathbf{p}^{(n)})^2}{2m_n}-\frac{(\boldsymbol{\eta}^{(n)}\cdot\mathbf{L}^{(n)})}{2m_n}+\frac{[\boldsymbol{\eta}^{(n)}\times\mathbf{x}^{(n)}]^2}{8m_n}\right)+ \\ &+ \hbar\omega_{osc}\left(\frac{(\tilde{p}^a)^2}{2}+\frac{\tilde{a}^2}{2}\right)+\hbar\omega_{osc}\left(\frac{(\tilde{p}^b)^2}{2}+\frac{\tilde{b}^2}{2}\right), \end{aligned} \tag{95}$$

where $\mathbf{L}^{(n)}$ reads

$$\mathbf{L}^{(n)}=[\mathbf{x}^{(n)}\times\mathbf{p}^{(n)}]. \tag{96}$$

In the case of a system of free particles, we have the following expressions for $H_0$ and $\Delta H$

$$\begin{aligned} H_0 &= \sum_n\left(\frac{(\mathbf{p}^{(n)})^2}{2m_n}+\frac{\langle(\eta^{(n)})^2\rangle(\mathbf{x}^{(n)})^2}{12m_n}\right)+ \\ &+ \hbar\omega_{osc}\left(\frac{(\tilde{p}^a)^2}{2}+\frac{\tilde{a}^2}{2}\right)+ \\ &+ \hbar\omega_{osc}\left(\frac{(\tilde{p}^b)^2}{2}+\frac{\tilde{b}^2}{2}\right), \end{aligned} \tag{97}$$

$$\begin{aligned} \Delta H &= \sum_n\left(-\frac{(\boldsymbol{\eta}^{(n)}\cdot\mathbf{L}^{(n)})}{2m_n}+\right. \\ &+ \left.\frac{[\boldsymbol{\eta}^{(n)}\times\mathbf{x}^{(n)}]^2}{8m_n}-\frac{\langle(\eta^{(n)})^2\rangle(\mathbf{x}^{(n)})^2}{12m}\right). \end{aligned} \tag{98}$$

So, up to the second order in the parameter of momentum noncommutativity we can study Hamiltonian $H_0$.

It is important that the following commutation relation is satisfied

$$\left[\sum_n\left(\frac{(\mathbf{p}^{(n)})^2}{2m_n}+\frac{\langle(\eta^{(n)})^2\rangle(\mathbf{x}^{(n)})^2}{12m_n}\right),H^a_{osc}+H^b_{osc}\right]=0. \tag{99}$$

Coordinates $x_i^{(n)}$ and momenta $p_i^{(n)}$ satisfy the ordinary commutation relations and therefore in the classical limit they satisfy the ordinary Poisson brackets. We have

$$\{x_i^{(n)},x_j^{(m)}\}=0, \tag{100}$$

$$\{x_i^{(n)},p_j^{(m)}\}=\delta_{ij}\delta_{nm}, \tag{101}$$

$$\{p_i^{(n)},p_j^{(m)}\}=0. \tag{102}$$

So, the Hamiltonian that describes a system of free particles reads

$$H_s=\sum_n\left(\frac{(\mathbf{p}^{(n)})^2}{2m_n}+\frac{\langle(\eta^{(n)})^2\rangle(\mathbf{x}^{(n)})^2}{12m_n}\right). \tag{103}$$

It corresponds to a Hamiltonian of a system of harmonic oscillators with frequencies determined by the parameters of momentum noncommutativity $\langle(\eta^{(n)})^2\rangle$ in the following way

$$\omega_n=\sqrt{\frac{\langle(\eta^{(n)})^2\rangle}{6m_n^2}}. \tag{104}$$

On the basis of expression (103) we can write the following equations

$$\begin{aligned} x_i^{(n)}(t) &= x_{0i}^{(n)}\cos\left(\sqrt{\frac{\langle(\eta^{(n)})^2\rangle}{6m_n^2}}t\right)+ \\ &+ \upsilon_{0i}^{(n)}\sqrt{\frac{6m_n^2}{\langle(\eta^{(n)})^2\rangle}}\sin\left(\sqrt{\frac{\langle(\eta^{(n)})^2\rangle}{6m_n^2}}t\right), \end{aligned} \tag{105}$$

where $x_{0i}^{(n)}$, $\upsilon_{0i}^{(n)}$ are the initial coordinates and initial velocity. It is important to stress that the trajectory of a free particle (151) depends on mass. This is because of the noncommutativity of momenta. As a result, even in the case when all particles have the same velocities $\upsilon_{0i}^{(n)}=\upsilon_{0i}$ the free particles fly away. For the trajectory of the center-of-mass and the relative motion we have the following expressions

$$\tilde{x}_i(t) = \sum_n \mu_n x_{0i}^{(n)} \cos\left(\sqrt{\frac{\langle(\eta^{(n)})^2\rangle}{6m_n^2}}t\right) + \sum_n \mu_n \upsilon_{0i}^{(n)} \sqrt{\frac{6m_n^2}{\langle(\eta^{(n)})^2\rangle}} \sin\left(\sqrt{\frac{\langle(\eta^{(n)})^2\rangle}{6m_n^2}}t\right), \tag{106}$$

$$\begin{aligned}\Delta x_i^{(n)}(t) &= x_{0i}^{(n)} \cos\left(\sqrt{\frac{\langle(\eta^{(n)})^2\rangle}{6m_n^2}}t\right) + \\ &+ \upsilon_{0i}^{(n)} \sqrt{\frac{6m_n^2}{\langle(\eta^{(n)})^2\rangle}} \sin\left(\sqrt{\frac{\langle(\eta^{(n)})^2\rangle}{6m_n^2}}t\right) + \\ &- \sum_l \mu_l x_{0i}^{(l)} \cos\left(\sqrt{\frac{\langle(\eta^{(l)})^2\rangle}{6m_l^2}}t\right) + \\ &+ \sum_l \mu_l \upsilon_{0i}^{(l)} \sqrt{\frac{6m_l^2}{\langle(\eta^{(l)})^2\rangle}} \sin\left(\sqrt{\frac{\langle(\eta^{(l)})^2\rangle}{6m_l^2}}t\right),\end{aligned} \tag{107}$$

where $\mu_n = m_n/\sum_l m_l$. It is important to stress that if the tensor of momentum noncommutativity is defined as

$$\eta_{ij}^{(n)} = \frac{\tilde{\alpha} m_n \hbar}{l_P^2} \sum_k \varepsilon_{ijk} \tilde{p}_k^b, \tag{108}$$

