TASK Quarterly 27 (3) 2023

Systems of free particles and harmonic oscillators in rotationally-invariant noncommutative phase space

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https://doi.org/10.34808/rjv5-9575

Abstract

In this paper, we introduce a rotationally-invariant noncommutative algebra that is equivalent to the canonical type. This algebra is built by extending the noncommutativity parameters to tensors. These tensors are defined with the help of additional coordinates and momenta corresponding to a rotationally-invariant system. In the frame of the rotationally-invariant noncommutative algebra we investigate a system of free particles, and systems of harmonic oscillators. The energy levels of these systems are found in noncommutative phase spave with preserved rotational symmetry.

Keywords:

noncommutative phase space, systems of harmonic oscillators, miminal length

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1. Introduction

One of the important predictions of the String Theory and the Quantum Gravity is the existence of the minimal length which is of the order of the Planck length (see, for instance, [1, 2, 3, 4, 5, 6, 7]). This feature of space structure can be described with modifications of the ordinary commutation relations for operators of coordinates and operators of momenta.

The first article in which the idea that coordinates may not commute was published by Snyder [8]. Before Snyder the idea was suggested by Heisenberg. The scientist proposed such a modification to solve the problem of ultraviolet divergences in quantum field theory.

Many different modifications of the commutation relations were proposed to describe features of space structure on the Planck scale. The most simple and well-known is algebra with noncommutativity of coordinates of canonical type. The algebra is characterized by the modification of commutation relation for operators of coordinates. It reads

$$
\begin{aligned}
[X_i,X_j] &= i\hbar\theta_{ij}, &(1)\\
[X_i,P_j] &= i\hbar\delta_{ij}, &(2)\\
[P_i,P_j] &= 0, &(3)
\end{aligned}
$$

where θ_{ij} are parameters of coordinates noncommutativity which are elements of the constant antisymmetric matrix. The algebra describes a space with minimal length. Note, that the noncommutativity of coordinates can be used to describe motion of a particle in a strong magnetic field (see, for instance, [9, 10, 11, 12, 13]). Various physical problems have been examined in the frame of noncommutative algebra of canonical type. Among the first papers on the subject it is worth mention [14, 15, 16, 17, 18].

It is important to note that in $2D$ case the noncommutative algebra of canonical type is rotationally invariant

$$
\begin{aligned}
[X_1,X_2] &= -[X_2,X_1] = i\hbar\theta, &(4)\\
[X_i,P_j] &= i\hbar\delta_{ij}, &(5)\\
[P_i,P_j] &= 0, &(6)
\end{aligned}
$$

where $i,j=(1,2)$, θ is a parameter of noncommutativity. But in 3D case of noncommutative algebra (1)-(3) one faces a problem of rotational symmetry breaking [19, 20].

It is evident that the same problem appears in more general case when the noncommutativity of momenta is also considered. The noncommutative phase space of canonical type is characterized by the following commutation relations for coordinates and momenta

$$
\begin{aligned}
[X_i,X_j] &= i\hbar\theta_{ij}, &(7)\\
[X_i,P_j] &= i\hbar(\delta_{ij}+\gamma_{ij}), &(8)\\
[P_i,P_j] &= i\hbar\eta_{ij}. &(9)
\end{aligned}
$$

Here θ_{ij}, η_{ij}, γ_{ij} are parameters of the algebra which in the case of noncommutative algebra of canonical type are considered to be elements of constant matrixes, θ_{ij} are parameters of coordinate noncommutativity, η_{ij} are parameters of momentum noncommutativity.

