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System of harmonic oscillators in a rotationally-invariant noncommutative phase space

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Abstract

Algebra with noncommutativity of coordinates and noncommutativity of momenta which is rotationally-invariant and equivalent to noncommutative algebra of the canonical type is considered. In the framework of algebra, the effect of space quantization on the spectrum of systems of harmonic oscillators is studied. Among them, two interacting oscillators, a system of three interacting oscillators, and a harmonic oscillator chain are examined. The energy levels of the systems are found up to the second orders in the parameters of noncommutativity. We conclude that space quantization has an effect on the frequencies of the harmonic oscillators.

Keywords:

noncommutative space, rotational symmetry, harmonic oscillator

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1. Introduction

To find new effects of the noncommutativity of coordinates and noncommutativity of momenta in the properties of a wide class of physical systems, it is important to examine many-particle systems. Studies of harmonic oscillator in noncommutative space have received much attention (see, for instance, [1–15]). Two coupled harmonic oscillators were studied in a noncommutative space [16, 17] and a noncommutative phase space [18, 19]. A system of free particles was examined in [20, 21] in a noncommutative phase space of the canonical type. Classical problems of many particles were examined in [22] in the case of space-time noncommutativity.

It is worth noting that systems of harmonic oscillators have various applications. Such studies have importance in nuclear physics [23–25], in quantum chemistry, and molecular spectroscopy [26–29]. Additionally, networks of harmonic oscillators are used in quantum information [30–32].

In this paper, we study a system of interacting oscillators in a uniform field in the framework of rotationally-invariant noncommutative algebra

$$[X_i, X_j] = i\hbar\theta_{ij}, \quad (1)$$

$$[X_i, P_j] = i\hbar\left(\delta_{ij} + \sum_k \frac{\theta_{ik}\eta_{jk}}{4}\right), \quad (2)$$

$$[P_i, P_j] = i\hbar_{ij}, \quad (3)$$

$$\theta_{ij} = \frac{c_\theta l_P^2}{\hbar}\sum_k \varepsilon_{ijk}\tilde{a}_k, \quad (4)$$

$$\eta_{ij} = \frac{c_\eta \hbar}{l_P^2}\sum_k \varepsilon_{ijk}\tilde{p}_k^b. \quad (5)$$

Here c_θ, c_η are constants and $\tilde{a}_k$, $\tilde{p}_k^b$ are additional coordinates and additional momenta that are governed by spherically symmetric systems, which can be harmonic oscillators; l_P is the Planck length [33]. The algebra (1)-(3) is equivalent to a noncommutative algebra of the canonical type in the sense that the noncommutative coordinates and noncommutative momenta, as well as tensors of noncommutativity, satisfy the same commutation relations as in the case of noncommutative algebra of the canonical type (the tensors of noncommutativity commute with coordinates and momenta) [33].

In order to solve the problem of description of composite system in a noncommutative phase space as well as the problem of violation of the weak equivalence principle, it was proposed to consider parameters of noncommutativity to be related with mass

$$c_\theta^{(n)} = \frac{\tilde{\gamma}}{m_n}, \quad (6)$$

$$c_\eta^{(n)} = \tilde{\alpha} m_n, \quad (7)$$

see [34].

In present paper, we consider systems of harmonic oscillators in a rotationally-invariant noncommutative phase space. The effect of noncommutativity of coordinates and noncommutativity of momenta on the energy levels of the system is analyzed.

The paper is organized as follows. A system of two interacting oscillators and three interacting oscillators are examined in Section 2 and Section 3 respectively. The effect of noncommutativity of coordinates and noncommutativity of momenta on the harmonic oscillator chain is studied in Section 4. Conclusions are presented in Section 5. Results presented in this paper are published in [35–37].

2. Energy levels of two interacting oscillators

We consider a system of two oscillators with masses m_1, m_2 and frequencies ω_1, ω_2. The Hamiltonian of the system reads

$$H_s = \frac{(\mathbf{P}^{(1)})^2}{2m_1} + \frac{(\mathbf{P}^{(2)})^2}{2m_2} + \frac{m_1\omega_1^2(\mathbf{X}^{(1)})^2}{2} + \frac{m_2\omega_2^2(\mathbf{X}^{(2)})^2}{2} + k(\mathbf{X}^{(1)} - \mathbf{X}^{(2)})^2. \quad (8)$$

Coordinates and momenta $\mathbf{X}^{(n)}$, $\mathbf{P}^{(n)}$ satisfy relations of noncommutative algebra.