(here constant $\tilde{\alpha}$ does not depend on mass), we can write

$$\frac{\langle(\eta^{(n)})^2\rangle}{m_n^2} = \frac{3\hbar^2\tilde{\alpha}^2}{2l_P^4} = B. \tag{109}$$

Here we use notation $B$ for a constant which is the same for particles with different masses. Taking into account (200), we have the following expression for the trajectory

$$x_i^{(n)}(t) = x_{0i}^{(n)} \cos\left(\sqrt{\frac{B}{6}}t\right) + \upsilon_{0i}^{(n)} \sqrt{\frac{6}{B}} \sin\left(\sqrt{\frac{B}{6}}t\right). \tag{110}$$

If the initial velocities are the same

$$\upsilon_{0i}^{(n)} = \upsilon_{0i}, \tag{111}$$

the trajectory of the center-of-mass reads

$$\tilde{x}_i(t) = \tilde{x}_{0i} \cos\left(\sqrt{\frac{B}{6}}t\right) + \upsilon_{0i} \sqrt{\frac{6}{B}} \sin\left(\sqrt{\frac{B}{6}}t\right). \tag{112}$$

Here

$$\tilde{x}_{0i} = \sum_n \mu_n x_{0i}^{(n)}, \tag{113}$$

and the relative coordinates of particles do not depend on time

$$\Delta x_i^{(n)}(t) = x_{0i}^{(n)} - \tilde{x}_{0i}. \tag{114}$$

So, the dependence of the parameter of momentum noncommutativity on mass is important for solving the problem of flying away from a system of free particles.

## 6. Exact results for energy and wavefunctions of a particle in a uniform field in noncommutative phase space

We examine a particle with mass $m$ in a uniform field. The Hamiltonian reads

$$H_p = \frac{P^2}{2m} - \alpha X_3, \tag{115}$$

$\alpha$ is a constant. Without loss of generality, we study the case when the field is pointed in the $X_3$ direction (in (115). The total Hamiltonian reads

$$H = \frac{P^2}{2m} - \alpha X_3 + \frac{(p^a)^2}{2m_{osc}} + \frac{m_{osc}\omega_{osc}^2 a^2}{2}. \tag{116}$$

Using representation for noncommutative coordinates and noncommutative momenta, we can write

$$\begin{aligned}H &= \frac{p^2}{2m} - \alpha x_3 - \frac{1}{2}[\boldsymbol{\theta} \times \mathbf{p}]_3 + \frac{(p^a)^2}{2m_{osc}} + \frac{m_{osc}\omega_{osc}^2 a^2}{2} = \\ &= \frac{p^2}{2m} - \alpha x_3 - \frac{\alpha c_\theta}{2\hbar}(p_1^a p_2 - p_2^a p_1) + \frac{(p^a)^2}{2m_{osc}} + \\ &+ \frac{m_{osc}\omega_{osc}^2 a^2}{2}.\end{aligned} \tag{117}$$

Let us rewrite Hamiltonian (117) as follows

$$H = \left(1-\frac{\alpha^2 c_\theta^2 m m_{osc}}{4\hbar^2}\right)\frac{p_1^2}{2m}+$$
$$+ \left(1-\frac{\alpha^2 c_\theta^2 m m_{osc}}{4\hbar^2}\right)\frac{p_2^2}{2m}+\frac{p_3^2}{2m}-\alpha x_3+$$
$$+ \frac{1}{2m_{osc}}\left(p_1^a-\frac{\alpha c_\theta m_{osc}}{2\hbar}p_2\right)^2+$$
$$+ \frac{1}{2m_{osc}}\left(p_2^a+\frac{\alpha c_\theta m_{osc}}{2\hbar}p_1\right)^2+\frac{(p_3^a)^2}{2m_{osc}}+$$
$$+ \frac{m_{osc}\omega_{osc}^2 a_1^2}{2}+\frac{m_{osc}\omega_{osc}^2 a_2^2}{2}+\frac{m_{osc}\omega_{osc}^2 a_3^2}{2}. \quad (118)$$

It is important to note that operators

$$\tilde{H}_p = \left(1-\frac{\alpha^2 c_\theta^2 m}{4\hbar\omega_{osc} l_P^2}\right)\frac{p_1^2}{2m}+$$
$$+ \left(1-\frac{\alpha^2 c_\theta^2 m}{4\hbar\omega_{osc} l_P^2}\right)\frac{p_2^2}{2m}+\frac{p_3^2}{2m}-\alpha x_3, \quad (119)$$
$$\tilde{H}_{osc} = \frac{1}{2m_{osc}}\left(p_1^a-\frac{\alpha c_\theta}{2\omega_{osc} l_P^2}p_2\right)^2+$$
$$+ \frac{1}{2m_{osc}}\left(p_2^a+\frac{\alpha c_\theta}{2\omega_{osc} l_P^2}p_1\right)^2+$$
$$+ \frac{(p_3^a)^2}{2m_{osc}}+\frac{m_{osc}\omega_{osc}^2 a_1^2}{2}+\frac{m_{osc}\omega_{osc}^2 a_2^2}{2}+$$
$$+ \frac{m_{osc}\omega_{osc}^2 a_3^2}{2}, \quad (120)$$

commute

$$[\tilde{H}_p,\tilde{H}_{osc}]=0. \quad (121)$$

The Hamiltonian of the particle $\tilde{H}_p$ can be rewritten as

$$\tilde{H}_p=\tilde{H}_1+\tilde{H}_2+\tilde{H}_3, \quad (122)$$

where

$$\tilde{H}_1=\frac{p_1^2}{2m_{eff}}, \quad (123)$$

$$\tilde{H}_2=\frac{p_2^2}{2m_{eff}}, \quad (124)$$

$$\tilde{H}_3=\frac{p_3^2}{2m}-\alpha x_3, \quad (125)$$

$$[\tilde{H}_1,\tilde{H}_2]=[\tilde{H}_2,\tilde{H}_3]=[\tilde{H}_1,\tilde{H}_3]=0, \quad (126)$$

with effective mass

$$m_{eff} = m\left(1-\frac{\alpha^2 c_\theta^2 m m_{osc}}{4\hbar^2}\right)^{-1} =$$
$$= m\left(1-\frac{\alpha^2 c_\theta^2 m}{4\hbar\omega_{osc} l_P^2}\right)^{-1}. \quad (127)$$