There are different ways of representation of the coordinates X_i and momenta P_i which do not commute (7), (9) Symmetrical representation is well known. It reads

$$
\begin{aligned}
X_i &= x_i-\frac{1}{2}\sum_j\theta_{ij}p_j, &(10)\\
P_i &= p_i+\frac{1}{2}\sum_j\eta_{ij}x_j, &(11)
\end{aligned}
$$

here x_i, p_i are coordinates and momenta that satisfy the ordinary algebra. We have

$$
\begin{aligned}
[x_i,x_j] &= 0, &(12)\\
[x_i,p_j] &= i\hbar\delta_{ij}, &(13)\\
[p_i,p_j] &= 0. &(14)
\end{aligned}
$$

On the basis of expressions (10), (11), one find [21]

$$
[X_i,P_j] = i\hbar\delta_{ij} + i\hbar\sum_k\frac{\theta_{ik}\eta_{jk}}{4}. \tag{15}
$$

So, from the symmetrical representation follows that parameters γ_{ij} read

$$
\gamma_{ij} = \sum_k\frac{\theta_{ik}\eta_{jk}}{4}. \tag{16}
$$

New classes of noncommutative algebras were developed to preserve the rotational symmetry. In paper, [22] the idea of foliating the space with concentric fuzzy spheres was proposed to preserve the rotational symmetry. Rotationally-invariant noncommutative space was constructed as a sequence of fuzzy spheres in [23]. Author of paper [24] introduced the curved noncommutative space. In [25] promotion of the parameter of noncommutativity to an operator in Hilbert space was implemented to construct rotationally-invariant noncommutative algebra. Rotation invariance in N dimensional case was studied in [26].

To find new effects of noncommutativity of coordinates and noncommutativity of momenta in the properties of a wide class of physical systems it is important to examine many-particle systems Studies of harmonic oscillator in noncommutative space have received much attention (see, for instance, [27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40]). Two coupled harmonic oscillators were studied in noncommutative space [41, 42], noncommutative phase space [43, 44]. System of free particles was examined in [45, 46] in noncommutative phase space of canonical type. Classical problems of many particles were examined in [47] in the case of space-time noncommutativity.

It is worth noting that system of harmonic oscillators has various applications. Such studies have importance in nuclei physics [48, 49, 50], in quantum chemistry and molecular spectroscopy [51, 52, 53, 54]. Also networks of harmonic oscillators are used in quantum information [55, 56, 57].

In the present paper, we present noncommutative algebra which is rotationally-invariant and besides it is equivalent to noncommutative algebra of canonical type. The algebra is constructed with the help of generalization of the parameters of noncommutativity to tensors. The tensors are defined by introducing additional coordinates and additional momenta governed by a system with rotational symmetry. The basis problems are studied in the frame of the algebra. They are free particle, harmonic oscillator, systems of harmonic oscillators. The spectrum of a system of harmonic oscillators is obtained up to the second order in the parameters of coordinate and momentum noncommutativity. Results presented in this paper are published in [40, 58, 59, 60].

The paper is organized as follows. In section 2 rotationally-invariant algebra with noncommutativity of coordinates and noncommutativity of momenta which is equivalent to algebra of canonical type is introduced. The spectrum of free particle in rotationally-invariant noncommutative phase space is examined in section 3. The harmonic oscillator in noncommutative phase space with preserved rotational symmetry is studied in section 4. In section 5 Hamiltonian of a system of interacting harmonic oscillators is analyzed in noncommutative phase space. Section 6 is devoted to studies of the energy levels of a system of interacting harmonic oscillators in uniform field in the frame of noncommutative algebra. Conclusions are presented in section 7.

2. Noncommutative phase space of canonical type with preserved rotational symmetry

To construct an algebra which is rotationally-invariant and describes a noncommutative phase space we propose to generalize parameters of noncommutativity θ_{ij}, η_{ij} to tensors. The tensors are considered to be constructed with the help of additional coordinates and additional momenta. Tensors of coordinate noncommutativity read

$$\theta_{ij} = \frac{l_0}{\hbar}\varepsilon_{ijk}a_k. \qquad (17)$$

Hare l_0 is a constant with the dimension of length and a_i are additional coordinates. For tensors of momentum noncommutativity we have the following expression

$$\eta_{ij} = \frac{p_0}{\hbar}\varepsilon_{ijk}p^b_k, \qquad (18)$$

here p_0 is a constant, p^b_k are additional momenta.