Coordinates and momenta of harmonic oscillators satisfy relations of rotationally-invariant noncommutative algebra

$$[X_i^{(n)}, X_j^{(m)}] = i\hbar\delta_{mn}\theta_{ij}^{(n)}, \quad (9)$$

$$[X_i^{(n)}, P_j^{(m)}] = i\hbar\delta_{mn}\left(\delta_{ij} + \sum_k \frac{\theta_{ik}^{(n)}\eta_{jk}^{(m)}}{4}\right), \quad (10)$$

$$[P_i^{(n)}, P_j^{(m)}] = i\hbar\delta_{mn}\eta_{ij}^{(n)}, \quad (11)$$

$$\theta_{ij}^{(n)} = \frac{c_\theta^{(n)} l_P^2}{\hbar}\sum_k \varepsilon_{ijk}\tilde{a}_k, \quad (12)$$

$$\eta_{ij}^{(n)} = \frac{c_\eta^{(n)}\hbar}{l_P^2}\sum_k \varepsilon_{ijk}\tilde{p}_k^b. \quad (13)$$

Here indexes $m,n=(1,...,N)$ label the oscillators.

It is worth noting that system of two coupled harmonic oscillators is considered as a model in molecular physics [26, 27]. It is also used for description of states of light in the framework of two-photon quantum optics [38, 39].

In the case of two interacting oscillators, we can write

$$\begin{aligned} H_0 &= \frac{(\mathbf{p}^{(1)})^2}{2m_{eff}^{(1)}}+\frac{(\mathbf{p}^{(2)})^2}{2m_{eff}^{(2)}}+ \\ &+ \frac{m_{eff}^{(1)}(\omega_{eff}^{(1)})^2(\mathbf{x}^{(1)})^2}{2}+\frac{m_{eff}^{(2)}(\omega_{eff}^{(2)})^2(\mathbf{x}^{(2)})^2}{2}+ \\ &+ k(\mathbf{x}^{(1)}-\mathbf{x}^{(2)})^2+\frac{k}{6}\Big(\langle(\theta^{(1)})^2\rangle(\mathbf{p}^{(1)})^2+ \\ &+ \langle(\theta^{(2)})^2\rangle(\mathbf{p}^{(2)})^2-2\langle\theta^{(1)}\theta^{(2)}\rangle(\mathbf{p}^{(1)}\cdot\mathbf{p}^{(2)})\Big)+ \\ &+ H_{osc}^a+H_{osc}^b. \end{aligned} \tag{14}$$

Here

$$m_{eff}^{(n)} = m_n\left(1+\frac{m_n^2\omega_n^2\langle(\theta^{(n)})^2\rangle}{6}\right)^{-1}, \tag{15}$$

$$\begin{aligned} \omega_{eff}^{(n)} &= \left(\omega_n^2+\frac{\langle(\eta^n)^2\rangle}{6m_n^2}\right)^{\frac{1}{2}}\times \\ &\times \left(1+\frac{m_n^2\omega_n^2\langle(\theta^{(n)})^2\rangle}{6}\right)^{\frac{1}{2}}, \end{aligned} \tag{16}$$

$$\begin{aligned} \langle\theta^{(n)}\theta^{(m)}\rangle &= \frac{c_\theta^{(n)}c_\theta^{(m)}l_P^4}{\hbar^2}\langle\psi_{0,0,0}^a|\tilde{a}^2|\psi_{0,0,0}^a\rangle= \\ &= \frac{3c_\theta^{(n)}c_\theta^{(m)}l_P^4}{2\hbar^2}, \end{aligned} \tag{17}$$

$$\begin{aligned} \langle(\eta^{(n)})^2\rangle &= \frac{\hbar^2(c_\eta^{(n)})^2}{l_P^4}\langle\psi_{0,0,0}^b|(\tilde{p}^b)^2|\psi_{0,0,0}^b\rangle= \\ &= \frac{3\hbar^2(c_\eta^{(n)})^2}{2l_P^4}. \end{aligned} \tag{18}$$