It is important to mention that $x_3$, $p_3$ in $\tilde{H}_3$ satisfy the ordinary commutation relations. So, Hamiltonian $\tilde{H}_3$ is the Hamiltonian of a particle in a uniform field in ordinary space. Let us introduce

$$\tilde{p}_1^a=p_1^a-\frac{\alpha c_\theta}{2\omega_{osc} l_P^2}p_2, \quad (128)$$

$$\tilde{p}_2^a=p_2^a+\frac{\alpha c_\theta}{2\omega_{osc} l_P^2}p_1, \quad (129)$$

$$\tilde{p}_3^a=p_3^a. \quad (130)$$

So, we can write (120) as follows

$$\tilde{H}_{osc}=\frac{(\tilde{p}^a)^2}{2m_{osc}}+\frac{m_{osc}\omega_{osc}^2 a^2}{2}. \quad (131)$$

For operators $a_i$ and $\tilde{p}_i^a$ we have the ordinary commutation relations

$$[a_i,a_j]=[\tilde{p}_i^a,\tilde{p}_j^a]=0, \quad (132)$$

$$[a_i,\tilde{p}_j^a]=i\hbar\delta_{ij}. \quad (133)$$

For operators $\tilde{H}_1$, $\tilde{H}_2$, $\tilde{H}_3$, $\tilde{H}_{osc}$ we have (121), (126). So, the exact expression for the spectrum of a particle in a uniform field reads

$$E = \frac{\hbar^2 k_1^2}{2m}\left(1-\frac{\alpha^2 c_\theta^2 m}{4\hbar\omega_{osc} l_P^2}\right)+$$
$$+ \frac{\hbar^2 k_2^2}{2m}\left(1-\frac{\alpha^2 c_\theta^2 m}{4\hbar\omega_{osc} l_P^2}\right)+E_3+\frac{3}{2}\hbar\omega_{osc}. \quad (134)$$

It is important to mention that we have free motion of a particle in the directions perpendicular to the field. Values $k_1$, $k_2$ are components of the wave vector that correspond to this free motion. Notation $E_3$ is used for denoting continuous eigenvalues of Hamiltonian $\tilde{H}_3$. In (134) the last term corresponds to the ground state of the harmonic oscillator.

The eigenfunctions of the total Hamiltonian (118) can be written as

$$\psi(\mathbf{x},\mathbf{a})=\tilde{\psi}_1(x_1)\tilde{\psi}_2(x_2)\tilde{\psi}_3(x_3)\tilde{\psi}(\mathbf{a}). \quad (135)$$

Here $\tilde{\psi}_i(x_i)$ are eigenfunctions of $\tilde{H}_i$ that are defined as

(123)-(125). The eigenfunction of a particle in the uniform field in the space with commutative coordinates and commutative momenta $\psi^{(3)}(x_3)$ reads

$$\psi^{(3)}(x_3) = \left(\frac{4m^2}{\pi^3\alpha\hbar^4}\right)^{\frac{1}{6}} \Phi\left(\left(\frac{2m\alpha}{\hbar^2}\right)^{\frac{1}{3}}\left(-x_3-\frac{E_3}{\alpha}\right)\right), \tag{136}$$

where $\Phi$ is the Airy function

$$\Phi(x) = \frac{1}{\sqrt{\pi}}\int_0^\infty \cos\left(\frac{t^3}{3}+tx\right)dt. \tag{137}$$

Functions $\tilde{\psi}(\mathbf{a})$ denote eigenfunctions of Hamiltonian

$$\begin{aligned} H'_{osc} &= \frac{1}{2m_{osc}}\left(p_1^a-\frac{\alpha c_\theta \hbar k_2}{2\omega_{osc}l_P^2}\right)^2+ \\ &+ \frac{1}{2m_{osc}}\left(p_2^a+\frac{\alpha c_\theta \hbar k_1}{2\omega_{osc}l_P^2}\right)^2+ \\ &+ \frac{(p_3^a)^2}{2m_{osc}}+\frac{m_{osc}\omega_{osc}^2a_1^2}{2}+\frac{m_{osc}\omega_{osc}^2a_2^2}{2}+ \\ &+ \frac{m_{osc}\omega_{osc}^2a_3^2}{2}. \end{aligned} \tag{138}$$

Note that expression for Hamiltonian (138) is obtained replacing $p_1$ by $\hbar k_1$ and $p_2$ by $\hbar k_2$, in (120). The ground state of harmonic oscillator (138) is as follows

$$\tilde{\psi}(\mathbf{a}) = \frac{1}{\pi^{\frac{3}{4}}l_P^{\frac{3}{2}}}e^{-\frac{a^2}{2l_P^2}-i\beta(k_1a_2-k_2a_1)}, \tag{139}$$

with

$$\beta = \frac{\alpha c_\theta}{2\omega_{osc}l_P^2}. \tag{140}$$

So, for the total Hamiltonian (118) we have the following eigenfunctions

$$\begin{aligned} &\psi(\mathbf{x},\mathbf{a}) = \\ &= Ce^{ik_1x_1}e^{ik_2x_2}\Phi\left(\left(\frac{2m\alpha}{\hbar^2}\right)^{\frac{1}{3}}\left(-x_3-\frac{E_3}{\alpha}\right)\right) \\ &e^{-\frac{a^2}{2l_P^2}-i\beta(k_1a_2-k_2a_1)}, \end{aligned} \tag{141}$$

where $C$ is the normalization constant.

It is important to stress that noncommutativity affects the motion of a particle in the directions perpendicular to the direction of the field. Namely, it affects the mass of the particle in uniform field.

