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Hydrogen and exotic atoms in rotationally-invariant space with noncommutativity of coordinates and noncommutativity of momenta

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Abstract

The effect of the noncommutativity of coordinates and noncommutativity of momenta on the spectrum of the hydrogen atom is studied. Corrections to the energy levels of the atom up to the second order in the parameter of noncommutativity are found. Based on the obtained results and the experimental data for the $1S-2S$ transition frequency, the upper bound for the minimal length is obtained. Also, a two-particle system with Coulomb interaction is examined and hydrogen-like exotic atoms are studied in rotationally-invariant noncommutative phase space.

Keywords:

noncommutative space, rotational symmetry, hydrogen atom, exotic atoms

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1. Introduction

Studies of the hydrogen atom in noncommutative space have received much attention (see [1–11]). In the paper [1], energy levels of hydrogen atom were obtained up to the first order in the parameter of noncommutativity. In the paper, the Lamb shift in noncommutative space was studied. In the paper [2], the case when particles of opposite charges feel opposite noncommutativity was examined. In the frame of such an algebra, the hydrogen atom as a two-particle system was considered. In [4], the quadratic Stark effect was studied. In [5], shifts in the spectrum of the hydrogen atom caused by space quantization were presented. In [6], the noncommutative Klein-Gordon equation was studied and the hydrogen atom energy levels were calculated. The influence of noncommutativity on the Dirac equation with a Coulomb field was studied in [7, 8].

The effect of the noncommutativity of coordinates and noncommutativity of momenta on the energy levels of the hydrogen atom was examined in [9–11]. The hydrogen atom problem in the frame of space-time noncommutativity was considered in [12–16].

In the present chapter, we examine the hydrogen atom and hydrogen-like exotic atoms in the frame of rotationally-invariant noncommutative algebra of the canonical type. The energy levels of the hydrogen atom are found up to the second order in the parameters of coordinate and momentum noncommutativity. Based on the obtained results, the upper bounds for the parameters are estimated. Also, a two-particle system with Coulomb interaction is studied in the frame of rotationally-invariant noncommutative algebra. We examine the influence of space quantization on the energy levels of the system. Based on the obtained results, the energy levels of muonic hydrogen and antiprotonic helium are examined.

The paper is organized as follows. In Section 2, rotationally-invariant noncommutative algebra is introduced. In Section 3, the Hamiltonian of a hydrogen atom is examined in the frame of rotationally-invariant noncommutative algebra. In Section 4, corrections to the energy levels of the hydrogen atom are found up to the second order in the perturbation theory. Section 5 is devoted to studies of the corrections to the *ns* energy levels of the hydrogen atom. The effect of noncommutativity on the energy levels of hydrogen-like atoms is examined in Section 6. Upper bounds for the parameters of coordinate and momentum noncommutativity are obtained in section 7. Section 8 is devoted to conclusions. The results presented in this paper are published in [17–20].

2. Rotationally-invariant noncommutative space of the canonical type

The noncommutative algebra

$$[X_i,X_j] = i\varepsilon_{ijk}l_0a_k, \tag{1}$$

$$[X_i,P_j] = i\hbar\left(\delta_{ij}+\frac{l_0p_0}{4\hbar^2}(\mathbf{a}\cdot\mathbf{p^b})\delta_{ij}-\frac{l_0p_0}{4\hbar^2}a_jp_i^b\right), \tag{2}$$

$$[P_i,P_j] = \varepsilon_{ijk}p_0p_k^b, \tag{3}$$

is rotationally-invariant and equivalent to a noncommutative algebra of the canonical type [20]. To construct the algebra, the parameters of noncommutativity θ_{ij}, η_{ij} are considered to be the tensors

$$\theta_{ij} = \frac{l_0}{\hbar}\varepsilon_{ijk}a_k, \tag{4}$$

$$\eta_{ij} = \frac{p_0}{\hbar}\varepsilon_{ijk}p_k^b. \tag{5}$$

Here, l_0, p_0 are constants and a_i, p_k^b are additional coordinates and momenta satisfying the ordinary commutation relations. They are governed by spherically symmetric systems, for instance, the harmonic oscillators

$$H_{osc}^a = \frac{(p^a)^2}{2m_{osc}}+\frac{m_{osc}\omega^2a^2}{2}, \tag{6}$$

$$H_{osc}^b = \frac{(p^b)^2}{2m_{osc}}+\frac{m_{osc}\omega^2b^2}{2}, \tag{7}$$

with $\sqrt{\hbar}/\sqrt{m_{osc}\omega}$ being equal to the Planck's length l_P. The frequency ω is assumed to be very large.

The algebra (1)-(3) is equivalent to a noncommutative algebra of the canonical type

$$[X_i,X_j] = i\hbar\theta_{ij}, \tag{8}$$

$$[X_i,P_j] = i\hbar(\delta_{ij}+\gamma_{ij}), \tag{9}$$

$$[P_i,P_j] = i\hbar\eta_{ij}, \tag{10}$$

in the sense that the following relations are satisfied

$$[\theta_{ij},\eta_{ij}] = [\theta_{ij},\gamma_{ij}] = [\gamma_{ij},\eta_{ij}] = 0. \tag{11}$$

3. The Hamiltonian of the hydrogen atom in noncommutative phase space with preserved rotational symmetry

Let us consider the hydrogen atom and find corrections to the energy levels of the atom in rotationally invariant noncommutative phase space (1)-(3). So, we consider the total Hamiltonian

$$H = H_h + H^a_{osc} + H^b_{osc}, \tag{12}$$

where

$$H_h = \frac{P^2}{2M} - \frac{e^2}{R}, \tag{13}$$

is the Hamiltonian of the hydrogen atom. Here $R = \sqrt{\sum_i X_i^2}$, coordinates X_i and momenta P_i satisfy the relations of the noncommutative algebra (1)-(3). The Hamiltonians H^a_{osc}, H^b_{osc} correspond to harmonic oscillators and are given by (6), (7).

Using representation for coordinates and momenta that satisfy the relations of the noncommutative algebra by coordinates and momenta satisfying the ordinary relations, we can write

$$\begin{aligned} H_h &= \frac{p^2}{2M} + \frac{(\boldsymbol{\eta}\cdot\mathbf{L})}{2M} + \frac{[\boldsymbol{\eta}\times\mathbf{r}]^2}{8M} - \\ &+ \frac{e^2}{\sqrt{r^2 - (\boldsymbol{\theta}\cdot\mathbf{L}) + \frac{1}{4}[\boldsymbol{\theta}\times\mathbf{p}]^2}}. \end{aligned} \tag{14}$$

To find the effect of the noncommutativity of the energy levels of the hydrogen atom, we expand the Hamiltonian of the hydrogen atom in the series over $\boldsymbol{\theta}$. For $1/R$ we obtain

$$\begin{aligned} \frac{1}{R} &= \frac{1}{\sqrt{r^2 - (\boldsymbol{\theta}\cdot\mathbf{L}) + \frac{1}{4}[\boldsymbol{\theta}\times\mathbf{p}]^2}} = \\ &= \frac{1}{r} + \frac{1}{2r^3}(\boldsymbol{\theta}\cdot\mathbf{L}) + \frac{3}{8r^5}(\boldsymbol{\theta}\cdot\mathbf{L})^2 - \\ &+ \frac{1}{16}\left(\frac{1}{r^2}[\boldsymbol{\theta}\times\mathbf{p}]^2\frac{1}{r} + \frac{1}{r}[\boldsymbol{\theta}\times\mathbf{p}]^2\frac{1}{r^2} + \frac{\hbar^2}{r^7}[\boldsymbol{\theta}\times\mathbf{r}]^2\right). \end{aligned} \tag{15}$$

To find the expansion for $1/R$, firstly we solve the problem of finding the expansion of R up to the second order in $\boldsymbol{\theta}$. The expression for the distance reads

$$R = \sqrt{(\mathbf{r} + \frac{1}{2}[\boldsymbol{\theta}\times\mathbf{p}])^2} = \sqrt{r^2 - (\boldsymbol{\theta}\cdot\mathbf{L}) + \frac{1}{4}[\boldsymbol{\theta}\times\mathbf{p}]^2}. \tag{16}$$

It is important to stress that the operators under the square root do not commute. Therefore, we introduce the unknown function $f(\mathbf{r})$ and find the expansion in the following form

$$\begin{aligned} R &= r - \frac{1}{2r}(\boldsymbol{\theta}\cdot\mathbf{L}) - \frac{1}{8r^3}(\boldsymbol{\theta}\cdot\mathbf{L})^2 + \\ &+ \frac{1}{16}\left(\frac{1}{r}[\boldsymbol{\theta}\times\mathbf{p}]^2 + [\boldsymbol{\theta}\times\mathbf{p}]^2\frac{1}{r} + \theta^2 f(\mathbf{r})\right). \end{aligned} \tag{17}$$

Then to obtain $f(\mathbf{r})$, we square the left- and right-hand sides of equation (17). Up to the second order in $\boldsymbol{\theta}$ we can write

$$\begin{aligned} & r^2 - (\boldsymbol{\theta}\cdot\mathbf{L}) + \frac{1}{4}[\boldsymbol{\theta}\times\mathbf{p}]^2 = r^2 - (\boldsymbol{\theta}\cdot\mathbf{L}) + \\ & + \frac{1}{16}\left(2[\boldsymbol{\theta}\times\mathbf{p}]^2 + r[\boldsymbol{\theta}\times\mathbf{p}]^2\frac{1}{r} + \right. \\ & \left. + \frac{1}{r}[\boldsymbol{\theta}\times\mathbf{p}]^2 r + 2r\theta^2 f(\mathbf{r})\right). \end{aligned} \tag{18}$$

From (18) we have

$$\frac{\hbar^2}{r^4}[\boldsymbol{\theta}\times\mathbf{r}]^2 - r\theta^2 f(\mathbf{r}) = 0. \tag{19}$$

And finally, function $f(\mathbf{r})$ reads

$$\theta^2 f(\mathbf{r}) = \frac{\hbar^2}{r^5}[\boldsymbol{\theta}\times\mathbf{r}]^2. \tag{20}$$

So, the expansion for the distance is as follows

$$\begin{aligned} R &= r - \frac{1}{2r}(\boldsymbol{\theta}\cdot\mathbf{L}) - \frac{1}{8r^3}(\boldsymbol{\theta}\cdot\mathbf{L})^2 + \\ &+ \frac{1}{16}\left(\frac{1}{r}[\boldsymbol{\theta}\times\mathbf{p}]^2 + [\boldsymbol{\theta}\times\mathbf{p}]^2\frac{1}{r} + \frac{\hbar^2}{r^5}[\boldsymbol{\theta}\times\mathbf{r}]^2\right). \end{aligned} \tag{21}$$