For coordinates and momenta $x_i^{(n)}$, $p_i^{(n)}$, we have the ordinary commutation relations. Therefore, the energy levels of H_0 are

$$\begin{aligned} E_{\{n_1\},\{n_2\},\{n_3\}} &= \hbar\omega_+\left(n_1^{(1)}+n_2^{(1)}+n_3^{(1)}+\frac{3}{2}\right)+ \\ &+ \hbar\omega_-\left(n_1^{(2)}+n_2^{(2)}+n_3^{(2)}+\frac{3}{2}\right)+3\hbar\omega_{osc}, \end{aligned} \tag{19}$$

with

$$\begin{aligned} \omega_\pm^2 &= \frac{1}{2}\sum_n\Bigg((\omega_{eff}^{(n)})^2+\frac{2k}{m_{eff}^{(n)}}+\frac{km_{eff}^{(n)}(\omega_{eff}^{(n)})^2\langle(\theta^{(n)})^2\rangle}{3}+ \\ &+ \frac{2k^2}{3}\left(\langle(\theta^{(n)})^2\rangle+\langle\theta^{(1)}\theta^{(2)}\rangle\right)\Bigg)\pm\frac{1}{2}\sqrt{D}, \end{aligned} \tag{20}$$

$$\begin{aligned} D &= \Bigg(\sum_n(\omega_{eff}^{(n)})^2+\sum_n\frac{2k}{m_{eff}^{(n)}}+\sum_n\frac{km_{eff}^{(n)}(\omega_{eff}^{(n)})^2\langle(\theta^{(n)})^2\rangle}{3}+ \\ &+ \sum_n\frac{2k^2}{3}\left(\langle(\theta^{(n)})^2\rangle+\langle\theta^{(1)}\theta^{(2)}\rangle\right)\Bigg)^2- \\ &+ 4\prod_n\Bigg((\omega_{eff}^{(n)})^2+\frac{2k}{m_{eff}^{(n)}}+\frac{km_{eff}^{(n)}(\omega_{eff}^{(n)})^2\langle(\theta^{(n)})^2\rangle}{3}+ \\ &+ \frac{2k^2}{3}\left(\langle(\theta^{(n)})^2\rangle+\langle\theta^{(1)}\theta^{(2)}\rangle\right)\Bigg)+ \\ &+ 4\Bigg(\frac{2k}{m_{eff}^{(2)}}+\frac{km_{eff}^{(1)}(\omega_{eff}^{(1)})^2\langle\theta^{(1)}\theta^{(2)}\rangle}{3}+ \\ &+ \frac{2k^2}{3}\left(\langle(\theta^{(2)})^2\rangle+\langle\theta^{(1)}\theta^{(2)}\rangle\right)\Bigg)\times \\ &\times \Bigg(\frac{2k}{m_{eff}^{(1)}}+\frac{km_{eff}^{(2)}(\omega_{eff}^{(2)})^2\langle\theta^{(1)}\theta^{(2)}\rangle}{3}+ \\ &+ \frac{2k^2}{3}\left(\langle(\theta^{(1)})^2\rangle+\langle\theta^{(1)}\theta^{(2)}\rangle\right)\Bigg). \end{aligned} \tag{21}$$

If the mass of the oscillators are the same $m_1=m_2$, we obtain

$$m_{eff}^{(n)} = m_{eff}, \tag{22}$$

$$\omega_{eff}^{(n)} = \omega_{eff}, \tag{23}$$

and

$$\omega_- = \omega_{eff}, \tag{24}$$

$$\begin{aligned} \omega_+ &= \Bigg(\omega_{eff}^2+\frac{4k}{m_{eff}}+\frac{2k\langle\theta^2\rangle m_{eff}\omega_{eff}^2}{3}+ \\ &+ \frac{8k^2\langle\theta^2\rangle}{3}\Bigg)^{\frac{1}{2}}. \end{aligned} \tag{25}$$

3. Effect of noncommutativity on the energy levels of a system of three interacting oscillators

We study three interacting oscillators with masses m_1, $m_2=m_3=m$, and frequencies ω_1, $\omega_2=\omega_3=\omega$ described with the following Hamiltonian

$$H_s = \frac{(\mathbf{P}^{(1)})^2}{2m_1}+\frac{(\mathbf{P}^{(2)})^2}{2m}+\frac{(\mathbf{P}^{(3)})^2}{2m}+ + \frac{m_1\omega_1^2(\mathbf{X}^{(1)})^2}{2}+\frac{m\omega^2(\mathbf{X}^{(2)})^2}{2}+\frac{m\omega^2(\mathbf{X}^{(3)})^2}{2}+ + k(\mathbf{X}^{(1)}-\mathbf{X}^{(2)})^2+k(\mathbf{X}^{(2)}-\mathbf{X}^{(3)})^2+k(\mathbf{X}^{(3)}-\mathbf{X}^{(3)})^2. \quad (26)$$