# 7. Motion of a particle in a uniform gravitational field in noncommutative phase space with preserved time reversal and rotational symmetries

Let us consider the motion of a particle of mass $m$ in rotationally-invariant and time-reversal invariant noncommutative phase space (74)-(76). The Hamiltonian of a particle in a uniform field is as follows

$$H_p = \frac{\mathbf{P}^2}{2m}+mgX_1. \tag{142}$$

In the Hamiltonian, we considered the $X_1$ axis to be directed along the field direction. The total Hamiltonian in terms of commuting coordinates and commuting momenta reads

$$\begin{aligned} H &= \frac{\mathbf{p}^2}{2m}+mgx_1-\frac{(\boldsymbol{\eta}\cdot\mathbf{L})}{2m}+\frac{mg}{2}[\boldsymbol{\theta}\times\mathbf{p}]_1+ \\ &+ \frac{[\boldsymbol{\eta}\times\mathbf{x}]^2}{8m}+H_{osc}^a+H_{osc}^b. \end{aligned} \tag{143}$$

This Hamiltonian can be represented as

$$H = H_0+\Delta H, \tag{144}$$

$$H_0 = \langle H_p\rangle_{ab}+H_{osc}^a+H_{osc}^b, \tag{145}$$

$$\Delta H = H-H_0 = H_p-\langle H_p\rangle_{ab}, \tag{146}$$

$$H_0 = \frac{\mathbf{p}^2}{2m}+mgx_1+\frac{\langle\eta^2\rangle\mathbf{x}^2}{12m}+H_{osc}^a+H_{osc}^b, \tag{147}$$

$$\begin{aligned} \Delta H &= -\frac{(\boldsymbol{\eta}\cdot\mathbf{L})}{2m}+\frac{mg}{2}[\boldsymbol{\theta}\times\mathbf{p}]_1 \\ &+ \frac{[\boldsymbol{\eta}\times\mathbf{x}]^2}{8m}-\frac{\langle\eta^2\rangle\mathbf{x}^2}{12m}. \end{aligned} \tag{148}$$

Up to the second order in $\Delta H$ one can study Hamiltonian $H_0$. In this approximation, one can write the equations of motion of the particle

$$\dot{x}_i = \frac{p_i}{m}, \tag{149}$$

$$\dot{p}_i = -mg\delta_{i,1}-\frac{\langle\eta^2\rangle x_i}{6m}. \tag{150}$$

The solution of the equations is as follows

$$x_i(t) = \left(x_{0i} + 6g\frac{m^2}{\langle\eta^2\rangle}\delta_{1,i}\right)\cos\left(\sqrt{\frac{\langle\eta^2\rangle}{6m^2}}t\right) + $$
$$+ \upsilon_{0i}\sqrt{\frac{6m^2}{\langle\eta^2\rangle}}\sin\left(\sqrt{\frac{\langle\eta^2\rangle}{6m^2}}t\right) - 6g\frac{m^2}{\langle\eta^2\rangle}\delta_{1,i}, \tag{151}$$

where we considered notations $x_{0i}$, $\upsilon_{0i}$ for initial coordinates and velocities of the particle. Note that only momentum noncommutativity affects the motion of a particle in a gravitational field. Considering limit $\langle\eta^2\rangle \to 0$ we find the well-known result in ordinary space

$$x_i(t) = \delta_{1,i}\frac{gt^2}{2} + x_{0i}. \tag{152}$$

Analyzing (151) we can see that the weak equivalence principle is violated because of noncommutativity. According to the principle, the velocity and position of a point mass in a gravitational field are independent of mass.

If we consider the parameter of momentum noncommutativity to be dependent on mass as

$$\frac{\langle\eta^2\rangle}{m^2} = \frac{3\hbar^2\tilde{\alpha}^2}{2l_P^4} = B = \text{const}, \tag{153}$$

where $B$ does not depend on mass, one obtains the following trajectory

$$x_i(t) = \left(x_{0i} + \frac{6g}{B}\delta_{1,i}\right)\cos\left(\sqrt{\frac{B}{6}}t\right) + $$
$$+ \upsilon_{0i}\sqrt{\frac{6}{B}}\sin\left(\sqrt{\frac{B}{6}}t\right) - \frac{6g}{B}\delta_{1,i}. \tag{154}$$

Let us consider a case when the parameters of noncommutativity are related with mass

$$c_\theta^{(n)} = \frac{\tilde{\gamma}}{m_n}, \tag{155}$$

$$c_\eta^{(n)} = \tilde{\alpha}m_n, \tag{156}$$

see [7]. Due to condition (156) the trajectory of a particle in the gravitational field does not depend on mass, and the weak equivalence principle is preserved.

Let us study a more general case. For a composite system in the gravitational field we have the following Hamiltonian

$$H_s = \frac{(\mathbf{P}^c)^2}{2M} + MgX_1^{(c)} + H_{rel}, \tag{157}$$

where $\mathbf{X}^{(\mathbf{c})}$, $\mathbf{P}^c$ are coordinates and momenta of the center-of-mass of the composite system. Hamiltonian $H_{rel}$ represents the relative motion. In the case when conditions (155), (156) are satisfied, we can represent the Hamiltonian as follows

$$H_0 = \frac{(\mathbf{p}^c)^2}{2M} + Mgx_1^c + \frac{\langle(\eta^c)^2\rangle(\mathbf{x}^c)^2}{12M} + \langle H_{rel}\rangle_{ab} + $$
$$+ H_{osc}^{(a)} + H_{osc}^{(b)}. \tag{158}$$

Taking into account that

$$[H_0, \langle H_{rel}\rangle_{ab}] = 0, \tag{159}$$

we can write

$$x_i^c(t) = \left(x_{0i}^c + 6g\frac{M^2}{\langle(\eta^c)^2\rangle}\delta_{1,i}\right)\cos\left(\sqrt{\frac{\langle(\eta^c)^2\rangle}{6M^2}}t\right) + $$
$$+ \upsilon_{0i}^c\sqrt{\frac{6M^2}{\langle(\eta^c)^2\rangle}}\sin\left(\sqrt{\frac{\langle(\eta^c)^2\rangle}{6M^2}}t\right) + $$
$$- 6g\frac{M^2}{\langle(\eta^c)^2\rangle}\delta_{1,i}, \tag{160}$$

Due to condition (156) the trajectory can be rewritten as

$$\frac{\langle(\eta^c)^2\rangle}{M^2} = \frac{3\hbar^2\tilde{\alpha}^2}{2l_P^4} = B = \text{const}, \tag{161}$$

$$x_i^c(t) = \left(x_{0i}^c + \frac{6g}{B}\delta_{1,i}\right)\cos\left(\sqrt{\frac{B}{6}}t\right) + $$
$$+ \upsilon_{0i}\sqrt{\frac{6}{B}}\sin\left(\sqrt{\frac{B}{6}}t\right) - \frac{6g}{B}\delta_{1,i}, \tag{162}$$

So, the weak equivalence principle is satisfied.