Then, on the basis of this result we can easily write (15). As a result, the total Hamiltonian reads

$$H = H_0 + V. \tag{22}$$

Here, V is the perturbation operator

$$V = \frac{(\boldsymbol{\eta}\cdot\mathbf{L})}{2M} + \frac{[\boldsymbol{\eta}\times\mathbf{r}]^2}{8M} - \frac{e^2}{2r^3}(\boldsymbol{\theta}\cdot\mathbf{L}) - \frac{3e^2}{8r^5}(\boldsymbol{\theta}\cdot\mathbf{L})^2 + $$
$$+ \frac{e^2}{16}\left(\frac{1}{r^2}[\boldsymbol{\theta}\times\mathbf{p}]^2\frac{1}{r} + \frac{1}{r}[\boldsymbol{\theta}\times\mathbf{p}]^2\frac{1}{r^2} + \frac{\hbar^2}{r^7}[\boldsymbol{\theta}\times\mathbf{r}]^2\right), \tag{23}$$

and H_0 contains the Hamiltonian of the hydrogen atom in the ordinary space and terms corresponding to the harmonic oscillators

$$H_0 = \frac{p^2}{2M} - \frac{e^2}{r} + H^a_{osc} + H^b_{osc}. \tag{24}$$

4. The effect of the noncommutativity of the energy levels of the hydrogen atom

Let us calculate corrections to the energy levels of the hydrogen atom caused by the noncommutativity of coordinates and noncommutativity of momenta. It is important that

$$\left[\frac{p^2}{2M} - \frac{e^2}{r}, H^a_{osc}\right] = \left[\frac{p^2}{2M} - \frac{e^2}{r}, H^b_{osc}\right] =$$
$$= \left[H^a_{osc}, H^b_{osc}\right] = 0. \tag{25}$$

So, the eigenvalues and eigenstates of the Hamiltonian H_0 can be written as follows

$$E^{(0)}_{n,\{n^a\},\{n^b\}} = -\frac{e^2}{2a_B n^2} + \hbar\omega(n^a_1 + n^a_2 + n^a_3 +$$
$$+ n^b_1 + n^b_2 + n^b_3 + 3), \tag{26}$$

$$\psi^{(0)}_{n,l,m,\{n^a\},\{n^b\}} = \psi_{n,l,m}\psi^a_{n^a_1,n^a_2,n^a_3}\psi^b_{n^b_1,n^b_2,n^b_3}. \tag{27}$$

Here, a_B is the Bohr radius, $\psi_{n,l,m}$ are well-known eigenfunctions of the hydrogen atom in the ordinary space ($\theta_{ij} = \eta_{ij} = 0$) and $\psi^a_{n^a_1,n^a_2,n^a_3}$, $\psi^b_{n^b_1,n^b_2,n^b_3}$ are eigenfunctions of the three-dimensional harmonic oscillators H^a_{osc}, H^b_{osc}. Using perturbation theory and taking into account the fact that the frequency of the oscillators is large and they are in the ground states, we can write

$$\Delta E^{(1)}_{n,l} = \langle\psi^{(0)}_{n,l,m,\{0\},\{0\}}|V|\psi^{(0)}_{n,l,m,\{0\},\{0\}}\rangle. \tag{28}$$

Note, that

$$\langle\psi^a_{0,0,0}|\theta_i|\psi^a_{0,0,0}\rangle = 0, \tag{29}$$
$$\langle\psi^b_{0,0,0}|\eta_i|\psi^b_{0,0,0}\rangle = 0. \tag{30}$$

So, we can write

$$\left\langle\psi^{(0)}_{n,l,m,\{0\},\{0\}}\left|\frac{(\boldsymbol{\eta}\cdot\mathbf{L})}{2M}\right|\psi^{(0)}_{n,l,m,\{0\},\{0\}}\right\rangle = 0, \tag{31}$$
$$\left\langle\psi^{(0)}_{n,l,m,\{0\},\{0\}}\left|\frac{e^2}{2r^3}(\boldsymbol{\theta}\cdot\mathbf{L})\right|\psi^{(0)}_{n,l,m,\{0\},\{0\}}\right\rangle = 0. \tag{32}$$

The effect of momentum noncommutativity is represented by the terms $[\boldsymbol{\eta}\times\mathbf{r}]^2/8M$. The correction caused by the term reads

$$\left\langle\psi^{(0)}_{n,l,m,\{0\},\{0\}}\left|\frac{[\boldsymbol{\eta}\times\mathbf{r}]^2}{8M}\right|\psi^{(0)}_{n,l,m,\{0\},\{0\}}\right\rangle =$$
$$= \left\langle\psi^{(0)}_{n,l,m,\{0\},\{0\}}\left|\frac{\eta^2 r^2}{8M} - \frac{(\boldsymbol{\eta}\cdot\mathbf{r})^2}{8M}\right|\psi^{(0)}_{n,l,m,\{0\},\{0\}}\right\rangle =$$
$$= \frac{a_B^2 n^2}{24M}(5n^2 + 1 - 3l(l+1))\langle\eta^2\rangle. \tag{33}$$

To write the expression, we use

$$\langle\psi^b_{0,0,0}|\eta_i\eta_j|\psi^b_{0,0,0}\rangle = \frac{m_{osc}\omega p_o^2}{2\hbar}\delta_{ij} = \frac{1}{3}\langle\eta^2\rangle\delta_{ij}, \tag{34}$$

where $\langle\eta^2\rangle$ is given by

$$\langle\eta^2\rangle = \frac{p_o^2}{\hbar^2}\langle\psi^b_{0,0,0}|(p^b)^2|\psi^b_{0,0,0}\rangle = \frac{3m_{osc}\omega p_o^2}{2\hbar} = \frac{3p_o^2}{2l_P^2}. \tag{35}$$

We also take into account the following result for the mean value (see, for example, [21])

$$\left\langle\psi_{n,l,m}\left|r^2\right|\psi_{n,l,m}\right\rangle = a_B^2\frac{n^2}{2}(5n^2 + 1 - 3l(l+1)). \tag{36}$$

To find the correction caused by the term $3e^2(\boldsymbol{\theta}\cdot\mathbf{L})^2/8r^5$, we take into account the following result for the mean value (see for instance [21])

$$\left\langle\psi_{n,l,m}\left|\frac{1}{r^5}\right|\psi_{n,l,m}\right\rangle =$$
$$= \frac{4(5n^2 - 3l(l+1) + 1)}{a_B^5 n^5 l(l+1)(l+2)(2l+1)(2l+3)(l-1)(2l-1)}. \tag{37}$$

We also calculate

$$\langle \psi^a_{0,0,0}\psi^b_{0,0,0}|\theta_i\theta_j|\psi^a_{0,0,0}\psi^b_{0,0,0}\rangle = \\ = \frac{1}{2}\left(\frac{\alpha}{m\omega}\right)^2\delta_{ij} = \frac{1}{3}\langle\theta^2\rangle\delta_{ij}. \quad (38)$$

As a result, on the basis of (37), (38), we find

$$\left\langle \psi^{(0)}_{n,l,m,\{0\},\{0\}}\left|\frac{3e^2}{8r^5}(\boldsymbol{\theta}\cdot\mathbf{L})^2\right|\psi^{(0)}_{n,l,m,\{0\},\{0\}}\right\rangle = \\ = \frac{\hbar^2 e^2(5n^2-3l(l+1)+1)\langle\theta^2\rangle}{2a_B^5 n^5(l+2)(2l+1)(2l+3)(l-1)(2l-1)}. \quad (39)$$

Then, let us rewrite last terms in the perturbation as follows

$$\frac{1}{r^2}[\boldsymbol{\theta}\times\mathbf{p}]^2\frac{1}{r}+\frac{1}{r}[\boldsymbol{\theta}\times\mathbf{p}]^2\frac{1}{r^2}+\frac{\hbar^2}{r^7}[\boldsymbol{\theta}\times\mathbf{r}]^2 = \\ = \theta^2\frac{1}{r^2}p^2\frac{1}{r}+\theta^2\frac{1}{r}p^2\frac{1}{r^2}+\theta^2\frac{\hbar^2}{r^5}-\frac{1}{r^2}(\boldsymbol{\theta}\cdot\mathbf{p})^2\frac{1}{r}- \\ +\frac{1}{r}(\boldsymbol{\theta}\cdot\mathbf{p})^2\frac{1}{r^2}-\frac{\hbar^2}{r^7}(\boldsymbol{\theta}\cdot\mathbf{r})^2. \quad (40)$$

So, after averaging over the eigenfunctions of the harmonic oscillators we find

$$\left\langle \psi^a_{0,0,0}\psi^b_{0,0,0}\left|\frac{1}{r^2}(\boldsymbol{\theta}\cdot\mathbf{p})^2\frac{1}{r}+\frac{1}{r}(\boldsymbol{\theta}\cdot\mathbf{p})^2\frac{1}{r^2}+\right.\right. \\ \left.\left.+\frac{\hbar^2}{r^7}(\boldsymbol{\theta}\cdot\mathbf{r})^2\right|\psi^a_{0,0,0}\psi^b_{0,0,0}\right\rangle = \\ = \frac{1}{3}\left(\frac{1}{r^2}p^2\frac{1}{r}+\frac{1}{r}p^2\frac{1}{r^2}+\frac{\hbar^2}{r^5}\right)\langle\theta^2\rangle. \quad (41)$$

So, we can write

$$\left\langle \psi^{(0)}_{n,l,m,\{0\},\{0\}}\left|\frac{1}{r^2}[\boldsymbol{\theta}\times\mathbf{p}]^2\frac{1}{r}+\right.\right. \\ \left.\left.+\frac{1}{r}[\boldsymbol{\theta}\times\mathbf{p}]^2\frac{1}{r^2}\right|\psi^{(0)}_{n,l,m,\{0\},\{0\}}\right\rangle+ \\ +\left\langle \psi^{(0)}_{n,l,m,\{0\},\{0\}}\left|\frac{\hbar^2}{r^7}[\boldsymbol{\theta}\times\mathbf{r}]^2\right|\psi^{(0)}_{n,l,m,\{0\},\{0\}}\right\rangle = \\ = \frac{2}{3}\left\langle\psi_{n,l,m}\left|\frac{1}{r^2}p^2\frac{1}{r}+\frac{1}{r}p^2\frac{1}{r^2}+\frac{\hbar^2}{r^5}\right|\psi_{n,l,m}\right\rangle\langle\theta^2\rangle. \quad (42)$$

We represent $\frac{1}{r^2}p^2\frac{1}{r}+\frac{1}{r}p^2\frac{1}{r^2}+\frac{\hbar^2}{r^5}$ as follows

$$\frac{1}{r^2}p^2\frac{1}{r}+\frac{1}{r}p^2\frac{1}{r^2}+\frac{\hbar^2}{r^5} = \frac{1}{r^3}p^2+p^2\frac{1}{r^3}+\frac{5\hbar^2}{r^5}. \quad (43)$$

So, the correction reads

$$\left\langle\psi_{n,l,m}\left|\frac{1}{r^2}p^2\frac{1}{r}+\frac{1}{r}p^2\frac{1}{r^2}+\frac{\hbar^2}{r^5}\right|\psi_{n,l,m}\right\rangle = \\ = -\frac{2\hbar^2}{a_B^2n^2}\left\langle\psi_{n,l,m}\left|\frac{1}{r^3}\right|\psi_{n,l,m}\right\rangle+ \\ +\frac{4\hbar^2}{a_B}\left\langle\psi_{n,l,m}\left|\frac{1}{r^4}\right|\psi_{n,l,m}\right\rangle+5\hbar^2\left\langle\psi_{n,l,m}\left|\frac{1}{r^5}\right|\psi_{n,l,m}\right\rangle. \quad (44)$$