If $\omega_n=0$, the model (26) is used for the description of confining forces between quarks [23–25]. Up to the second order in the parameters of noncommutativity, we can study the Hamiltonian

$$H_0 = \sum_n\frac{(\mathbf{p}^{(n)})^2}{2m^{(n)}_{eff}}+\sum_n\frac{m^{(n)}_{eff}(\omega^{(n)}_{eff})^2(\mathbf{x}^{(n)})^2}{2}+ + \frac{k}{2}\sum_{\substack{m,n\\ m\neq n}}(\mathbf{x}^{(n)}-\mathbf{x}^{(m)})^2+ + \frac{k}{12}\sum_{\substack{m,n\\ m\neq n}}\Big(\langle(\theta^{(n)})^2\rangle(\mathbf{p}^{(n)})^2+\langle(\theta^{(m)})^2\rangle(\mathbf{p}^{(m)})^2- + 2\langle\theta^{(n)}\theta^{(m)}\rangle(\mathbf{p}^{(n)}\cdot\mathbf{p}^{(m)})\Big)+H^a_{osc}+H^b_{osc}, \quad (27)$$

with $m^{(n)}_{eff}$, $\omega^{(n)}_{eff}$, $\langle\theta^{(n)}\theta^{(m)}\rangle$ given by (15)-(17).

The energy levels of the Hamiltonian (27) are the following

$$E_{\{n_1\},\{n_2\},\{n_3\}} = \sum_{a=1}^{3}\hbar\tilde{\omega}_a\left(n_1^{(a)}+n_2^{(a)}+n_3^{(a)}+\frac{3}{2}\right)+ + 3\hbar\omega_{osc}, \quad (28)$$

$$\tilde{\omega}_1 = \frac{1}{\sqrt{2}}\Bigg(\omega^2_{eff}+(\omega^{(1)}_{eff})^2+\frac{2k}{m_{eff}}+\frac{4k}{m^{(1)}_{eff}}+ + A_1-\sqrt{D}\Bigg)^{\frac{1}{2}}, \quad (29)$$

$$\tilde{\omega}_2 = \frac{1}{\sqrt{2}}\Bigg(\omega^2_{eff}+(\omega^{(1)}_{eff})^2+\frac{2k}{m_{eff}}+\frac{4k}{m^{(1)}_{eff}}+ + A_1+\sqrt{D}\Bigg)^{\frac{1}{2}}, \quad (30)$$

$$\tilde{\omega}_3 = \left(\omega^2_{eff}+\frac{6k}{m_{eff}}\right)^{\frac{1}{2}}\left(1+km_{eff}\langle\theta^2\rangle\right)^{\frac{1}{2}}, \quad (31)$$

where

$$D = \left(\omega^2_{eff}-(\omega^{(1)}_{eff})^2+\frac{4k}{m_{eff}}-\frac{4k}{m^{(1)}_{eff}}+A_2\right)^2+ + \left(\frac{2k}{m}+A_3\right)\Bigg(2(\omega^{(1)}_{eff})^2-2\omega^2_{eff}-\frac{6k}{m}+\frac{8k}{m^{(1)}_{eff}}+ + 8\left(\frac{2k}{m}+A_4\right)\left(\frac{2k}{m_1}+A_5\right)\left(\frac{2k}{m}+A_3\right)^{-1}+A_6\Bigg), \quad (32)$$

$$A_1 = \left(\frac{km_{eff}\omega^2_{eff}}{3}+\frac{2k^2}{3}\right)\langle\theta^2\rangle+ + \left(\frac{2km^{(1)}_{eff}(\omega^{(1)}_{eff})^2}{3}+\frac{8k^2}{3}\right)\langle(\theta^{(1)})^2\rangle+\frac{8k^2}{3}\langle\theta\theta^{(1)}\rangle, \quad (33)$$

$$A_2 = \left(\frac{2km_{eff}\omega^2_{eff}}{3}+\frac{10k^2}{3}\right)\langle\theta^2\rangle- + \left(\frac{2km^{(1)}_{eff}(\omega^{(1)}_{eff})^2}{3}+\frac{8k^2}{3}\right)\langle(\theta^{(1)})^2\rangle-\frac{2k^2}{3}\langle\theta\theta^{(1)}\rangle, \quad (34)$$