Using (151) for the trajectory of the center-of-mass of a system of $N$ non-interacting particles in a uniform gravitational field we have

$$x_i^c(t) = \sum_a \mu_a x_i^{(a)}(t) = -\sum_a 6g\mu_a\frac{m_a^2}{\langle(\eta^{(a)})^2\rangle}\delta_{1,i} + $$
$$+ \sum_a \mu_a\left(x_{0i}^{(a)} + 6g\frac{m_a^2}{\langle(\eta^{(a)})^2\rangle}\delta_{1,i}\right)\times $$
$$\times \cos\left(\sqrt{\frac{\langle(\eta^{(a)})^2\rangle}{6m_a^2}}t\right) + $$
$$+ \sum_a \mu_a\upsilon_{0i}^{(a)}\sqrt{\frac{6m_a^2}{\langle(\eta^a)^2\rangle}}\sin\left(\sqrt{\frac{\langle(\eta^a)^2\rangle}{6m_a^2}}t\right), \tag{163}$$

Here $m_a$ is the mass of particle $a$, $x_{0i}^{(a)}$, $\upsilon_{0i}^{(a)}$ are initial coordinates and initial velocities. Note, that due to condition (156), taking into account

$$x_{0i}^{(c)} = \sum_a \mu_a x_{0i}^{(a)}, \tag{164}$$

$$\upsilon_{0i}^{(c)} = \sum_a \mu_a \upsilon_{0i}^{(a)}, \tag{165}$$

one finds that expression (163) reduces to (160).

# 8. Motion in a non-uniform gravitational field in rotationally- and time-reversal invariant noncommutative phase space

For a particle in a non-uniform gravitational field, we have the following Hamiltonian

$$H_p = \frac{P^2}{2m} - \frac{G\tilde{M}m}{X}, \tag{166}$$

where $m$ is the mass of the particle,

$$X = |\mathbf{X}| = \sqrt{\sum_i X_i^2}. \tag{167}$$

Similarly as in the previous sections, up to the second order in the parameters of noncommutativity, we can consider the Hamiltonian as follows

$$\begin{aligned} H_0 &= \frac{p^2}{2m} - \frac{G\tilde{M}m}{x} + \frac{\langle\eta^2\rangle x^2}{12m} - \frac{G\tilde{M}mL^2\langle\theta^2\rangle}{8x^5} + \\ &+ \frac{G\tilde{M}m\langle\theta^2\rangle}{24}\left(\frac{2}{x^3}p^2 + \frac{6i\hbar}{x^5}(\mathbf{x}\cdot\mathbf{p}) - \frac{\hbar^2}{x^5}\right) + \\ &+ H_{osc}^a + H_{osc}^b. \end{aligned} \tag{168}$$

So, in this approximation of a particle in a non-uniform gravitational field, we have the following equations of motion

$$\dot{\mathbf{x}} = \frac{\mathbf{p}}{m} - \frac{G\tilde{M}m\langle\theta^2\rangle}{12}\left(\frac{1}{x^3}\mathbf{p} - \frac{3\mathbf{x}}{x^5}(\mathbf{x}\cdot\mathbf{p})\right), \tag{169}$$

$$\begin{aligned} \dot{\mathbf{p}} &= -\frac{G\tilde{M}m\mathbf{x}}{x^3} - \frac{\langle\eta^2\rangle\mathbf{x}}{6m} - \frac{G\tilde{M}m\langle\theta^2\rangle}{4}\left(\frac{1}{x^5}(\mathbf{x}\cdot\mathbf{p})\mathbf{p} + \right. \\ &- \left.\frac{2\mathbf{x}}{x^5}p^2 + \frac{5\mathbf{x}}{2x^7}L^2 + \frac{5\hbar^2\mathbf{x}}{6x^7} - \frac{5i\hbar}{x^7}\mathbf{x}(\mathbf{x}\cdot\mathbf{p})\right). \end{aligned} \tag{170}$$

In the limit $\hbar \to 0$ we can write

$$\dot{\mathbf{x}} = \boldsymbol{\upsilon} - \frac{G\tilde{M}m^2\langle\theta^2\rangle}{12}\left(\frac{1}{x^3}\boldsymbol{\upsilon} - \frac{3\mathbf{x}}{x^5}(\mathbf{x}\cdot\boldsymbol{\upsilon})\right), \tag{171}$$

$$\begin{aligned} \dot{\boldsymbol{\upsilon}} &= -\frac{G\tilde{M}\mathbf{x}}{x^3} - \frac{\langle\eta^2\rangle\mathbf{x}}{6m^2} + \\ &- \frac{G\tilde{M}m^2\langle\theta^2\rangle}{4}\left(\frac{1}{x^5}(\mathbf{x}\cdot\boldsymbol{\upsilon})\boldsymbol{\upsilon} - \frac{2\mathbf{x}}{x^5}\upsilon^2 + \frac{5\mathbf{x}}{2x^7}[\mathbf{x}\times\boldsymbol{\upsilon}]^2\right). \end{aligned} \tag{172}$$

Here we use notation

$$\boldsymbol{\upsilon} = \frac{\mathbf{p}}{m} \tag{173}$$

Note that the obtained results depend on $m^2\langle\theta^2\rangle$ and $\langle\eta^2\rangle/m^2$. So, if we consider conditions (155), (156) we can write

$$\dot{\mathbf{x}} = \boldsymbol{\upsilon} - \frac{G\tilde{M}A}{12}\left(\frac{1}{x^3}\boldsymbol{\upsilon} - \frac{3\mathbf{x}}{x^5}(\mathbf{x}\cdot\boldsymbol{\upsilon})\right), \tag{174}$$

$$\begin{aligned} \dot{\boldsymbol{\upsilon}} &= -\frac{G\tilde{M}\mathbf{x}}{x^3} - \frac{B\mathbf{x}}{6} - \\ &- \frac{G\tilde{M}A}{4}\left(\frac{1}{x^5}(\mathbf{x}\cdot\boldsymbol{\upsilon})\boldsymbol{\upsilon} - \frac{2\mathbf{x}}{x^5}\upsilon^2 + \frac{5\mathbf{x}}{2x^7}[\mathbf{x}\times\boldsymbol{\upsilon}]^2\right). \end{aligned} \tag{175}$$

Here we take into account (153), (155) and

$$\langle\theta^2\rangle m^2 = \frac{3\alpha^2 l_P^4 m^2}{2\hbar^2} = A = \text{const}. \tag{176}$$

Constant $A$ does not depend on mass.

Results for the equations of motion (174), (175) depend on constants $A$, $B$. The constants are the same for different particles. So, conditions (155), (156) open a possibility to recover the weak equivalence principle.