Taking into account well-known results for the mean values (see for instance [21])

$$\left\langle\psi_{n,l,m}\left|\frac{1}{r^3}\right|\psi_{n,l,m}\right\rangle = \\ = \frac{2}{a_B^3n^3l(l+1)(2l+1)}, \quad (45)$$

$$\left\langle\psi_{n,l,m}\left|\frac{1}{r^4}\right|\psi_{n,l,m}\right\rangle = \\ = \frac{4(3n^2-l(l+1))}{a_B^4n^5l(l+1)(2l+1)(2l+3)(2l-1)}, \quad (46)$$

and all the obtained results, we can write an explicit expression for the corrections to the energy levels caused by the coordinates noncommutativity. It reads

$$\left\langle\psi^{(0)}_{n,l,m,\{0\},\{0\}}\left|-\frac{3e^2}{8r^5}(\boldsymbol{\theta}\cdot\mathbf{L})^2+\right.\right. \\ \left.\left.+\frac{e^2}{16r^2}[\boldsymbol{\theta}\times\mathbf{p}]^2\frac{1}{r}\right|\psi^{(0)}_{n,l,m,\{0\},\{0\}}\right\rangle+ \\ \left\langle\psi^{(0)}_{n,l,m,\{0\},\{0\}}\left|\frac{e^2}{16}\frac{1}{r}[\boldsymbol{\theta}\times\mathbf{p}]^2\frac{1}{r^2}+\right.\right. \\ \left.\left.+\frac{e^2}{16}\frac{\hbar^2}{r^7}[\boldsymbol{\theta}\times\mathbf{r}]^2\right|\psi^{(0)}_{n,l,m,\{0\},\{0\}}\right\rangle = \\ -\frac{\hbar^2e^2\langle\theta^2\rangle}{a_B^5n^5}\left(\frac{1}{6l(l+1)(2l+1)}-\right. \\ -\frac{6n^2-2l(l+1)}{3l(l+1)(2l+1)(2l+3)(2l-1)}+ \\ \frac{5n^2-3l(l+1)+1}{2(l+2)(2l+1)(2l+3)(l-1)(2l-1)}- \\ \left.\frac{5}{6}\frac{5n^2-3l(l+1)+1}{l(l+1)(l+2)(2l+1)(2l+3)(l-1)(2l-1)}\right), \quad (47)$$

where $\langle\theta^2\rangle$ is given by

$$\langle\theta^2\rangle = \frac{l_0^2}{\hbar^2}\langle\psi^a_{0,0,0}|a^2|\psi^a_{0,0,0}\rangle = \frac{3l_0^2}{2\hbar}\left(\frac{1}{m_{osc}\omega}\right) = \frac{3l_0^2 l_P^2}{2\hbar^2}. \quad (48)$$

So, using (31), (32), (33) and (47) in the first order of perturbation theory, the effect of noncommutativity on the energy levels of the hydrogen atom is as follows

$$\Delta E^{(1)}_{n,l} = \Delta E^{(\eta)}_{n,l} + \Delta E^{(\theta)}_{n,l}, \quad (49)$$

where

$$\Delta E^{(\eta)}_{n,l} = \frac{a_B^2 n^2\langle\eta^2\rangle}{24M}(5n^2+1-3l(l+1)), \quad (50)$$

are corrections to the spectrum caused by the noncommutativity of momenta and

$$\Delta E^{(\theta)}_{n,l} = -\frac{\hbar^2 e^2\langle\theta^2\rangle}{a_B^5 n^5}\times$$
$$\times\left(-\frac{6n^2-2l(l+1)}{3l(l+1)(2l+1)(2l+3)(2l-1)}+\right.$$
$$+\frac{1}{6l(l+1)(2l+1)}+$$
$$+\frac{5n^2-3l(l+1)+1}{2(l+2)(2l+1)(2l+3)(l-1)(2l-1)}-$$
$$\left.+\frac{5}{6}\frac{5n^2-3l(l+1)+1}{l(l+1)(l+2)(2l+1)(2l+3)(l-1)(2l-1)}\right), \quad (51)$$

being corrections caused by the coordinates noncommutativity.

Note that in the second order of the perturbation theory we have

$$\Delta E^{(2)}_{n,l,m,\{0\}} = \sum_{n',l',m',\{n^a\},\{n^b\}}\times$$
$$\times\left|\left\langle\psi^{(0)}_{n',l',m',\{n^a\},\{n^b\}}|V|\,\psi^{(0)}_{n,l,m,\{0\},\{0\}}\right\rangle\right|^2\times$$
$$\times(E^{(0)}_n - E^{(0)}_{n'} - \hbar\omega(n_1^a+n_2^a+n_3^a+n_1^b+n_2^b+n_3^b))^{-1}, \quad (52)$$

$$E^{(0)}_n = -\frac{e^2}{2a_B n^2}. \quad (53)$$

In the limit $\omega\to\infty$, this correction vanishes

$$\lim_{\omega\to\infty}\Delta E^{(2)}_{n,l,m,\{0\}} = 0. \quad (54)$$

So, up to the second order in the parameters of the coordinates noncommutativity and parameters of momentum noncommutativity, the corrections to the energy levels of the hydrogen atom are as follows

$$\Delta E_{n,l} = \Delta E^{(1)}_{n,l}. \quad (55)$$

It is important to stress that the obtained corrections to the energy levels of the hydrogen atom (55) are divergent for $l=0$ and $l=1$. From this, it follows that we cannot use expansion of the Hamiltonian into the series over the parameter of coordinate noncommutativity. In the next section, we find finite result for corrections to the *ns* energy levels of the hydrogen atom. We are interested in the corrections because on the basis of the results, a stringent upper bound for the minimal length can be found.

5. Corrections to the *ns* energy levels of the hydrogen atom in noncommutative phase space

To calculate corrections to the *ns* energy levels, we rewrite perturbation V caused by noncommutativity of coordinates and noncommutativity of momenta as follows

$$V = \frac{(\boldsymbol{\eta}\cdot\mathbf{L})}{2M}+\frac{[\boldsymbol{\eta}\times\mathbf{r}]^2}{2M}-\frac{e^2}{R}+\frac{e^2}{r} = -\frac{e^2}{\sqrt{r^2-(\boldsymbol{\theta}\cdot\mathbf{L})+\frac{1}{4}[\boldsymbol{\theta}\times\mathbf{p}]^2}}+\frac{e^2}{r}. \quad (56)$$

So, the corrections read

$$\Delta E_{ns} = \left\langle\psi^{(0)}_{n,0,0,\{0\},\{0\}}\right|\frac{(\boldsymbol{\eta}\cdot\mathbf{L})}{2M}+$$
$$+\frac{[\boldsymbol{\eta}\times\mathbf{r}]^2}{8M}\left|\psi^{(0)}_{n,0,0,\{0\},\{0\}}\right\rangle+\left\langle\psi^{(0)}_{n,0,0,\{0\},\{0\}}\right|-\frac{e^2}{r}-$$
$$+\frac{e^2}{\sqrt{r^2-(\boldsymbol{\theta}\cdot\mathbf{L})+\frac{1}{4}[\boldsymbol{\theta}\times\mathbf{p}]^2}}\left|\psi^{(0)}_{n,0,0,\{0\},\{0\}}\right\rangle. \quad (57)$$

It is important to note that

$$[(\boldsymbol{\theta}\cdot\mathbf{L}),[\boldsymbol{\theta}\times\mathbf{p}]^2] = [(\boldsymbol{\theta}\cdot\mathbf{L}),r^2] = 0. \quad (58)$$

Also, we have

$$(\boldsymbol{\theta}\cdot\mathbf{L})\psi^{(0)}_{n,0,0,\{0\},\{0\}}(\mathbf{r},\mathbf{a},\mathbf{b}) = 0. \quad (59)$$

So, we can write

$$\Delta E_{ns} = \frac{a_B^2 n^2 \langle \eta^2 \rangle}{24M}(5n^2+1) + + \left\langle \psi^{(0)}_{n,0,0,\{0\},\{0\}}(\mathbf{r},\mathbf{a},\mathbf{b}) \right| \frac{e^2}{r} - + \frac{e^2}{\sqrt{r^2+\frac{1}{4}[\boldsymbol{\theta}\times\mathbf{p}]^2}} \left| \psi^{(0)}_{n,0,0,\{0\},\{0\}}(\mathbf{r},\mathbf{a},\mathbf{b}) \right\rangle. \tag{60}$$

We introduce $\mathbf{a}' = \mathbf{a}/l_p$, $\mathbf{b}' = \mathbf{b}/l_p$,

$$\mathbf{r}' = \sqrt{\frac{2}{\alpha}}\frac{\mathbf{r}}{l_p}, \tag{61}$$

with l_p being the Planck length. So, we can rewrite $\boldsymbol{\theta}$ as

$$\boldsymbol{\theta} = \frac{\alpha l_p^2}{\hbar}\boldsymbol{\theta}', \tag{62}$$

$$\boldsymbol{\theta}' = [\mathbf{a}'\times\mathbf{b}']. \tag{63}$$

So, the correction caused by noncommutativity of coordinates $\Delta E^{(\theta)}_{ns}$ reads

$$\Delta E^{(\theta)}_{ns} = \frac{\chi^2 e^2}{a_B} I_{ns}(\chi), \tag{64}$$

where

$$I_{ns}(\chi) = = \int d\mathbf{a}'\tilde{\psi}^a_{0,0,0}(\mathbf{a}') \int d\mathbf{b}'\tilde{\psi}^b_{0,0,0}(\mathbf{b}') \int d\mathbf{r}'\tilde{\psi}_{n,0,0}(\chi\mathbf{r}') \times \times \left(\frac{1}{r'} - \frac{1}{\sqrt{(r')^2+[\boldsymbol{\theta}'\times\mathbf{p}']^2}}\right) \times \times \tilde{\psi}_{n,0,0}(\chi\mathbf{r}')\tilde{\psi}^a_{0,0,0}(\mathbf{a}')\tilde{\psi}^b_{0,0,0}(\mathbf{b}'), \tag{65}$$

and

$$\chi = \sqrt{\frac{\alpha}{2}}\frac{l_p}{a_B}. \tag{66}$$

The Eigenfunctions of the harmonic oscillators and hydrogen atom read

$$\tilde{\psi}^a_{0,0,0}(\mathbf{a}') = \pi^{-\frac{3}{4}} e^{-\frac{(a')^2}{2}}, \tag{67}$$

$$\tilde{\psi}^b_{0,0,0}(\mathbf{b}') = \pi^{-\frac{3}{4}} e^{-\frac{(b')^2}{2}}, \tag{68}$$

$$\tilde{\psi}_{n,0,0}(\chi\mathbf{r}') = \sqrt{\frac{1}{\pi n^5}} e^{-\frac{\chi r'}{n}} L^1_{n-1}\left(\frac{2\chi r'}{n}\right), \tag{69}$$

$L^1_{n-1}\left(\frac{2\chi r'}{n}\right)$ are the generalized Laguerre polynomials.