We consider the additional coordinates a_i, b_i and momenta p^a_i, p^b_i to satisfy the ordinary commutation relations. Namely, we have

$$[a_i,a_j] = [b_i,b_j] = [a_i,b_j] = 0, \qquad (19)$$

$$[a_i,p^a_j] = [b_i,p^b_j] = i\hbar\delta_{ij}, \qquad (20)$$

$$[p^a_i,p^a_j] = [p^b_i,p^b_j] = [p^a_i,p^b_j] = 0, \qquad (21)$$

$$[a_i,p^b_j] = [b_i,p^a_j] = 0. \qquad (22)$$

To preserve the rotational symmetry the additional coordinates and additional momenta are assumed to be governed by spherically-symmetric systems. For simplicity they are considered to be harmonic oscillators

$$H^a_{osc} = \frac{(p^a)^2}{2m_{osc}} + \frac{m_{osc}\omega^2 a^2}{2}, \qquad (23)$$

$$H^b_{osc} = \frac{(p^b)^2}{2m_{osc}} + \frac{m_{osc}\omega^2 b^2}{2}. \qquad (24)$$

Parameters of the oscillators are assumed to be as follows

$$\sqrt{\frac{\hbar}{m_{osc}\omega}} = l_P, \qquad (25)$$

where l_P is the Planck's length. We also consider the frequency ω to be very large. In this case because of large distance between energy levels $\hbar\omega$ the oscillators are in the ground state.

Taking into account (16), (17), (18), we can write

$$\gamma_{ij} = \frac{l_0 p_0}{4\hbar^2}\left((\mathbf{a}\cdot\mathbf{p^b})\delta_{ij} - a_j p^b_i\right). \qquad (26)$$

As a result, the noncommutative algebra is characterized by the following relations

$$[X_i,X_j] = i\varepsilon_{ijk}l_0 a_k, \qquad (27)$$

$$[X_i,P_j] = i\hbar\left(\delta_{ij} + \frac{l_0 p_0}{4\hbar^2}(\mathbf{a}\cdot\mathbf{p^b})\delta_{ij} - \frac{l_0 p_0}{4\hbar^2}a_j p^b_i\right), \qquad (28)$$

$$[P_i,P_j] = \varepsilon_{ijk}p_0 p^b_k. \qquad (29)$$

Additional coordinates a_i, b_i can be treated as some internal coordinates of a particle. Quantum fluctuations of a_i, b_i lead effectively to a non-point-like particle. The size of the particle is of the order of the Planck scale.

It is important to note that γ_{ij}, θ_{ij}, η_{ij} commute with each other

$$[\theta_{ij},\eta_{ij}]=[\theta_{ij},\gamma_{ij}]=[\gamma_{ij},\eta_{ij}]=0. \tag{30}$$

Also, we have that the following commutation relations are satisfied

$$\begin{aligned}[\theta_{ij},X_k] &= [\theta_{ij},P_k]=[\eta_{ij},X_k]= \\ &= [\eta_{ij},P_k]=[\gamma_{ij},X_k]=[\gamma_{ij},P_k]=0. \end{aligned} \tag{31}$$

So, similarly as in the case when θ_{ij}, η_{ij}, and γ_{ij} are constants, tensors θ_{ij}, η_{ij} and γ_{ij} commute with coordinates and momenta. In this sense we have that the constructed algebra (27)-(29) is equivalent to noncommutative algebra of canonical type (7)-(9).

Commutation relations of algebra (27)-(29) remain the same after rotation

$$[X_i',X_j']=i\varepsilon_{ijk}l_0a_k', \tag{32}$$

$$[X_i',P_j']=i\hbar\left(\delta_{ij}+\frac{l_0p_0}{4\hbar^2}(\mathbf{a}'\cdot\mathbf{p}^{\mathbf{b}\prime})\delta_{ij}-\frac{l_0p_0}{4\hbar^2}a_j'p_i^{b\prime}\right), \tag{33}$$

$$[P_i',P_j']=\varepsilon_{ijk}p_0p_k^{b\prime}. \tag{34}$$

Here we use the following notations

$$\begin{aligned} X_i' &= U(\varphi)X_iU^+(\varphi), &(35)\\ P_i' &= U(\varphi)P_iU^+(\varphi), &(36)\\ a_i' &= U(\varphi)a_iU^+(\varphi), &(37)\\ p_i^{b\prime} &= U(\varphi)p_i^bU^+(\varphi). &(38)\end{aligned}$$