Let us also consider a quantum case. If relations (155), (156) are satisfied, the equations (169), (170) can be rewritten as

$$\dot{\mathbf{x}} = \boldsymbol{\upsilon} - \frac{G\tilde{M}B}{12}\left(\frac{1}{x^3}\boldsymbol{\upsilon} - \frac{3\mathbf{x}}{x^5}(\mathbf{x}\cdot\boldsymbol{\upsilon})\right), \tag{177}$$

$$\begin{aligned} \dot{\boldsymbol{\upsilon}} &= -\frac{G\tilde{M}\mathbf{x}}{x^3} - \frac{B\mathbf{x}}{6} - \frac{G\tilde{M}A}{4}\left(\frac{1}{x^5}(\mathbf{x}\cdot\boldsymbol{\upsilon})\boldsymbol{\upsilon} - \frac{2\mathbf{x}}{x^5}\upsilon^2 + \right. \\ &+ \left.\frac{5\mathbf{x}}{2x^7}[\mathbf{x}\times\boldsymbol{\upsilon}]^2 + \frac{5\hbar^2\mathbf{x}}{6m^2x^7} - \frac{5i\hbar}{mx^7}\mathbf{x}(\mathbf{x}\cdot\boldsymbol{\upsilon})\right). \end{aligned} \tag{178}$$

Note, that these equations depend on $\hbar/m$, as it has to be. This is due to commutation relation

$$[\mathbf{x}, \boldsymbol{\upsilon}] = i\hbar\frac{\hat{I}}{m}. \tag{179}$$

(see [8] for the details).

So, if relations (155), (156) hold, the motion of a particle in a gravitational field is independent of its mass, and the weak equivalence principle is preserved.

The same conclusion can be made in the case of motion of a composite system. We have

$$H_s = \frac{(P^c)^2}{2M} - \frac{G\tilde{M}M}{(X^c)^2} + H_{rel}, \tag{180}$$

$$\begin{aligned} H_0 &= \frac{(p^c)^2}{2M} - \frac{G\tilde{M}M}{x^c} + \frac{\langle(\eta^c)^2\rangle (x^c)^2}{12M} + \\ &- \frac{G\tilde{M}M (L^c)^2\langle\theta^2\rangle}{8(x^c)^5} + \frac{G\tilde{M}M\langle(\theta^c)^2\rangle}{24}\left(\frac{2}{(x^c)^3}(p^c)^2 + \right. \\ &+ \left. \frac{6i\hbar}{(x^c)^5}(\mathbf{x}^c\cdot\mathbf{p}^c) - \frac{\hbar^2}{(x^c)^5}\right) + \langle H_{rel}\rangle_{ab} + H^a_{osc} + H^b_{osc}. \end{aligned} \tag{181}$$

In the case when conditions (155), (156) are satisfied, we can write

$$\dot{\mathbf{x}}^c = \boldsymbol{\upsilon}^c - \frac{G\tilde{M}B}{12}\left(\frac{1}{(x^c)^3}\boldsymbol{\upsilon}^c - \frac{3\mathbf{x}^c}{(x^c)^5}(\mathbf{x^c}\cdot\boldsymbol{\upsilon}^c)\right), \tag{182}$$

$$\begin{aligned} \dot{\boldsymbol{\upsilon}}^c &= -\frac{G\tilde{M}\mathbf{x}^c}{(x^c)^3} - \frac{B\mathbf{x}^c}{6} - \frac{G\tilde{M}A}{4}\left(\frac{1}{(x^c)^5}(\mathbf{x}^c\cdot\boldsymbol{\upsilon}^c)\boldsymbol{\upsilon}^c - \right. \\ &- \left. \frac{2\mathbf{x}^c}{(x^c)^5}(\upsilon^c)^2 + \frac{5\mathbf{x}^c}{2(x^c)^7}[\mathbf{x}^c\times\boldsymbol{\upsilon}^c]^2\right). \end{aligned} \tag{183}$$

It is important to stress that if relations (155), (156) are not preserved, the equations of motion of a composite system depend on its mass and parameters $\langle(\theta^c)^2\rangle$, $\langle(\eta^c)^2\rangle$. The parameters are defined as

$$\theta^c_{ij} = \sum_n \mu_n^2\theta^{(n)}_{ij}, \tag{184}$$

$$\eta^c_{ij} = \sum_n \eta^{(n)}_{ij}, \tag{185}$$

and depend on the composition. So, this in addition causes violation of the weak equivalence principle in quantum space.

# 9. Studies of the effect of space quantization on the motion of Mercury

Let us first consider a particle of mass $m$ in the gravitational field $-k/X$ in noncommutative phase space with preserved rotational and time-reversal symmetries (74)-(76). So, the total Hamiltonian reads

$$H = H_p + H^a_{osc} + H^b_{osc}, \tag{186}$$

$$H_p = \frac{P^2}{2m} - \frac{mk}{X}. \tag{187}$$

Here $X_i$, $P_i$ satisfy relations (74)-(76), terms $H^a_{osc}$, $H^b_{osc}$ are Hamiltonians of harmonic oscillators. Up to the second order in the parameters of noncommutativity, we can consider Hamiltonian as follows

$$\begin{aligned} \langle H_p\rangle_{ab} &= \frac{p^2}{2m} - \frac{mk}{x} + \frac{\langle\eta^2\rangle x^2}{12m} - \frac{\langle\theta^2\rangle mkL^2}{8x^5} + \\ &+ \frac{\langle\theta^2\rangle mkp^2}{12x^3}. \end{aligned} \tag{188}$$

Noncommutativity of coordinates and noncommutativity of momenta cause the precession of the orbit of the particle. To find the precession rate of the orbit, we consider

$$\mathbf{u} = \frac{\mathbf{p}}{m} - \frac{mk[\mathbf{L}\times\mathbf{x}]}{xL^2}, \tag{189}$$

and calculate

$$\boldsymbol{\Omega} = \frac{[\mathbf{u}\times\dot{\mathbf{u}}]}{u^2}. \tag{190}$$