$$A_3 = \left(\frac{8k^2}{3}+\frac{km_{eff}\omega^2_{eff}}{3}\right)\langle\theta^2\rangle-\frac{2k^2}{3}\langle\theta\theta^{(1)}\rangle, \quad (35)$$

$$A_4 = \left(\frac{km^{(1)}_{eff}(\omega^{(1)}_{eff})^2}{3}+\frac{4k^2}{3}\right)\langle\theta\theta^{(1)}\rangle+\frac{2k^2}{3}\langle,\theta^2\rangle, \quad (36)$$

$$A_5 = \left(\frac{k m_{eff}(\omega^2_{eff})}{3} + \frac{2k^2}{3}\right)\langle\theta\theta^{(1)}\rangle + \frac{4k^2}{3}\langle(\theta^{(1)})^2\rangle, \quad (37)$$

$$A_6 = -\left(k m_{eff}\omega^2_{eff} + 4k^2\right)\langle\theta^2\rangle + \\ + \left(\frac{4k m^{(1)}_{eff}(\omega^{(1)}_{eff})^2}{3} + \frac{16k^2}{3}\right)\langle(\theta^{(1)})^2\rangle + \frac{2k^2}{3}\langle\theta\theta^{(1)}\rangle. \quad (38)$$

For convenience, we introduce the notations

$$m_{eff} = m^{(2)}_{eff} = m^{(3)}_{eff}, \omega_{eff} = \omega^{(2)}_{eff} = \omega^{(3)}_{eff}, \quad (39)$$

$$\theta = \theta^{(2)} = \theta^{(3)}. \quad (40)$$

Considering $m_1 = m$, $\omega_1 = \omega$ we can write

$$\tilde{\omega}_1 = \omega_{eff}, \quad (41)$$

$$\tilde{\omega}_2 = \tilde{\omega}_3 = \\ = \left(\omega^2_{eff} + \frac{6k}{m_{eff}} + k\langle\theta^2\rangle m_{eff}\omega^2_{eff} + 6k^2\langle\theta^2\rangle\right)^{\frac{1}{2}}. \quad (42)$$

If $\omega_n = 0$ in the Hamiltonian (26) , the spectrum is given by (28) with (29), (30), (31) and $m^{(1)}_{eff} = m_1, m_{eff} = m$,

$$\omega^{(1)}_{eff} = \frac{\sqrt{\langle(\eta^1)^2\rangle}}{\sqrt{6m_1^2}}, \quad (43)$$

$$\omega_{eff} = \frac{\sqrt{\langle(\eta)^2\rangle}}{\sqrt{6m^2}}. \quad (44)$$

It is worth mentioning that the spectrum of the center-of-mass of the system is discrete. It has the form of the spectrum of a harmonic oscillator with the frequency $\tilde{\omega}_1$ (29).

If we consider algebra with commutation relations (9), (10) and commutative momenta $[P^{(n)}_i, P^{(m)}_j] = 0)$, the spectrum of a system (26) with $\omega_n = 0$ reads (28), where $\tilde{\omega}_i$ are given by

$$\tilde{\omega}_1 = 0, \quad (45)$$

$$\tilde{\omega}_2 = \frac{1}{\sqrt{2}}\left(\frac{2k}{m} + \frac{4k}{m^{(1)}} + \frac{2k^2}{3}\langle\theta^2\rangle + \right. \\ \left. + \frac{8k^2}{3}\langle(\theta^{(1)})^2\rangle + \frac{8k^2}{3}\langle\theta\theta^{(1)}\rangle + \sqrt{D}\right)^{\frac{1}{2}}, \quad (46)$$

$$\tilde{\omega}_3 = \left(\frac{6k}{m} + 6k^2\langle\theta^2\rangle\right)^{\frac{1}{2}}. \quad (47)$$