The rotation operator reads

$$U(\varphi)=e^{\frac{i}{\hbar}\varphi(\mathbf{n}\cdot\tilde{\mathbf{L}})}, \tag{39}$$

where $\tilde{\mathbf{L}}$ is the total angular momentum defined as

$$\tilde{\mathbf{L}}=[\mathbf{r}\times\mathbf{p}]+[\mathbf{a}\times\mathbf{p}^a]+[\mathbf{b}\times\mathbf{p}^b], \tag{40}$$

$\mathbf{r}=(x_1,x_2,x_3)$. It is easy to show that $\tilde{\mathbf{L}}$ satisfies the following relations

$$\begin{aligned}[\tilde{L}_i,(\mathbf{a}\cdot\mathbf{p})] &= [\tilde{L}_i,(\mathbf{b}\cdot\mathbf{p})]= \\ &= [\tilde{L}_i,(\mathbf{a}\cdot\mathbf{b})]=0, &(41)\\ [\tilde{L}_i,(\mathbf{r}\cdot\mathbf{a})] &= [\tilde{L}_i,(\mathbf{r}\cdot\mathbf{b})]=0, &(42)\\ [\tilde{L}_i,(\mathbf{a}\cdot\mathbf{L})] &= [\tilde{L}_i,(\mathbf{b}\cdot\mathbf{L})]= \\ &= [\tilde{L}_i,(\mathbf{p^a}\cdot\mathbf{L})]= \\ &= [\tilde{L}_i,(\mathbf{p^b}\cdot\mathbf{L})]=0, &(43)\\ [\tilde{L}_i,r^2] &= [\tilde{L}_i,p^2]=[\tilde{L}_i,a^2]= \\ &= [\tilde{L}_i,b^2]=0, &(44)\\ [\tilde{L}_i,(p^a)^2] &= [\tilde{L}_i,(p^b)^2]=0. &(45)\end{aligned}$$

Here for convenience we introduce notation $L=[\mathbf{r}\times\mathbf{p}]$. So, taking these relations into account we have that

$$[\tilde{L}_i,R]=0, \tag{46}$$

where R is the operator of distance. This operator on the basis of (10), (11), (17), (18) can be rewritten as

$$R=\sqrt{\sum_i X_i^2}=$$

$$=\sqrt{r^2+\frac{l_0^2}{4\hbar^2}a^2p^2-\frac{l_0}{4\hbar^2}(\mathbf{a}\cdot\mathbf{p})^2-\frac{l_0}{\hbar}(\mathbf{a}\cdot\mathbf{L})}. \tag{47}$$

So, after rotation, we obtain the same distance

$$R'=U(\varphi)RU^+(\varphi)=R. \tag{48}$$

Also, the operator of the total angular momentum commutes with momentum $P=\sqrt{\sum_i P_i^2}$. We have

$$[\tilde{L}_i,P]=0, \tag{49}$$

$$P=\sqrt{p^2+\frac{p_0^2}{4\hbar^2}r^2(p^b)^2-\frac{p_0^2}{4\hbar^2}(\mathbf{r}\cdot\mathbf{p^b})^2+\frac{p_0}{\hbar}(\mathbf{p^b}\cdot\mathbf{L})}. \tag{50}$$