We obtain

$$\left\{\mathbf{u}, \frac{p^2}{2m} - \frac{mk}{x}\right\} = 0, \tag{191}$$

$$\begin{aligned} \dot{\mathbf{u}} &= \left\{\mathbf{u}, \frac{\langle\eta^2\rangle x^2}{12m} - \frac{\langle\theta^2\rangle mkL^2}{8x^5} + \frac{\langle\theta^2\rangle mkp^2}{12x^3}\right\} = \\ &= -\frac{\langle\eta^2\rangle\mathbf{x}}{6m^2} - \frac{k\langle\theta^2\rangle}{4}\left(\frac{(\mathbf{x}\cdot\mathbf{p})\mathbf{p}}{x^5} - \frac{2p^2\mathbf{x}}{x^5} + \frac{5L^2\mathbf{x}}{2x^7}\right) + \\ &+ \frac{m^2k^2\langle\theta^2\rangle[\mathbf{L}\times\mathbf{p}]}{12L^2x^4} - \frac{m^2k^2\langle\theta^2\rangle(\mathbf{x}\cdot\mathbf{p})[\mathbf{L}\times\mathbf{x}]}{12L^2x^6}. \end{aligned} \tag{192}$$

It is known that in ordinary space

$$u^2 = \frac{m^2k^2e^2}{L^2}, \tag{193}$$

where $e$ is the eccentricity of the orbit. So, we find

$$\begin{aligned} \Omega &= \langle\theta^2\rangle\left(\frac{5L^4}{8km^3x^7e^2} - \frac{p^2L^2}{2m^3x^5ke^2} + \right. \\ &+ \left. \frac{p^2}{4me^2x^4} - \frac{7L^2}{24mx^6e^2} - \frac{mk}{12x^5e^2}\right)L + \\ &+ \langle\eta^2\rangle\left(\frac{L^2}{6m^5k^2e^2} - \frac{x}{6m^3ke^2}\right)L. \end{aligned} \tag{194}$$

For the perihelion shift per revolution, we can write

$$\begin{aligned}\Delta\phi_p &= \int_0^T \Omega dt = \int_0^{2\pi} \frac{\Omega}{\dot{\phi}} d\phi = \\ &= \langle\theta^2\rangle \frac{\pi k m^2(4+e^2)}{8a^3(1-e^2)^3} + \\ &- \langle\eta^2\rangle \frac{\pi a^3\sqrt{1-e^2}}{2m^2 k},\end{aligned} \tag{195}$$

with $a$ being the semi-major axis, $\phi$ being the polar angle. To find (195) we take into account that in ordinary space

$$L = mx^2\dot{\phi}, \tag{196}$$

$$x = \frac{a(1-e^2)}{1+e\cos\phi}, \tag{197}$$

$$\frac{p^2}{2m} - \frac{mk}{x} = -\frac{mk}{2a}. \tag{198}$$

It is important to stress that the perihelion shift depends on the mass of the particle $m$. If relations (155), (156) hold, we obtain

$$\langle\theta^2\rangle m^2 = \frac{3\tilde{\gamma}^2}{2l_P^2} = A, \tag{199}$$

$$\frac{\langle\eta^2\rangle}{m^2} = \frac{3\tilde{\alpha}^2}{2l_P^2} = B, \tag{200}$$

where $A, B$ are constants that do not depend on the masses of particles.

Taking into account (195), (199), and (200) we find

$$\Delta\phi_p = A\frac{\pi k(4+e^2)}{8a^3(1-e^2)^3} - B\frac{\pi a^3\sqrt{1-e^2}}{2k}. \tag{201}$$

It is worth mentioning that the proposed conditions (155), (156) are important for solving the problem of violation of the weak equivalence principle in quantum space.

For a composite system with mass $M$ in gravitational field, we have

$$H_s = H_{cm} + H_{rel}, \tag{202}$$

$$H_{cm} = \frac{(P^c)^2}{2M} - \frac{Mk}{X^c}, \tag{203}$$

$X_i^c$, $P_i^c$ are coordinates and momenta of the center-of-mass, $H_{rel}$ describes the relative motion. If relations (155), (156) are satisfied, commutators for coordinates and momenta correspond to noncommutative algebra (74), (76). The coordinates and momenta of the center-of-mass can be represented as

$$X_i^c = x_i^c - \frac{\theta_{ij}^c p_j^c}{2}, \tag{204}$$

$$P_i^c = p_i^c + \frac{\eta_{ij}^c x_j^c}{2}. \tag{205}$$

So, up to the second order in the parameters of noncommutativity we can study the Hamiltonian as follows

$$\begin{aligned} H_0 &= \langle H_s\rangle_{ab} + H_{osc}^a + H_{osc}^b = \\ &= \frac{(p^c)^2}{2M} - \frac{Mk}{x^c} + \frac{\langle(\eta^c)^2\rangle(x^c)^2}{12M} - \\ &- \frac{\langle(\theta^c)^2\rangle Mk(L^c)^2}{8(x^c)^5} + \\ &+ \frac{\langle(\theta^c)^2\rangle Mk}{24}\left(\frac{1}{(x^c)^2}(p^c)^2\frac{1}{x^c} + \right. \\ &+ \left. \frac{1}{x^c}(p^c)^2\frac{1}{(x^c)^2} + \frac{\hbar^2}{(x^c)^5}\right) + \\ &+ \langle H_{rel}\rangle_{ab} + H_{osc}^a + H_{osc}^b. \end{aligned} \tag{206}$$

Here

$$\mathbf{L}^c = [\mathbf{x}^c \times \mathbf{p}^c]. \tag{207}$$

Using definitions

$$\Delta\mathbf{X}^{(n)} = \mathbf{X}^{(n)} - \mathbf{X}^c, \tag{208}$$

$$\Delta\mathbf{P}^{(n)} = \mathbf{P}^{(n)} - \mu_n\mathbf{P}^c, \tag{209}$$

and taking into account (155), (156), we have

$$\Delta X_i^{(n)} = \Delta x_i^{(n)} - \frac{\theta_{ij}^{(n)}\Delta p_j^{(n)}}{2}, \tag{210}$$

$$\Delta P_i^{(n)} = \Delta p_i^{(n)} + \frac{\eta_{ij}^{(n)}\Delta x_j^{(n)}}{2}. \tag{211}$$

Here coordinates and momenta

$$\Delta\mathbf{x}^{(n)} = \mathbf{x}^{(n)} - \mathbf{x}^c, \tag{212}$$

$$\Delta\mathbf{p}^{(n)} = \mathbf{p}^{(n)} - \mu_n\mathbf{p}^c, \tag{213}$$

satisfy the ordinary commutation relations. It is important that $\langle H_{rel}\rangle_{ab}$ commutes with $H_0$. So, one can consider the following Hamiltonian

$$\langle H_{cm}\rangle_{ab} = \frac{(p^c)^2}{2M} - \frac{Mk}{x^c} + \frac{\langle(\eta^c)^2\rangle(x^c)^2}{12M} + - \frac{\langle(\theta^c)^2\rangle Mk(L^c)^2}{8(x^c)^5} + \frac{\langle(\theta^c)^2\rangle Mk(p^c)^2}{12(x^c)^3}. \tag{214}$$