Here we have

$$D = \left(\frac{4k}{m} - \frac{4k}{m^{(1)}} + \right. \\ \left. + \frac{10k^2}{3}\langle\theta^2\rangle - \frac{8k^2}{3}\langle(\theta^{(1)})^2\rangle - \frac{2k^2}{3}\langle\theta\theta^{(1)}\rangle\right)^2 + \\ + \left(\frac{2k}{m} + \frac{8k^2}{3}\langle\theta^2\rangle - \frac{2k^2}{3}\langle\theta\theta^{(1)}\rangle\right) \times \\ \times \left(-\frac{6k}{m} + \frac{8k}{m^{(1)}} + 8\left(\frac{2k}{m} + \frac{4k^2}{3}\langle\theta\theta^{(1)}\rangle + \frac{2k^2}{3}\langle\theta^2\rangle\right) \times \right. \\ \times \left(\frac{2k}{m_1} + \frac{2k^2}{3}\langle\theta\theta^{(1)}\rangle + \frac{4k^2}{3}\langle(\theta^{(1)})^2\rangle\right) \times \\ \times \left(\frac{2k}{m} + \frac{8k^2}{3}\langle\theta^2\rangle - \frac{2k^2}{3}\langle\theta\theta^{(1)}\rangle\right)^{-1} - \\ \left. + 4k^2\langle\theta^2\rangle + \frac{16k^2}{3}\langle(\theta^{(1)})^2\rangle + \frac{2k^2}{3}\langle\theta\theta^{(1)}\rangle\right). \quad (48)$$

It is worth mentioning that noncommutativity of coordinates does not affect the spectrum of the center-of-mass of the system (45). Space quantization affects the frequencies of the relative motion (46), (47).

4. Harmonic oscillator chain in a noncommutative phase space with preserved rotational symmetry

Let us study the Hamiltonian as follows

$$H_s = \sum_{n=1}^{N}\frac{(\mathbf{P}^{(n)})^2}{2m} + \sum_{n=1}^{N}\frac{m\omega^2(\mathbf{X}^{(n)})^2}{2} + \\ + k\sum_{n=1}^{N}(\mathbf{X}^{(n+1)} - \mathbf{X}^{(n)})^2 \quad (49)$$

with periodic boundary conditions $\mathbf{X}^{(N+1)} = \mathbf{X}^{(1)}$, k is a constant. The Hamiltonian corresponds to the N interacting harmonic oscillator chain, m are the masses of oscillators and ω are frequencies.

The Hamiltonian H_s can be represented as

$$
\begin{aligned}
H_s &= \sum_{n=1}^{N}\left(\frac{(\mathbf{p}^{(n)})^2}{2m}+\frac{m\omega^2(\mathbf{x}^{(n)})^2}{2}+\right.\\
&+ k(\mathbf{x}^{(n+1)}-\mathbf{x}^{(n)})^2-\frac{(\boldsymbol{\eta}\cdot[\mathbf{x}^{(n)}\times\mathbf{p}^{(n)}])}{2m}-\\
&+ \frac{m\omega^2(\boldsymbol{\theta}\cdot[\mathbf{x}^{(n)}\times\mathbf{p}^{(n)}])}{2}-\\
&+ k(\boldsymbol{\theta}\cdot[(\mathbf{x}^{(n+1)}-\mathbf{x}^{(n)})\times(\mathbf{p}^{(n+1)}-\mathbf{p}^{(n)})])+\\
&+ \frac{[\boldsymbol{\eta}\times\mathbf{x}^{(n)}]^2}{8m}+\frac{m\omega^2}{8}[\boldsymbol{\theta}\times\mathbf{p}^{(n)}]^2+\\
&+ \left.\frac{k}{4}[\boldsymbol{\theta}\times(\mathbf{p}^{(n+1)}-\mathbf{p}^{(n)})]^2\right). \qquad (50)
\end{aligned}
$$

Also, for the harmonic oscillator chain we can write

$$
\begin{aligned}
\Delta H &= \sum_{n=1}^{N}\left(\frac{[\boldsymbol{\eta}\times\mathbf{x}^{(n)}]^2}{8m}+\frac{m\omega^2}{8}[\boldsymbol{\theta}\times\mathbf{p}^{(n)}]^2-\right.\\
&+ \frac{m\omega^2(\boldsymbol{\theta}\cdot[\mathbf{x}^{(n)}\times\mathbf{p}^{(n)}])}{2}-\frac{(\boldsymbol{\eta}\cdot[\mathbf{x}^{(n)}\times\mathbf{p}^{(n)}])}{2m}-\\
&+ k\boldsymbol{\theta}\cdot[(\mathbf{x}^{(n+1)}-\mathbf{x}^{(n)})\times(\mathbf{p}^{(n+1)}-\mathbf{p}^{(n+1)})]+\\
&+ \frac{k}{4}[\boldsymbol{\theta}\times(\mathbf{p}^{(n+1)}-\mathbf{p}^{(n)})]^2-\frac{\langle\eta^2\rangle(\mathbf{x}^{(n)})^2}{12m}-\\
&+ \left.\frac{\langle\theta^2\rangle m\omega^2(\mathbf{p}^{(n)})^2}{12}-\frac{k}{6}\langle\theta^2\rangle(\mathbf{p}^{(n+1)}-\mathbf{p}^{(n)})^2\right). \qquad (51)
\end{aligned}
$$