Integral (65) is finite for $\chi = 0$. So, the asymptotic of $\Delta E^{(\theta)}_{ns}$ for $\chi \to 0$ reads

$$\Delta E^{(\theta)}_{ns} = \frac{\chi^2 e^2}{a_B} I_{ns}(0). \tag{70}$$

So, to obtain the asymptotic of $\Delta E^{(\theta)}_{ns}$, we have to calculate integral $I_{ns}(0)$. As the first step we consider the integral over $\mathbf{r}'$. We have

$$I_{ns}(\chi,\boldsymbol{\theta}') = \int d\mathbf{r}'\tilde{\psi}_{n,0,0}(\chi\mathbf{r}') \times \times \left(\frac{1}{r'} - \frac{1}{\sqrt{(r')^2+[\boldsymbol{\theta}'\times\mathbf{p}']^2}}\right)\tilde{\psi}_{n,0,0}(\chi\mathbf{r}'). \tag{71}$$

In the momentum representation, the integral reads

$$I_{ns}(\chi,\boldsymbol{\theta}') = \frac{1}{\chi^6}\int d\mathbf{p}'\tilde{\psi}_{n,0,0}\left(\frac{\mathbf{p}'}{\chi}\right) \times \times \left(\frac{1}{\sqrt{-\nabla^2_{p'}}} - \frac{1}{\sqrt{-\nabla^2_{p'}+[\boldsymbol{\theta}'\times\mathbf{p}']^2}}\right)\tilde{\psi}_{n,0,0}\left(\frac{\mathbf{p}'}{\chi}\right), \tag{72}$$

where

$$\nabla^2_{p'} = \sum_i \frac{\partial^2}{(\partial p'_i)^2}. \tag{73}$$

Integral $I_{ns}(\chi,\boldsymbol{\theta}')$ does not depend on the direction of the vector $\boldsymbol{\theta}'$. So, we can rewrite the integral as

$$I_{ns}(\chi,\boldsymbol{\theta}') = I_{ns}(\chi,\theta') = = \frac{1}{4\pi\chi^6}\int d\Omega\int d\mathbf{p}'\tilde{\psi}_{n,0,0}\left(\frac{\mathbf{p}'}{\chi}\right)\left(\frac{1}{\sqrt{-\nabla^2_{p'}}} - + \frac{1}{\sqrt{-\nabla^2_{p'}+[\boldsymbol{\theta}'\times\mathbf{p}']^2}}\right)\tilde{\psi}_{n,0,0}\left(\frac{\mathbf{p}'}{\chi}\right) = = \frac{1}{4\pi\chi^6}\int d\Omega\int d\mathbf{p}'\tilde{\psi}_{n,0,0}\left(\frac{\mathbf{p}'}{\chi}\right)\left(\frac{1}{\sqrt{-\nabla^2_{p'}}} - + \frac{1}{\sqrt{-\nabla^2_{p'}+(\theta')^2(p')^2\sin^2\Theta}}\right)\tilde{\psi}_{n,0,0}\left(\frac{\mathbf{p}'}{\chi}\right), \tag{74}$$

where Θ is the angle between vectors $\boldsymbol{\theta}'$ and $\mathbf{p}'$, $\theta' = |\boldsymbol{\theta}'|$, and $d\Omega = \sin\Theta d\Theta d\Phi$.

Let us use the substitution

$$\tilde{\mathbf{p}} = \kappa \mathbf{p}', \quad (75)$$

$$\kappa = \sqrt{\theta' \sin\Theta}, \quad (76)$$

and return to the coordinate representation. So, we have

$$I_{ns}(\chi,\theta') = \frac{\theta'}{2}\int_0^{\pi} d\Theta \sin^2\Theta \int d\tilde{\mathbf{r}} \tilde{\psi}_{n,0,0}(\kappa\chi\tilde{\mathbf{r}}) \times$$

$$\times \left(\frac{1}{\tilde{r}} - \frac{1}{\sqrt{\tilde{r}^2+\tilde{p}^2}}\right) \tilde{\psi}_{n,0,0}(\kappa\chi\tilde{\mathbf{r}}) =$$

$$= \frac{\theta'}{2}\int_0^{\pi} d\Theta \sin^2\Theta \int_0^{\infty} d\tilde{r} \times$$

$$\times \tilde{r}^2 \tilde{R}_{n,0}(\kappa\chi\tilde{r}) \left(\frac{1}{\tilde{r}} - \frac{1}{\sqrt{\tilde{r}^2+p_{\tilde{r}}^2}}\right) \tilde{R}_{n,0}(\kappa\chi\tilde{r}), \quad (77)$$

with $\tilde{R}_{n,0}(\kappa\chi\tilde{r})$ being the radial wave function of the hydrogen atom

$$\tilde{R}_{n,0}(\kappa\chi\tilde{r}) = \sqrt{\frac{4}{n^5}} e^{-\frac{\kappa\chi\tilde{r}}{n}} L^1_{n-1}\left(\frac{2\kappa\chi\tilde{r}}{n}\right), \quad (78)$$

and

$$p_{\tilde{r}} = -i\frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}}\tilde{r}. \quad (79)$$

Then for convenience, we use the notation

$$S_{ns}(\kappa\chi) = 4\int_0^{\infty} d\tilde{r}\tilde{r}^2 e^{-\frac{\kappa\chi\tilde{r}}{n}} L^1_{n-1}\left(\frac{2\kappa\chi\tilde{r}}{n}\right) \times$$

$$\times \left(\frac{1}{\tilde{r}} - \frac{1}{\sqrt{\tilde{r}^2+p_{\tilde{r}}^2}}\right) e^{-\frac{\kappa\chi\tilde{r}}{n}} L^1_{n-1}\left(\frac{2\kappa\chi\tilde{r}}{n}\right). \quad (80)$$

So, for $I_{ns}(\chi,\theta')$ we obtain

$$I_{ns}(\chi,\theta') = \frac{\theta'}{2n^5}\int_0^{\pi} d\Theta \sin^2\Theta S_{ns}(\kappa\chi). \quad (81)$$

Taking into account

$$I_{ns}(0) = \langle I_{ns}(0,\theta')\rangle_{\mathbf{a}',\mathbf{b}'}, \quad (82)$$

$$I_{ns}(0,\theta') = \frac{\theta'}{2n^5}\int_0^{\pi} d\Theta \sin^2\Theta S_{ns}(0) =$$

$$= \frac{\pi\theta'}{4n^5} S_{ns}(0), \quad (83)$$

we have

$$\Delta E^{(\theta)}_{ns} = \frac{\pi\langle\theta'\rangle\chi^2 e^2}{4a_B n^5} S_{ns}(0), \quad (84)$$

$$\langle\theta'\rangle = \langle\tilde{\psi}^a_{0,0,0}(\mathbf{a}')\tilde{\psi}^b_{0,0,0}(\mathbf{b}')| \sqrt{\sum_i (\theta'_i)^2} |\tilde{\psi}^a_{0,0,0}(\mathbf{a}')\tilde{\psi}^b_{0,0,0}(\mathbf{b}')\rangle = 1. \quad (85)$$

Note, that

$$S_{ns}(0) = S_{1s}(0) n^2. \quad (86)$$

On the basis of (85), (86), we find the expression for the leading term in the asymptotic expansion of the corrections to the *ns* energy levels

$$\Delta E_{ns} = \frac{\pi\chi^2 e^2}{4a_B n^3} S_{1s}(0). \quad (87)$$

So, we have to calculate the integral

$$S_{1s}(0) = 4\int_0^{\infty} d\tilde{r}\tilde{r}^2 \left(\frac{1}{\tilde{r}} - \frac{1}{\sqrt{\tilde{r}^2+p_{\tilde{r}}^2}}\right). \quad (88)$$

We expand 1 over the eigenfunctions of operator $\tilde{r}^2 + p^2_{\tilde{r}}$. They read

$$\phi_k = \sqrt{\frac{2k!}{\Gamma(k+\frac{3}{2})}} e^{-\frac{\tilde{r}^2}{2}} L_k^{\frac{1}{2}}(\tilde{r}^2). \quad (89)$$

We have

$$1 = \sum_{k=0}^{\infty} C_k \phi_k, \quad (90)$$

C_k are the expansion coefficients, which are as follows

$$C_k = \sqrt{\frac{2k!}{\Gamma(k+\frac{3}{2})}} \int_0^{\infty} d\tilde{r}\tilde{r}^2 e^{-\frac{\tilde{r}^2}{2}} L_k^{\frac{1}{2}}(\tilde{r}^2) =$$

$$= (-1)^k \sqrt{\frac{4\Gamma(k+\frac{3}{2})}{k!}}. \quad (91)$$

So, for the second term in (88) we obtain

$$\int_0^{\infty} d\tilde{r}\tilde{r}^2 \frac{1}{\sqrt{\tilde{r}^2+p^2_{\tilde{r}}}} = \sum_{k=0}^{\infty} \frac{C_k^2}{\sqrt{\lambda_k}}, \quad (92)$$

where

$$\lambda_k = 2\left(2k+\frac{3}{2}\right), \quad (93)$$

are the eigenvalues of operator $\tilde{r}^2 + p^2_{\tilde{r}}$.

Let us represent the first term in (88) as follows

$$\int_0^\infty d\tilde{r}\tilde{r} = \sum_{k=0}^{\infty} C_k I_k, \tag{94}$$

$$I_k = \sqrt{\frac{2k!}{\Gamma(k+\frac{3}{2})}} \int_0^\infty d\tilde{r}\tilde{r}e^{-\frac{\tilde{r}^2}{2}} L_k^{\frac{1}{2}}\left(\tilde{r}^2\right) = $$
$$= (-1)^k \sqrt{\frac{8k!}{\pi\Gamma(k+\frac{3}{2})}} {}_2F_1\left(-k,\frac{1}{2};\frac{3}{2};2\right), \tag{95}$$

where ${}_2F_1\left(-k,\frac{1}{2};\frac{3}{2};2\right)$ is the hypergeometric function. Using (92), (94), we obtain

$$S_{1s}(0) = 4\sum_{k=0}^{\infty}\left(C_k I_k - \frac{C_k^2}{\sqrt{\lambda_k}}\right) =$$
$$= 16\sqrt{\frac{2}{\pi}}\sum_{k=0}^{\infty}\frac{\Gamma(k+\frac{3}{2})}{k!}\times$$
$$\times\left({}_2F_1\left(-k,\frac{1}{2};\frac{3}{2};2\right) - \sqrt{\frac{\pi}{8k+6}}\right). \tag{96}$$

It is important to mention that the two sums in $S_{1s}(0)$

$$16\sqrt{\frac{2}{\pi}}\sum_{k=0}^{\infty}\frac{\Gamma(k+\frac{3}{2})}{k!}{}_2F_1\left(-k,\frac{1}{2};\frac{3}{2};2\right), \tag{97}$$

$$16\sqrt{\frac{2}{\pi}}\sum_{k=0}^{\infty}\frac{\Gamma(k+\frac{3}{2})}{k!}\sqrt{\frac{\pi}{8k+6}}, \tag{98}$$

are divergent. But the value of $S_{1s}(0)$ is finite. To study the sums (97), (98) separately we consider the additional multiplier η^k ($\eta < 1$)

$$16\sqrt{\frac{2}{\pi}}\sum_{k=0}^{\infty}\frac{\Gamma(k+\frac{3}{2})}{k!}{}_2F_1\left(-k,\frac{1}{2};\frac{3}{2};2\right)\eta^k, \tag{99}$$

$$16\sqrt{\frac{2}{\pi}}\sum_{k=0}^{\infty}\frac{\Gamma(k+\frac{3}{2})}{k!}\sqrt{\frac{\pi}{8k+6}}\eta^k. \tag{100}$$

In the case of $\eta = 1$, we find (97), (98).