Using (195), for the perihelion shift of orbit of a macroscopic body we can write

$$\Delta\phi_{nc} = \langle(\theta^c)^2\rangle \frac{\pi k M^2(4+e^2)}{8a^3(1-e^2)^3} - \langle(\eta^c)^2\rangle \frac{\pi a^3\sqrt{1-e^2}}{2M^2k}, \tag{215}$$

where

$$\langle(\theta^c)^2\rangle = \frac{3\tilde{\gamma}^2}{2l_P^2M^2} = \frac{A}{M^2}, \tag{216}$$

$$\langle(\eta^c)^2\rangle = \frac{3\tilde{\alpha}^2M^2}{2l_P^2} = BM^2. \tag{217}$$

# 10. Upper bounds on the parameters of noncommutativity

We apply the obtained result for the perihelion shift for the planet Mercury. We compare the perihelion shift caused by space quantization (215) with

$$\Delta\phi_{obs} - \Delta\phi_{GR} = = 2\pi(-0.00049 \pm 0.00017)\times \times 10^{-8}\text{radians/revolution} \tag{218}$$

(here $\Delta\phi_{GR}$ is the perihelion precession rate from General Relativity predictions, $\Delta\phi_{obs}$ is the result of observations). We assume that $|\Delta\phi_{nc}|$ is less than $|\Delta\phi_{obs} - \Delta\phi_{GR}|$ at $3\sigma$ and write the following inequality

$$|\Delta\phi_{nc}| \leq 2\pi\cdot 10^{-11}\text{radians/revolution}, \tag{219}$$

Parameter $\theta^c_{ij}$ or parameter $\eta^c_{ij}$ could be equal to zero. Therefore, it is sufficient to consider the following inequalities

$$\left|\langle(\theta^c)^2\rangle \frac{\pi G M_\odot M^2(4+e^2)}{8a^3(1-e^2)^3}\right| \leq 2\pi\cdot 10^{-11}\text{radians/revolution}, \tag{220}$$

$$\left|\langle(\eta^c)^2\rangle \frac{\pi a^3\sqrt{1-e^2}}{2GM_\odot M^2}\right| \leq 2\pi\cdot 10^{-11}\text{radians/revolution}, \tag{221}$$

where $M$ is the mass of Mercury, $a$, $e$ are parameters of its orbit. So, we find

$$\hbar\sqrt{\langle(\theta^c)^2\rangle} < 2.3\cdot 10^{-57}\text{m}^2, \tag{222}$$

$$\hbar\sqrt{\langle(\eta^c)^2\rangle} < 1.8\cdot 10^{-22}\text{kg}^2\text{m}^2/\text{s}^2. \tag{223}$$

Taking into account (199), (200), (216), 217), we have

$$\langle(\theta^c)^2\rangle M^2 = \langle(\theta^{(n)})^2\rangle m_n^2, \tag{224}$$

$$\frac{\langle(\eta^c)^2\rangle}{M^2} = \frac{\langle(\theta^{(n)})^2\rangle}{m_n^2}, \tag{225}$$

where parameters $\langle(\theta^{(n)})^2\rangle$, $\langle(\eta^{(n)})^2\rangle$ correspond to a particle of mass $m_n$.

Based on relations (222), (223), (224), (225) one can find upper bounds on the parameters of noncommutativity of different particles. In the case of the electron, we have

$$\hbar\sqrt{\langle(\theta^{(e)})^2\rangle} < 8.3\cdot 10^{-4}\text{m}^2, \tag{226}$$

$$\hbar\sqrt{\langle(\eta^{(e)})^2\rangle} < 5.1\cdot 10^{-76}\text{kg}^2\text{m}^2/\text{s}^2. \tag{227}$$

We do not obtain a strong upper bound for the parameter of coordinate noncommutativity. This is because the influence of the noncommutativity of coordinates on the motion of macroscopic bodies is less than on the motion of particles. So, for strong upper bounds on the parameters of coordinate noncommutativity, data of high accuracy are needed.

The result (227) is quite strong. This result is at least ten orders less than that obtained based on studies of the hydrogen and exotic atoms [7, 9, 10]. Using (227), we can also estimate the minimal momentum

$$p_{min} = \sqrt[4]{\frac{3\hbar^2\langle(\eta^{(e)})^2\rangle}{2}} < 2.5\cdot 10^{-38}\text{kg}\cdot\text{m/s} \tag{228}$$

In the case of nucleons, we have

$$\frac{\langle(\eta^c)^2\rangle}{M^2} = \frac{\langle(\theta^{(nuc)})^2\rangle}{m_{nuc}^2}, \tag{229}$$

$$\hbar\sqrt{\langle(\eta^{(nuc)})^2\rangle} < 9.3\cdot 10^{-73}\text{kg}^2\text{m}^2/\text{s}^2, \tag{230}$$

where $m_{nuc}$ is the mass of the nucleon. The obtained result (230) is 6 orders less than that estimated on the basis of studies of neutrons in a gravitational quantum well [11].

# 11. Conclusions

The noncommutative phase space of the canonical type, with preserved rotational and time-reversal symmetries, has been considered (74)-(76). A corresponding noncommutative algebra (74)-(76) is constructed by generalizing the parameters of noncommutativity to tensors, which are defined with the help of additional momenta. These momenta are governed by harmonic oscillators.

We have analyzed a particle in a uniform field within the framework of the noncommutative algebra. The energy and wave functions of the particle have been detemined precisely (134), (141). It was obtained that noncommutativity affects the mass of the particle in directions perpendicular to the field. However, the motion of the particle along the field direction remains the same as in ordinary space.

The effect of space quantization on a particle in a Coulomb potential has also been studied. We derived an expression for the perihelion shift of the particle's orbit up to the second order in the parameters of noncommutativity. This result was generalized to the case of macroscopic body motion. Upper bounds (222), (223) have been estimated based on the perihelion shift of the planet Mercury in quantum space (215) and the precession data from MESSENGER spacecraft ranging.

The obtained upper bounds for the parameters of momentum noncommutativity (227), (230), and the minimal momentum (52) are stringent. For the parameter of momentum noncommutativity for an electron, we obtained an upper bound (227) that is at least 10 orders of magnitude smaller than the one based on studies of the hydrogen atom [10, 12].

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