So, up to the second order in the parameters of non-commutativity one can study the Hamiltonian H_0 as follows

$$
\begin{aligned}
H_0 &= \sum_{n=1}^{N}\left(\frac{(\mathbf{p}^{(n)})^2}{2m_{eff}}+\frac{m_{eff}\omega_{eff}^2(\mathbf{x}^{(n)})^2}{2}+\right.\\
&+ k(\mathbf{x}^{(n+1)}-\mathbf{x}^{(n)})^2+\\
&+ \left.\frac{k}{6}\langle\theta^2\rangle(\mathbf{p}^{(n+1)}-\mathbf{p}^{(n)})^2+H^a_{osc}+H^b_{osc}\right), \qquad (52)
\end{aligned}
$$

where

$$
m_{eff} = m\left(1+\frac{m^2\omega^2\langle\theta^2\rangle}{6}\right)^{-1}, \qquad (53)
$$

$$
\omega_{eff} = \left(\omega^2+\frac{\langle\eta^2\rangle}{6m^2}\right)^{\frac{1}{2}}\left(1+\frac{m^2\omega^2\langle\theta^2\rangle}{6}\right)^{\frac{1}{2}}. \qquad (54)
$$

Note that $[H^a_{osc}+H^b_{osc},H_0]=0$. Coordinates and momenta $\mathbf{x}^{(n)}$, $\mathbf{p}^{(n)}$ satisfy the ordinary commutation relations. It is convenient to rewrite the Hamiltonian as follows

$$
\begin{aligned}
H_0 &= \frac{\hbar\omega_{eff}}{2}\sum_n\left(1+\frac{4km_{eff}\langle\theta^2\rangle}{3}\sin^2\frac{\pi n}{N}\right)\tilde{\mathbf{p}}^{(n)}(\tilde{\mathbf{p}}^{(n)})^{\dagger}+\\
&+ \frac{\hbar\omega_{eff}^2}{2}\sum_n\left(1+\frac{8k}{m_{eff}\omega_{eff}^2}\sin^2\frac{\pi n}{N}\right)\tilde{\mathbf{x}}^{(n)}(\tilde{\mathbf{x}}^{(n)})^{\dagger}, \qquad (55)
\end{aligned}
$$

where

$$
\mathbf{x}^{(n)} = \sqrt{\frac{\hbar}{Nm_{eff}\omega_{eff}}}\sum_{l=1}^{N}\exp\left(\frac{2\pi inl}{N}\right)\tilde{\mathbf{x}}^{(l)}, \qquad (56)
$$

$$
\mathbf{p}^{(n)} = \sqrt{\frac{\hbar m_{eff}\omega_{eff}}{N}}\sum_{l=1}^{N}\exp\left(-\frac{2\pi inl}{N}\right)\tilde{\mathbf{p}}^{(l)}, \qquad (57)
$$

(see, for example, [32]). Introducing

$$
a_j^{(n)} = \frac{1}{\sqrt{2w_n}}\left(w_n\tilde{x}_j^{(n)}+i\tilde{p}_j^{(n)}\right), \qquad (58)
$$

$$
\begin{aligned}
w_n &= \left(1+\frac{8k}{m_{eff}\omega_{eff}^2}\sin^2\frac{\pi n}{N}\right)^{\frac{1}{2}}\times\\
&\times \left(1+\frac{4km_{eff}\langle\theta^2\rangle}{3}\sin^2\frac{\pi n}{N}\right)^{-\frac{1}{2}}, \qquad (59)
\end{aligned}
$$

we obtain

$$
\begin{aligned}
H_0 &= \hbar\omega_{eff}\sum_{n=1}^{N}\sum_{j=1}^{3}\left(1+\frac{4km_{eff}\langle\theta^2\rangle}{3}\sin^2\frac{\pi n}{N}\right)^{\frac{1}{2}}\times\\
&\times \left(1+\frac{8k}{m_{eff}\omega_{eff}^2}\sin^2\frac{\pi n}{N}\right)^{\frac{1}{2}}\left((a_j^{(n)})^{\dagger}a_j^{(n)}+\frac{1}{2}\right). \qquad (60)
\end{aligned}
$$