First let us calculate (100). It is easy to show that

$$\sqrt{\frac{\pi}{k+\frac{3}{4}}} = 2\int_0^\infty dz e^{-(k+\frac{3}{4})z^2}. \tag{101}$$

Also, we can write

$$\sum_{k=0}^{\infty}\frac{\Gamma(k+\frac{3}{2})}{k!}t^k = \frac{\sqrt{\pi}}{2(1-t)^{\frac{3}{2}}}. \tag{102}$$

As a result, using (101), (102), we find

$$16\sqrt{2}\sum_{k=0}^{\infty}\frac{\Gamma(k+\frac{3}{2})}{k!\sqrt{8k+6}}\eta^k =$$
$$= 16\sum_{k=0}^{\infty}\frac{\Gamma(k+\frac{3}{2})}{k!\sqrt{\pi}}\eta^k\int_0^\infty dz e^{-(k+\frac{3}{4})z^2} =$$
$$= 8\int_0^\infty dz\frac{e^{-\frac{3}{4}z^2}}{(1-\eta e^{-z^2})^{\frac{3}{2}}}. \tag{103}$$

To calculate (99) we represent the hypergeometric function as

$${}_2F_1\left(-k,\frac{1}{2};\frac{3}{2};2\right) = \sum_{q=0}^{k}\frac{(-1)^q C_k^q 2^q}{2q+1}, \tag{104}$$

where C_k^q are the binomial coefficients. We can write

$$\frac{1}{2q+1} = \int_0^1 dz z^{2q}. \tag{105}$$

So, taking into account (104), (105), we find

$${}_2F_1\left(-k,\frac{1}{2},\frac{3}{2},2\right) = \sum_{q=0}^{k}\int_0^1 dz C_k^q(-2)^q z^{2q} =$$
$$= \int_0^1 dz(1-2z^2)^k. \tag{106}$$

Then, using (102) and (106), we rewrite (99) as

$$16\sqrt{\frac{2}{\pi}}\sum_{k=0}^{\infty}\frac{\Gamma(k+\frac{3}{2})}{k!}{}_2F_1\left(-k,\frac{1}{2},\frac{3}{2},2\right)\eta^k =$$
$$= 16\sqrt{\frac{2}{\pi}}\sum_{k=0}^{\infty}\frac{\Gamma(k+\frac{3}{2})}{k!}\eta^k\int_0^1 dz(1-2z^2)^k =$$
$$= 8\sqrt{2}\int_0^1\frac{dz}{(1-\eta(1-2z^2))^{\frac{3}{2}}}. \tag{107}$$

We split the integral (107) into two integrals as

$$\int_0^1\frac{dz}{(1-\eta(1-2z^2))^{\frac{3}{2}}} = I_1(\eta) + I_2(\eta), \tag{108}$$

$$I_1(\eta) = \int_0^{\frac{1}{\sqrt{2}}}\frac{dz}{(1-\eta(1-2z^2))^{\frac{3}{2}}}, \tag{109}$$

$$I_2(\eta) = \int_{\frac{1}{\sqrt{2}}}^1\frac{dz}{(1-\eta(1-2z^2))^{\frac{3}{2}}}. \tag{110}$$

The integral $I_2(\eta)$ has a finite value even for $\eta = 1$, we find

$$I_2(1)=\frac{\sqrt{2}}{8}. \qquad (111)$$

Let us represent (109) in the form close to (103). We use substitution $e^{-t^2}=1-2z^2$, and obtain

$$I_1(\eta)=\frac{\sqrt{2}}{2}\int_0^\infty dt\frac{te^{-t^2}}{(1-e^{-t^2})^{\frac{1}{2}}(1-\eta e^{-t^2})^{\frac{3}{2}}}. \qquad (112)$$

Using (103), (107), (111), (112), we can write

$$16\sqrt{\frac{2}{\pi}}\sum_{k=0}^{\infty}\frac{\Gamma(k+\frac{3}{2})}{k!}{}_2F_1\left(-k,\frac{1}{2};\frac{3}{2};2\right)\eta^k-$$
$$+16\sqrt{\frac{2}{\pi}}\sum_{k=0}^{\infty}\frac{\Gamma(k+\frac{3}{2})}{k!}\sqrt{\frac{\pi}{8k+6}}\eta^k=$$
$$=8\sqrt{2}I_2(\eta)+8\int_0^\infty dt\frac{te^{-t^2}-e^{-\frac{3}{4}t^2}(1-e^{-t^2})^{\frac{1}{2}}}{(1-e^{-t^2})^{\frac{1}{2}}(1-\eta e^{-t^2})^{\frac{3}{2}}}. \qquad (113)$$

The integral (113) is finite for $\eta=1$. So, substituting $\eta=1$ into (113), and using (96), (111), we obtain

$$S_{1s}(0) = 2+8\int_0^\infty dt\frac{te^{-t^2}-e^{-\frac{3}{4}t^2}\sqrt{1-e^{-t^2}}}{(1-e^{-t^2})^2}= 1.72006\ldots \qquad (114)$$

Taking into account (87), we find

$$\Delta E_{ns}^{(\theta)}\simeq 1.72\frac{\pi\chi^2e^2}{4a_Bn^3}. \qquad (115)$$

So, on the basis of (62) and (66), we can write

$$\Delta E_{ns}^{(\theta)}\simeq 1.72\frac{\hbar\langle\theta\rangle\pi e^2}{8a_B^3n^3}, \qquad (116)$$

$$\langle\theta\rangle= = \langle\psi^a_{0,0,0}(\mathbf{a})\psi^b_{0,0,0}(\mathbf{b})|\sqrt{\sum_i\theta_i^2}|\psi^a_{0,0,0}(\mathbf{a})\psi^b_{0,0,0}(\mathbf{b})\rangle= =\frac{\alpha l_p^2}{\hbar}. \qquad (117)$$

Finally, corrections to the ns energy levels of the hydrogen atom caused by noncommutativity of coordinates and noncommutativity of momenta read

$$\Delta E_{ns}=\frac{a_B^2n^2\langle\eta^2\rangle}{24M}(5n^2+1)+1.72\frac{\hbar\langle\theta\rangle\pi e^2}{8a_B^3n^3}. \qquad (118)$$

Let us analyze the corrections (55), (118). There is an important difference between the influences of the coordinates noncommutativity and momentum noncommutativity on the spectrum of the hydrogen atom. In the case of large quantum numbers n , corrections caused by noncommutativity of momenta $\Delta E^{(\eta)}_{n,l}$ (176) are proportional to n^4, and corrections caused by noncommutativity of coordinates $\Delta E^{(\theta)}_{n,l}$ (175) are proportional to $1/n^3$. So, we can conclude that the energy levels with large quantum numbers n are more sensitive to the momentum noncommutativity than to noncommutativity of coordinates. Energy levels with small quantum numbers n are more sensitive to the coordinates noncommutativity

Note also that ns energy levels are more sensitive to the noncommutativity of coordinates (1) Namely corrections to the ns energy levels (118) contain terms with $\langle\theta\rangle$ and $\langle\eta^2\rangle$. Corrections to other energy levels ($l>1$) include terms proportional to $\langle\theta^2\rangle$ and $\langle\eta^2\rangle$.

6. Energy levels of hydrogen-like exotic atoms in quantum space

We examine two particles with masses m_1, m_2 with Coulomb interaction in the frame of a rotationally-invariant noncommutative algebra of the canonical type (1)-(3). In this case, the total Hamiltonian reads

$$H=\frac{(\mathbf{P}^{(1)})^2}{2m_1}+\frac{(\mathbf{P}^{(2)})^2}{2m_2}-\frac{\kappa}{|\mathbf{X}^{(1)}-\mathbf{X}^{(2)}|}+H^a_{osc}+H^b_{osc}. \qquad (119)$$

Here, κ is a constant.

In the general case, coordinates of different particles may satisfy commutation relations of noncommutative algebra with different tensors of noncommutativity $\theta^{(n)}_{ij}$, $\eta^{(n)}_{ij}$ (n labels the particles, $n=(1,2)$). So, in this case the relations of the algebra read

$$[X_i^{(n)},X_j^{(m)}] = i\hbar\delta_{mn}\theta^{(n)}_{ij}, \qquad (120)$$
$$[X_i^{(n)},P_j^{(m)}] = i\hbar\delta_{mn}\left(\delta_{ij}+\sum_k\frac{\theta^{(n)}_{ik}\eta^{(m)}_{jk}}{4}\right), \qquad (121)$$
$$[P_i^{(n)},P_j^{(m)}] = i\hbar\delta_{mn}\eta^{(n)}_{ij}, \qquad (122)$$

$n,m=(1,2)$. Note that we also suppose that commutators for coordinates and the momenta of different particles equal zero.

Let us introduce coordinates and momenta of the center-of-mass and coordinates and momenta of the relative motion as in ordinary space

$$\mathbf{X}^c=\mu_1\mathbf{X}^{(1)}+\mu_2\mathbf{X}^{(2)}, \tag{123}$$

$$\mathbf{P}^c=\mathbf{P}^{(1)}+\mathbf{P}^{(2)}, \tag{124}$$

$$\mathbf{X}^r=\Delta\mathbf{X}^{(1)}-\Delta\mathbf{X}^{(2)}=\mathbf{X}^{(1)}-\mathbf{X}^{(2)}, \tag{125}$$

$$\mathbf{P}^r=\frac{1}{2}(\Delta\mathbf{P}^{(1)}-\Delta\mathbf{P}^{(2)})=\mu_2\mathbf{P}^{(1)}-\mu_1\mathbf{P}^{(2)}. \tag{126}$$

So, we can rewrite the Hamiltonian of the system as

$$H_s=\frac{(\mathbf{P}^c)^2}{2M}+\frac{(\mathbf{P}^r)^2}{2\mu}-\frac{\kappa}{|\mathbf{X}^r|}, \tag{127}$$

where $M=m_1+m_2$, $\mu=m_1m_2/M$ are the total and the reduced masses respectively, $\mu_i=m_i/M$.

Coordinates and momenta of the center-of mass X_i^c, P_i^c satisfy the following relations

$$[X_i^c,X_j^c]=i\hbar\sum_{n=1}^{2}\mu_n^2\theta_{ij}^{(n)}=i\hbar\theta_{ij}^c, \tag{128}$$

$$[P_i^c,P_j^c]=i\hbar\sum_{n=1}^{2}\eta_{ij}^{(n)}=i\hbar\eta_{ij}^c, \tag{129}$$

$$[X_i^c,P_j^c]=i\hbar(\delta_{ij}+\sum_{n=1}^{2}\sum_{k=1}^{2}\mu_n\frac{\theta_{ik}^{(n)}\eta_{jk}^{(n)}}{4}). \tag{130}$$

where

$$\theta_{ij}^c = \mu_1^2\theta_{ij}^{(1)}+\mu_2^2\theta_{ij}^{(2)}, \tag{131}$$

$$\eta_{ij}^c = \eta_{ij}^{(1)}+\eta_{ij}^{(2)}. \tag{132}$$

It is important to stress that

$$i\hbar(\delta_{ij}+\sum_n\sum_k\mu_n\frac{\theta_{ik}^{(n)}\eta_{jk}^{(n)}}{4})\neq i\hbar(\delta_{ij}+\sum_k\frac{\theta_{ik}^c\eta_{jk}^c}{4}). \tag{133}$$

So, commutators (128)-(130) do not correspond to noncommutative algebra (1)-(3).

It is worth mentioning that in the case when parameters of noncommutativity depend on mass as

$$c_\theta^{(n)} = \frac{\tilde{\gamma}}{m_n}, \tag{134}$$

$$c_\eta^{(n)} = \tilde{\alpha}m_n, \tag{135}$$

the tensors of noncommutativity can be rewritten as

$$\theta_{ij}^{(n)} = \frac{\tilde{\gamma}l_P^2}{\hbar m_n}\sum_k\varepsilon_{ijk}\tilde{a}_k, \tag{136}$$

$$\eta_{ij}^{(n)} = \frac{\tilde{\alpha}\hbar m_n}{l_P^2}\sum_k\varepsilon_{ijk}\tilde{p}_k^b, \tag{137}$$

and the effective tensors of noncommutativity do not depend on the masses of particles in the system. They read

$$\theta_{ij}^c = \frac{\tilde{\gamma}l_P^2}{\hbar M}\sum_k\varepsilon_{ijk}\tilde{a}_k, \tag{138}$$

$$\eta_{ij}^c = \frac{\tilde{\alpha}\hbar M}{l_P^2}\sum_k\varepsilon_{ijk}\tilde{p}_k^b. \tag{139}$$

Also, due to conditions (134), (135) we can write

$$[X_i^c,P_j^c] = i\hbar(\delta_{ij}+\tilde{\gamma}\tilde{\alpha}\sum_{k,l,m}\frac{\varepsilon_{ikl}\varepsilon_{jkm}\tilde{a}_l\tilde{p}_m^b}{4})= = i\hbar(\delta_{ij}+\sum_k\frac{\theta_{ik}^c\eta_{jk}^c}{4}). \tag{140}$$

In the case when conditions (134), (135) hold, coordinates and momenta of the relative motion satisfy the following relations

$$[X_i^r,X_j^r] = i\hbar\theta_{ij}^r, \tag{141}$$

$$[P_i^r,P_j^r] = i\hbar\eta_{ij}^r, \tag{142}$$

$$[X_i^r,P_j^r] = i\hbar(\delta_{ij}+\frac{1}{4}\sum_k\theta_{ik}^r\eta_{jk}^r), \tag{143}$$

where

$$\theta_{ij}^r = \theta_{ij}^{(1)}+\theta_{ij}^{(2)}, \tag{144}$$

$$\eta_{ij}^r = \mu_2^2\eta_{ij}^{(1)}+\mu_1^2\eta_{ij}^{(2)}. \tag{145}$$

It is also important to stress that coordinates and momenta of the center-of-mass commute with coordinates and momenta of the relative motion due to conditions (134), (135)

$$[X_i^c,X_j^r]=[P_i^c,P_j^r]=0. \tag{146}$$

Taking into account (136), (137), (144), (145) we can write

$$\theta_{ij}^r=\frac{c_\theta^r l_P^2}{\hbar}\varepsilon_{ijk}\tilde{a}_k=\frac{\tilde{\gamma}l_P^2}{\mu\hbar}\varepsilon_{ijk}\tilde{a}_k, \tag{147}$$

$$\eta_{ij}^r=\frac{c_\eta^r\hbar}{l_P^2}\varepsilon_{ijk}\tilde{p}_k^b=\frac{\tilde{\alpha}\mu\hbar}{l_P^2}\varepsilon_{ijk}\tilde{p}_k^b, \tag{148}$$

where

$$c^r_\theta = c^{(1)}_\theta + c^{(2)}_\theta, c^r_\eta = \mu_2^2 c^{(1)}_\eta + \mu_1^2 c^{(2)}_\eta. \quad (149)$$

We also have

$$\theta^c_{ij} = \frac{c^c_\theta l_P^2}{\hbar}\varepsilon_{ijk}\tilde{a}_k = \frac{\tilde{\gamma} l_P^2}{M\hbar}\varepsilon_{ijk}\tilde{a}_k, \quad (150)$$

$$\eta^c_{ij} = \frac{c^c_\eta \hbar}{l_P^2}\varepsilon_{ijk}\tilde{p}^b_k = \frac{\tilde{\alpha} M\hbar}{l_P^2}\varepsilon_{ijk}\tilde{p}^b_k, \quad (151)$$

with

$$c^c_\theta = \mu_1^2 c^{(1)}_\theta + \mu_2^2 c^{(2)}_\theta, \quad (152)$$

$$c^c_\eta = c^{(1)}_\eta + c^{(2)}_\eta. \quad (153)$$

So, from (147)-(151) we can conclude that in the case when conditions (134), (135) are satisfied, the tensors of noncommutativity describing the center-of-mass θ^c_{ij}, η^c_{ij} and relative motion θ^r_{ij}, η^r_{ij} depend on the total and reduced masses, respectively.

Note also that conditions (134), (135) are also satisfied for constants c^c_θ, c^c_η, c^r_θ, c^r_η. Namely, we can write

$$c^c_\theta M = c^r_\theta \mu = c^{(1)}_\theta m_1 = c^{(2)}_\theta m_2 = \tilde{\gamma} = \text{const}, \quad (154)$$

$$\frac{c^c_\eta}{M} = \frac{c^r_\eta}{\mu} = \frac{c^{(1)}_\eta}{m_1} = \frac{c^{(2)}_\eta}{m_2} = \tilde{\alpha} = \text{const}. \quad (155)$$

The noncommutative coordinates and noncommutative momenta of the center-of-mass and the noncommutative coordinates and noncommutative momenta of the relative motion can be represented as

$$X^c_i = x^c_i - \frac{1}{2}\theta_{ij}p^c_j = x^c_i + \frac{1}{2}[\boldsymbol{\theta}^c \times \mathbf{p}^c]_i, \quad (156)$$

$$P^c_i = p^c_i + \frac{1}{2}\eta^c_{ij}x^c_j = p^c_i - \frac{1}{2}[\boldsymbol{\eta}^c \times \mathbf{x}^c]_i, \quad (157)$$

$$X^r_i = x^r_i - \frac{1}{2}\theta^r_{ij}p^r_j = x^r_i + \frac{1}{2}[\boldsymbol{\theta}^r \times \mathbf{p}^r]_i, \quad (158)$$

$$P^r_i = p^r_i + \frac{1}{2}\eta^r_{ij}x^r_j = p^r_i - \frac{1}{2}[\boldsymbol{\eta}^r \times \mathbf{x}^r]_i. \quad (159)$$

For coordinates x^r_i, x^c_i and momenta p^r_i, p^c_i we have the ordinary commutation relations

$$[x^c_i, x^c_j] = [p^c_i, p^c_j] = [x^r_i, x^r_j] = [p^r_i, p^r_j] = 0, \quad (160)$$

$$[x^c_i, x^r_j] = [p^c_i, p^r_j] = [x^r_i, p^c_j] = [p^r_i, x^c_j] = 0, \quad (161)$$

$$[x^c_i, p^c_j] = [x^r_i, p^r_j] = i\hbar\delta_{ij}. \quad (162)$$

So, the Hamiltonian in the representation (156)-(159) reads

$$\begin{aligned} H_s &= \frac{(\mathbf{p}^c)^2}{2M} + \frac{(\mathbf{p}^r)^2}{2\mu} + \frac{(\boldsymbol{\eta}^c \cdot \mathbf{L}^\mathbf{c})}{2M} + \frac{[\boldsymbol{\eta}^c \times \mathbf{x}^\mathbf{c}]^\mathbf{2}}{8M} + \\ &+ \frac{(\boldsymbol{\eta}^r \cdot \mathbf{L}^\mathbf{r})}{2\mu} + \frac{[\boldsymbol{\eta}^r \times \mathbf{x}^\mathbf{r}]^\mathbf{2}}{8\mu} - \\ &+ \frac{\kappa}{\sqrt{(x^r)^2 - (\boldsymbol{\theta}^r \cdot \mathbf{L}^r) + \frac{1}{4}[\boldsymbol{\theta}^r \times \mathbf{p}^r]^2}}. \end{aligned} \quad (163)$$

Here, we consider the notation

$$\mathbf{L}^c = [\mathbf{x}^\mathbf{c} \times \mathbf{p}^\mathbf{c}], \quad (164)$$

$$\mathbf{L}^r = [\mathbf{x}^r \times \mathbf{p}^r]. \quad (165)$$

Up to the second order in the parameters of noncommutativity, the Hamiltonian of the system reads

$$\begin{aligned} H_s &= \frac{(\mathbf{p}^c)^2}{2M} + \frac{(\mathbf{p}^r)^2}{2\mu} - \frac{\kappa}{x^r} + \frac{(\boldsymbol{\eta}^c \cdot \mathbf{L}^c)}{2M} + \\ &+ \frac{[\boldsymbol{\eta}^c \times \mathbf{x}^c]^2}{8M} + \frac{(\boldsymbol{\eta}^r \cdot \mathbf{L}^r)}{2\mu} + \frac{[\boldsymbol{\eta}^r \times \mathbf{x}^r]^2}{8\mu} - \\ &+ \frac{\kappa}{2(x^r)^3}(\boldsymbol{\theta}^r \cdot \mathbf{L}^r) - \frac{3\kappa}{8(x^r)^5}(\boldsymbol{\theta}^r \cdot \mathbf{L}^r)^2 + \\ &+ \frac{\kappa}{16}\left(\frac{1}{(x^r)^2}[\boldsymbol{\theta}^r \times \mathbf{p}^r]^2\frac{1}{x^r} + \frac{1}{x^r}[\boldsymbol{\theta}^r \times \mathbf{p}^r]^2\frac{1}{(x^r)^2} + \right. \\ &+ \left. \frac{\hbar^2}{(x^r)^7}[\boldsymbol{\theta}^r \times \mathbf{x}^r]^2\right). \end{aligned} \quad (166)$$

After averaging over the eigenfunctions of the harmonic oscillators $\psi^a_{0,0,0}$, $\psi^b_{0,0,0}$ we find

$$\begin{aligned} \langle H_s \rangle_{ab} &= \frac{(\mathbf{p}^c)^2}{2M} + \frac{(x^c)^2\langle(\eta^c)^2\rangle}{12M} + \frac{(\mathbf{p}^r)^2}{2\mu} - \\ &+ \frac{\kappa}{x^r} + \frac{(x^r)^2\langle(\eta^r)^2\rangle}{12\mu} - \frac{\kappa(L^r)^2\langle(\theta^r)^2\rangle}{8(x^r)^5} + \\ &+ \frac{\kappa}{24}\left(\frac{1}{(x^r)^2}(p^r)^2\frac{1}{x^r} + \frac{1}{x^r}(p^r)^2\frac{1}{(x^r)^2} + \right. \\ &+ \left. \frac{\hbar^2}{(x^r)^5}\right)\langle(\theta^r)^2\rangle. \end{aligned} \quad (167)$$

Up to the second order in the parameters of noncommutativity, we can examine H_0

$$H_0 = \langle H_c\rangle_{ab} + \langle H_r\rangle_{ab} + H^a_{osc} + H^b_{osc}, \tag{168}$$

$$\langle H_c\rangle_{ab} = \frac{(\mathbf{p}^c)^2}{2M} + \frac{(x^c)^2\langle(\eta^c)^2\rangle}{12M}, \tag{169}$$

$$\langle H_r\rangle_{ab} = \frac{(\mathbf{p}^r)^2}{2\mu} - \frac{\kappa}{x^r} + \frac{(x^r)^2\langle(\eta^r)^2\rangle}{12\mu} - $$
$$+\frac{\kappa(L^r)^2\langle(\theta^r)^2\rangle}{8(x^r)^5} + \frac{\kappa}{24}\left(\frac{1}{(x^r)^2}(p^r)^2\frac{1}{x^r} + \right.$$
$$\left. + \frac{1}{x^r}(p^r)^2\frac{1}{(x^r)^2} + \frac{\hbar^2}{(x^r)^5}\right)\langle(\theta^r)^2\rangle. \tag{170}$$

Operators $\langle H_c\rangle_{ab}$, $\langle H_r\rangle_{ab}$ describe the motion of the center-of-mass and the relative motion.