So, the energy levels of H_0 are given by

$$
\begin{aligned}
E_{\{n_1\},\{n_2\},\{n_3\}} &= \hbar\sum_{a=1}^{N}\left(\omega_{eff}^2+\frac{8k}{m_{eff}}\sin^2\frac{\pi a}{N}\right)^{\frac{1}{2}}\times\\
&\times \left(1+\frac{4km_{eff}\langle\theta^2\rangle}{3}\sin^2\frac{\pi a}{N}\right)^{\frac{1}{2}}\times\\
&\times \left(n_1^{(a)}+n_2^{(a)}+n_3^{(a)}+\frac{3}{2}\right)=\\
&= \sum_{a=1}^{N}\hbar\omega_a\left(n_1^{(a)}+n_2^{(a)}+n_3^{(a)}+\frac{3}{2}\right). \qquad (61)
\end{aligned}
$$

Here $n_i^{(a)}$ are quantum numbers ($n_i^{(a)}=0,1,2...$). Using (53), (54), we have the following expressions for the frequencies

$$\omega_a^2 = \left(\omega^2+\frac{\langle\eta^2\rangle}{6m^2}\right)\left(1+\frac{m^2\omega^2\langle\theta^2\rangle}{6}+ + \frac{4k^2m\langle\theta^2\rangle}{3}\sin^2\frac{\pi a}{N}\right)+\frac{8k}{m}\sin^2\frac{\pi a}{N}+ + \frac{32k^2\langle\theta^2\rangle}{3}\sin^4\frac{\pi a}{N}. \quad (62)$$

Let us also study a particular case of $\omega = 0$. So, up to the second order in the parameters of noncommutativity for a system of particles with harmonic oscillator interaction we have

$$E_{\{n_1\},\{n_2\},\{n_3\}} = = \sum_{a=1}^{N}\hbar\omega_a\left(n_1^{(a)}+n_2^{(a)}+n_3^{(a)}+\frac{3}{2}\right), \quad (63)$$

where

$$\omega_a^2 = \frac{8k}{m}\sin^2\frac{\pi a}{N}+\frac{\langle\eta^2\rangle}{6m^2}+\frac{32k^2\langle\theta^2\rangle}{3}\sin^4\frac{\pi a}{N}. \quad (64)$$

If momenta commutes $\eta_{ij} = 0$ the spectrum of a chain of particles with harmonic oscillator interaction in a space with noncommutativity of coordinates has the form (63) with frequencies

$$\omega_a^2 = \frac{8k}{m}\sin^2\frac{\pi a}{N}+\frac{32k^2\langle\theta^2\rangle}{3}\sin^4\frac{\pi a}{N}. \quad (65)$$

From (63), (64) we have that the spectrum of the center-of-mass of the system is the spectrum of the harmonic oscillator with the frequency

$$\omega_N^2 = \frac{\langle\eta^2\rangle}{6m^2}. \quad (66)$$

Note that in the limit $\langle\theta^2\rangle \to 0$, $\langle\eta^2\rangle \to 0$ on the basis of (62) we have

$$\omega_a^2 = \omega^2+\frac{8k}{m}\sin^2\frac{\pi a}{N}. \quad (67)$$

which is a well-known result in ordinary space.

5. Conclusions

We have analyzed the energy levels of a system consisting of N harmonic oscillators interacting through harmonic oscillator potentials in a uniform field, within a rotationally-invariant noncommutative phase space of the canonical type.

In the second-order approximation in the parameters of noncommutativity, we determined the influence of noncommutativity on the system's energy levels. Our findings indicate that the space quantization affects the frequencies of the system. The particular case of a system of two interacting oscillators and a system of three interacting oscillators have been examined. We have found the energy levels of the systems in a rotationally-invariant noncommutative phase space (19), (28).

Additionally, a harmonic oscillator chain has been studied. We have determined that noncommutativity of coordinates and noncommutativity of momenta do not change the form of the spectrum of the system (61). The frequencies of the system are affected by space quantization as (62). In the particular case of a system of particles with harmonic oscillator interaction which corresponds to $\omega = \mathbf{0}$, we have analyzed the effect of noncommutativity on the energy levels of the system.

We have found that because of momentum noncommutativity the spectrum of the center-of-mass of the system corresponds to the the spectrum of a harmonic oscillator. The frequency of the oscillator depends on the parameter of momentum noncommutativity as it is given by (66).

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