It is important that

$$[\langle H_c\rangle_{ab}, \langle H_r\rangle_{ab}] = [\langle H_c\rangle_{ab}, H^a_{osc} + H^b_{osc}] = 0. \tag{171}$$

So, we can study $\langle H_c\rangle_{ab}$ independently. Operator $\langle H_c\rangle_{ab}$ corresponds to the Hamiltonian of the three-dimensional harmonic oscillator of mass M and frequency

$$\omega = \frac{\sqrt{2\langle(\eta^c)^2\rangle}}{\sqrt{3}M} \tag{172}$$

The spectrum of the oscillator is well known

$$E_{n^c_1,n^c_2,n^c_3} = \frac{\hbar\sqrt{2\langle(\eta^c)^2\rangle}}{\sqrt{3}M}\left(n^c_1 + n^c_2 + n^c_3 + \frac{3}{2}\right), \tag{173}$$

where n^c_1, n^c_2, n^c_3 are quantum numbers.

According to the perturbation theory, we have the following corrections to the energy levels caused by noncommutativity of coordinates and noncommutativity of momenta

$$\Delta E^{(\theta\eta)}_{n,l} = \langle\psi^{(0)}_{n,l,m}|V|\psi^{(0)}_{n,l,m}\rangle =$$
$$\Delta E^{(\eta)}_{n,l} + \Delta E^{(\theta)}_{n,l}, \tag{174}$$

$$\Delta E^{(\eta)}_{n,l} = \langle\psi^{(0)}_{n,l,m}|V^\eta|\psi^{(0)}_{n,l,m}\rangle =$$
$$= \frac{\kappa a^3 n^2\langle(\eta^r)^2\rangle}{24\hbar^2}(5n^2 + 1 - 3l(l+1)), \tag{175}$$

$$\Delta E^{(\theta)}_{n,l} = \langle\psi^{(0)}_{n,l,m}|V^\theta|\psi^{(0)}_{n,l,m}\rangle = -\frac{\hbar^2\kappa\langle(\theta^r)^2\rangle}{a^5n^5}\times$$
$$\times\left(-\frac{6n^2 - 2l(l+1)}{3l(l+1)(2l+1)(2l+3)(2l-1)} + \right.$$
$$\frac{1}{6l(l+1)(2l+1)} +$$
$$+\frac{5n^2 - 3l(l+1) + 1}{2(l+2)(2l+1)(2l+3)(l-1)(2l-1)} -$$
$$\left. + \frac{5}{6}\frac{5n^2 - 3l(l+1) + 1}{l(l+1)(l+2)(2l+1)(2l+3)(l-1)(2l-1)}\right), \tag{176}$$

where

$$a = \frac{\hbar^2}{\mu\kappa}. \tag{177}$$

The correction to the energy levels with $l = 0$ reads

$$\Delta E^{(\theta\eta)}_{n,0} = \frac{a^3\kappa\langle(\eta^r)^2\rangle}{24\hbar^2}n^2(5n^2+1) + 1.72\frac{\hbar\langle\theta^r\rangle\pi\kappa}{8a^3n^3}. \tag{178}$$

Let us examine the effect of noncommutativity on hydrogen-like atoms. Corrections caused by noncommutativity of momenta (175) are proportional to $\langle(\eta^r)^2\rangle a^3$. From (148) it follows that

$$\langle(\eta^r)^2\rangle a^3 \sim \frac{1}{\mu}. \tag{179}$$

In corrections to the energy levels caused by noncommutativity of coordinates we have proportionality to $\langle\theta^r\rangle/a^3$ in the case of ns energy levels, or proportionality to $\langle(\theta^r)^2\rangle/a^5$ for energy levels with $l > 1$, (176). From (147), we can write

$$\frac{\langle\theta^r\rangle}{a^3} \sim \mu^2, \tag{180}$$
$$\frac{\langle\theta^r\rangle}{a^5} \sim \mu^3. \tag{181}$$

So, the effect of the coordinates noncommutativity can be better examined in the spectrum of atoms with large reduced masses, especially for energy levels with $l = 0$ and small quantum numbers n. The effect of momentum noncommutativity better appears in energy levels with large quantum numbers of atoms with small reduced masses. Also, it is worth mentioning that in the case of atoms with large reduced masses, the differences in the effects of momentum and the coordinates noncommutativity appear better.

Let us examine muonic hydrogen, which is a system of a proton and muon. We have that

$$\frac{\mu_{\mu p}}{\mu_H} \simeq \frac{m_\mu}{m_e} = 206.8 \quad (182)$$

where $\mu_{\mu p}$, μ_H are the reduced mass of muonic hydrogen and hydrogen atoms, m_e, m_μ are the mass of the electron and the mass of the muon. Because of this ratio, the corrections to the energy levels of muonic hydrogen in the case of $l > 1$ (176) are $(m_\mu/m_e)^3 = 8.8 \cdot 10^6$ times larger than that for the hydrogen atom. So, noncommutativity of coordinates can be better examined in the case of muonic hydrogen. Corrections (175) are 206.8 times smaller in the case of muonic hydrogen than in the case of the hydrogen atom.

7. Upper bounds on the parameters of coordinates and momentum noncommutativity obtained based on studies of the hydrogen atom and antiprotonic helium

To find upper bounds for the parameters of coordinate and momentum noncommutativity, we assume that corrections to the hydrogen atom transition energies in quantum space do not exceed the accuracy of the transition measurements. In paper [22], the authors presented experimental results for the $1s-2s$ transition frequency $f_{1s-2s} = 2466061413187018(11)$Hz with a relative uncertainty of 4.5×10^{-15}. So, we can write the following inequality

$$\left|\frac{\Delta^\theta_{1,2} + \Delta^\eta_{1,2}}{E^{(0)}_2 - E^{(0)}_1}\right| \leq 4.5 \times 10^{-15}, \quad (183)$$

where $E^{(0)}_n$ are well-known energy levels of the hydrogen atom in the ordinary space. To estimate the order of the upper bounds for the parameters of noncommutativity, we consider

$$\left|\frac{\Delta^\theta_{1,2}}{E^{(0)}_2 - E^{(0)}_1}\right| \leq 2.25 \times 10^{-15}, \quad (184)$$

$$\left|\frac{\Delta^\eta_{1,2}}{E^{(0)}_2 - E^{(0)}_1}\right| \leq 2.25 \times 10^{-15}. \quad (185)$$

Using (118) we have

$$\Delta^\theta_{1,2} = -\frac{3\hbar\langle\theta\rangle\pi e^2}{16 a_B^3}, \quad (186)$$

$$\Delta^\eta_{1,2} = \frac{13 a_B^2\langle\eta^2\rangle}{4M}. \quad (187)$$

So, the upper bounds read

$$\hbar\langle\theta\rangle \leq 10^{-36}\,\mathrm{m}^2, \quad (188)$$

$$\hbar\sqrt{\langle\eta^2\rangle} \leq 10^{-61}\,\mathrm{kg}^2\mathrm{m}^2/\mathrm{s}^2. \quad (189)$$

The obtained results are in agreement with those obtained on the basis of studies of the spectrum of a gravitation quantum well [23]. The are also in agreement with the results obtained from the spectrum of the hydrogen atom considered in noncommutative space of the canonical type [24], and examining the Lamb shift [1]. Note that the ratio $m_p/m_e = 1836$, therefore $\mu \simeq m_e$. Therefore the orders of the upper bounds do not change if we take into account the effect of the reduced mass of the hydrogen atom.

Let us examine the exotic atom known as antiprotonic helium $\bar{p}^4He^+$. It is composed of an antiproton, an electron and a helium nucleus. In papers [25, 26], it was shown that the transition frequency of the atom can be approximately written as transitions of the hydrogen atom effective nuclear charge $Z_{eff} < 2$. The charge describes the shielding of the nuclear charge by the electron. Of course the difference of masses of hydrogen and antiprotonic helium atoms has to be taken into consideration. So, the obtained results for the effect of noncommutativity of coordinate and noncommutativity of momenta (175), (176) can be used for estimation of the upper bounds. Atom $\bar{p}^4He^+$ has a large reduced mass. So, the effect of coordinate noncommutativity on the spectrum of the exotic atom is larger than on the hydrogen atom. So, the antiprotonic helium is an attractive candidate for studies of noncommutativity of coordinates

The experimental result for transition frequency $(n,l) = (36,34) \to (34,32)$ of antiprotonic helium reads $f = 1522107062$ MHz. The result is obtained with the total experimental error 3.5 MHz [27]. Assuming that the effect of noncommutativity on the energy levels is smaller than the accuracy of measurements, we have

$$|\Delta^{(\theta)} + \Delta^{(\eta)}| \leq 3.5\text{MHz}, \quad (190)$$

$$\Delta^\theta = \Delta E^{(\theta)}_{36,34} - \Delta E^{(\theta)}_{34,32}, \quad (191)$$

$$\Delta^\eta = \Delta E^{(\eta)}_{36,34} - \Delta E^{(\eta)}_{34,32}, \quad (192)$$

and $\Delta E^{(\theta)}_{n,l}$, $\Delta E^{(\eta)}_{n,l}$ read (175), (176). To estimate the upper bounds, we write

$$|\Delta^{\theta}| \leq 1.75\text{MHz}, \quad (193)$$

$$|\Delta^{\eta}| \leq 1.75\text{MHz}. \quad (194)$$

We also consider $Z = 2$, $a = m_e a_B / m_{\bar{p}}$, where $m_{\bar{p}}$ is the mass of the antiproton, a_B is the Bohr radius of the hydrogen atom in (175), (176). As a result, we find

$$\hbar\langle\theta^r\rangle \leq 10^{-27}\,\text{m}^2, \quad (195)$$

$$\hbar\sqrt{\langle(\eta^r)^2\rangle} \leq 10^{-50}\,\text{kg}^2\text{m}^2/\text{s}^2. \quad (196)$$

Because of not high precision of the measurements of the spectrum of $\bar{p}^4He^+$, the obtained upper bounds do not lead to strong restriction on the values of parameters of noncommutativity. But, it is worth stressing that the effect of the coordinates noncommutativity on $\bar{p}^4He^+$ is three orders larger than that on the hydrogen atom. So, improvement of precision of measurements of the spectrum of the exotic atom opens the possibility to find stringent upper bounds for the parameter of the coordinates noncomutativity.

8. Conclusions

The hydrogen atom spectrum has been examined in noncommutative phase space with preserved rotational symmetry (1)-(3). The effect of noncommutativity of coordinates and noncommutativity of momenta on the energy levels of the atom has been obtained (55). We conclude that the effect of momentum noncommutativity is larger in the case of energy levels with large principal quantum numbers.

The effect of the coordinates noncommutativity can be better studied on the basis of energy levels of the hydrogen atom with small quantum numbers n. We have also found that corrections to the ns-energy levels (118) are proportional to $\langle\theta\rangle$. For energy levels with $l > 1$ (55), we have proportionality to $\langle\theta^2\rangle$. So, ns energy levels of the hydrogen atom are more sensitive to the coordinates noncommutativity.

We have also studied effect of noncommutativity of coordinates and noncommutativity of momenta on the spectrum of hydrogen-like atoms.

We have examined a general case when different particles feel the effects of space quantization with different tensors of noncommutativity. The problem of description of a system of particles in rotationally-invariant noncommutative phase space has been considered.

It has been shown that in the case when tensors of noncommutativity corresponding to different particles are determined by their masses for coordinates and momenta of the center-of-mass of a system, we have a noncommutative algebra with the effective tensors of noncommutativity. Also, in the case when the conditions hold, the effective tensors of noncommutativity do not depend on the composition of the system and are instead determined by their total mass (138), (139).

It is important to stress that idea of the relation of parameters of noncommutativity with mass opens the possibility to solve fundamental principles in noncommutative space of the canonical type [28, 29], noncommutative phase phase of the canonical type [30, 31], deformed space with minimal length [32–34]. The proposed conditions on the tensors of noncommutativity (134), (135) are similar to those $\theta m = \gamma = \text{const}$, $\eta/m = \alpha = \text{const}$ proposed in the noncommutative phase space of the canonical type [30, 31]. They lead to solving of the problem of violation of the properties of kinetic energy, and violation of the weak equivalence principle in space.

We have obtained corrections to the spectrum of a two-particle system with Colomb interaction caused by noncommutativity of coordinates and noncommutativity of momenta. It has been determined that the corrections caused by noncommutativity of coordinates and corrections caused by noncommutativity of momenta have different dependencies on the reduced mass μ and parameter of interaction κ. So, one can choose a system with good sensitivity to the particular type of noncommutativity. We have found that the effect of momentum noncommutativity can be better examined for the ns energy levels with large quantum numbers of atoms with small reduced masses. Studies of ns energy levels with small quantum numbers of atoms with large reduced masses are important for finding the effect of coordinate noncommutativity. We have also shown that antiprotonic helium is an attractive candidate for studies of the effect of coordinate noncommutativity.

Upper bounds for parameters of noncommutativity have been found on the basis of studies of the hydrogen atom and antiprotonic helium. The upper bounds obtained based on studies of the hydrogen atom are in agreement with those presented in the literature.

References

[1] M. Chaichian, M. M. Sheikh-Jabbari, and A. Tureanu, "Hydrogen atom spectrum and the Lamb shift in noncommutative QED," *Phys. Rev. Lett.*, vol. 86, no. 13, pp. 2716–2719, 2001.

[2] P.-M. Ho and H.-C. Kao, "Noncommutative quantum mechanics from noncommutative quantum field theory," *Phys. Rev. Lett.*, vol. 88, no. 15, 2002.

[3] M. Chaichian, M. M. Sheikh-Jabbari, and A. Tureanu, "Noncommutativity of space-time and the hydrogen atom spectrum," *Eur. Phys. J. C*, vol. 36, no. 2, pp. 251–252, 2004.

[4] N. Chair and M. A. Dalabeeh, "The noncommutative quadratic Stark effect for the H-atom," *J. Phys. A*, vol. 38, no. 7, pp. 1553–1558, 2005.

[5] A. Stern, "Noncommutative point sources," *Phys. Rev. Lett.*, vol. 100, no. 6, 2008.

[6] S. Zaim, L. Khodja, and Y. Delenda, "Second-order corrections to the non-commutative Klein-Gordon equation with a Coulomb potential," *Int. J. Mod. Phys. A*, vol. 26, no. 23, pp. 4133–4144, 2011.

[7] T. C. Adorno, M. C. Baldiotti, M. Chaichian, D. Gitman, and T. A., "Dirac equation in noncommutative space for hydrogen atom," *Phys. Lett. B*, vol. 682, no. 2, pp. 235–239, 2009.

[8] L. Khodja and S. Zaim, "New treatment of the noncommutative Dirac equation with a Coulomb potential," *Int. J. Mod. Phys. A*, vol. 27, no. 19, 2012.

[9] A. E. F. Djemai and H. Smail, "On quantum mechanics on noncommutative quantum phase space," *Commun. Theor. Phys.*, vol. 41, no. 6, pp. 837–844, 2004.

[10] K. Li and N. Chamoun, "Hydrogen atom spectrum in noncommutative phase space," *Chin. Phys. Lett.*, vol. 23, no. 5, pp. 1122–1123, 2006.

[11] S. A. Alavi, "Lamb shift and Stark effect in simultaneous space-space and momentum-momentum noncommutative quantum mechanics and θ-deformed $su(2)$ algebra," *Mod. Phys. Lett. A*, vol. 22, no. 5, pp. 377–383, 2007.

[12] A. P. Balachandran and A. Pinzul, "On time-space noncommutativity for transition processes and noncommutative symmetries," *Mod. Phys. Lett. A*, vol. 20, no. 27, pp. 2023–2034, 2005.

[13] A. Stern, "Particlelike solutions to classical noncommutative gauge theory," *Phys. Rev. D*, vol. 78, no. 6, 2008.

[14] M. Moumni, A. BenSlama, and S. Zaim, "A new limit for the noncommutative spacetime parameter," *J. Geom. Phys.*, vol. 61, no. 1, pp. 151–156, 2011.

[15] M. Moumni, A. BenSlama, and S. Zaim, "Spectrum of hydrogen atom in space-time non-commutativity," *The African Review of Physics*, vol. 7, no. 0010, pp. 83–94, 2012.

[16] S. Zaim and Y. Delenda, "Noncommutative of space-time and the relativistic hydrogen atom," *J. Phys.: Conf. Ser.*, vol. 435, 2013.

[17] Kh. P. Gnatenko and V. M. Tkachuk, "Hydrogen atom in rotationally invariant noncommutative space," *Phys. Lett. A*, vol. 378, no. 47, pp. 3509–3515, 2014.

[18] Kh.. P. Gnatenko, Yu.. S. Krynytskyi, and V. Tkachuk, "Perturbation of the ns levels of the hydrogen atom in rotationally invariant noncommutative space," *Mod. Phys. Lett. A*, vol. 30, no. 8. – Art. 1550033. – 12 p., 2015.

[19] Kh. P. Gnatenko and V. M. Tkachuk, "Composite system in rotationally invariant noncommutative phase space," *Int. J. Mod. Phys. A.*, vol. 33, no. 7, p. 1850037, 2018.

[20] Kh. P. Gnatenko and V. M. Tkachuk, "Noncommutative phase space with rotational symmetry and hydrogen atom," *Int. J. Mod. Phys. A.*, vol. 32, no. 26, p. 1750161, 2017.

[21] W.-C. Qiang and S.-H. Dong, "An alternative approach to calculating the mean values r^k for hydrogen-like atoms," *Phys. Scripta*, vol. 70, no. 5, pp. 276–279, 2004.

[22] A. Matveev, C. G. Parthey, K. Predehl, J. Alnis, A. Beyer, R. Holzwarth, T. Udem, T. Wilken, N. Kolachevsky, M. Abgrall, D. Rovera, C. Salomon, P. Laurent, G. Grosche, O. Terra, T. Legero, H. Schnatz, S. Weyers, B. Altschul, and T. W. H´ansch, "Precision measurement of the hydrogen $1s$-$2s$ frequency via a 920-km fiber link," *Phys. Rev. Lett.*, vol. 110, no. 23, 2013.

[23] O. Bertolami, J. G. Rosa, C. M. L. Aragão, P. Castorina, and D. Zappalà, "Noncommutative gravitational quantum well," *Phys. Rev. D*, vol. 72, no. 2. – Art. – 025010. – 9 p., 2005.

[24] O. Bertolami and R. Queiroz, "Phase-space noncommutativity and the Dirac equation," *Phys. Lett. A*, vol. 375, no. 2011, pp. 4116–4119, 2011.

[25] J. Polchinski, *Precision Spectroscopy of Antiprotonic Helium Atoms and Ions – Weighing the Antiproton. In: Karshenboim S.G. (eds) Precision Physics of Simple Atoms and Molecules. Lecture Notes in Physics vol 745. Springer, Berlin, Heidelberg.* Springer, Berlin, Heidelberg.

[26] A. Matveev, C. G. Parthey, K. Predehl, J. Alnis, A. Beyer, R. Holzwarth, T. Udem, T. Wilken, N. Kolachevsky, M. Abgrall, D. Rovera, C. Salomon, P. Laurent, G. Grosche, O. Terra, T. Legero, H. Schnatz, S. Weyers, B. Altschul, and T. W. Hänsch, "Spectroscopy of antiprotonic helium atoms and its contribution to the fundamental physical constants," *Proc. Jpn. Acad. Ser. B Phys. Biol. Sci.*, vol. 86, no. 1, pp. 1–10, 2010.

[27] M. Hori, A. Soter, D. Barn, A. Dax, R. Hayano, S. Friedreich, B. Juh:'asz, T. Pask, E. Widmann, D. Horv:'ath, L. Venturelli, and N. Zurlo, "Two-photon laser spectroscopy of antiprotonic helium and the antiproton-to-electron mass ratio," *Nature*, vol. 475, pp. 484–488, 2011.

[28] Kh. P. Gnatenko, "Composite system in noncommutative space and the equivalence principle," *Phys. Lett. A*, vol. 377, no. 43, pp. 3061–3066, 2013.

[29] Kh. P. Gnatenko and V. M. Tkachuk, "Effect of coordinate noncommutativity on the mass of a particle in a uniform field and the equivalence principle," *Mod. Phys. Lett. A*, vol. 31, no. 5, p. 1650026, 2016.

[30] Kh. P. Gnatenko and V. M. Tkachuk, "Weak equivalence principle in noncommutative phase space and the parameters of noncommutativity," *Phys. Lett. A*, vol. 381, no. 31, p. 24632469, 2017.

[31] Kh. P. Gnatenko, "Kinematic variables in noncommutative phase space and parameters of noncommutativity," *Mod. Phys. Lett. A*, vol. 32, no. 31, p. 1750166, 2017.

[32] C. Quesne and V. M. Tkachuk, "Composite system in deformed space with minimal length," *Phys. Rev. A*, vol. 81, no. 1, p. 012106, 2010.

[33] V. M. Tkachuk, "Deformed Heisenberg algebra with minimal length and the equivalence principle," *Phys. Rev. A*, vol. 86, no. 6, p. 062112, 2012.

[34] V. M. Tkachuk, "Galilean and Lorentz transformations in a space with generalized uncertainty principle," *Found. Phys.*, vol. 46, no. 12, p. 16661679, 2016.