So, the absolute value of momentum does not change after rotation

$$P'=U(\varphi)PU^+(\varphi)=P. \tag{51}$$

Commutators for coordinates and total angular momentum are the same as in the ordinary space (space with ordinary commutation relations for operators of coordinates and momenta)

$$\begin{aligned}[X_i,\tilde{L}_j] &= i\hbar\varepsilon_{ijk}X_k, &(52)\\ [P_i,\tilde{L}_j] &= i\hbar\varepsilon_{ijk}P_k, &(53)\\ [a_i,\tilde{L}_j] &= i\hbar\varepsilon_{ijk}a_k, &(54)\\ [p_i^a,\tilde{L}_j] &= i\hbar\varepsilon_{ijk}p_k^a, &(55)\\ [b_i,\tilde{L}_j] &= i\hbar\varepsilon_{ijk}b_k, &(56)\\ [p_i^b,\tilde{L}_j] &= i\hbar\varepsilon_{ijk}p_k^b. &(57)\end{aligned}$$

Using (10), (11), (17), (18), for noncommutative coordinates and noncommutative momenta we have the following representation

$$X_i = x_i + \frac{l_0}{2\hbar}[\mathbf{a}\times\mathbf{p}]_i, \quad (58)$$
$$P_i = p_i - \frac{p_0}{2\hbar}[\mathbf{r}\times\mathbf{p^b}]_i. \quad (59)$$

The existence of such a representation guarantees that the Jacobi identity is satisfied for all possible triplets of operators. Also, it is important to note that from (58), (59) follows the following relations

$$[X_i, p^a_j] = i\varepsilon_{ijk}\frac{l_0}{2}p_k, \quad (60)$$
$$[P_i, b_j] = i\varepsilon_{ijk}\frac{l_0}{2}x_k, \quad (61)$$
$$[X_i, a_j] = [X_i, b_j] = [X_i, p^b_j] = 0, \quad (62)$$
$$[P_i, a_j] = [P_i, p^a_j] = [P_i, p^b_j] = 0. \quad (63)$$

3. Free particle in rotationally-invariant quantum phase space

Let us consider a free particle of mass m

$$H_p = \sum_i \frac{P_i^2}{2m}, \quad (64)$$

and study its energy levels in the frame of rotationally-invariant noncommutative algebra (27)-(29). So, momenta in the Hamiltonian do not commute, we have (29).

To construct algebra (27)-(29) we involve additional coordinates and additional momenta $\tilde{a}_i$, $\tilde{b}_i$, $\tilde{p}^a_i$, $\tilde{p}^b_i$, So, to find energy levels of free particle in noncommutative phase space we have to consider the total Hamiltonian as follows

$$H = \sum_i \frac{P_i^2}{2m} + H^a_{osc} + H^b_{osc}. \quad (65)$$

Here H^a_{osc}, H^b_{osc} are Hamiltonians of harmonic oscillators, that are given by (23), (24). For convenience, we introduce the following operator

$$\Delta H = H_p - \langle H_p\rangle_{ab}. \quad (66)$$

Here $\langle\ldots\rangle_{ab}$ denotes averaging over the eigenstates of oscillators (23), (24) in the ground states $\psi^a_{0,0,0}$, $\psi^b_{0,0,0}$.

$$\langle\ldots\rangle_{ab} = \langle\psi^a_{0,0,0}\psi^b_{0,0,0}|\ldots|\psi^a_{0,0,0}\psi^b_{0,0,0}\rangle. \quad (67)$$

So, we can rewrite Hamiltonian (64) as follows

$$H = H_0 + \Delta H, \quad (68)$$
$$H_0 = \langle H_p\rangle_{ab} + H^a_{osc} + H^b_{osc}. \quad (69)$$

Up to the second order in ΔH in the rotationally-invariant noncommutative phase space we can study (69). To show this we find corrections caused by the term ΔH to the energy levels of the total Hamiltonian

$$H = H_s + H^a_{osc} + H^b_{osc}, \quad (70)$$

here H_s is a Hamiltonian of a system. It is important that

$$[\langle H_s\rangle_{ab}, H^a_{osc} + H^b_{osc}] = 0. \quad (71)$$

So, the eigenfunctions and the eigenvalues of Hamiltonian H_0 read

$$\psi^{(0)}_{\{n_s\},\{0\},\{0\}} = \psi^s_{\{n_s\}}\psi^a_{0,0,0}\psi^b_{0,0,0}, \quad (72)$$
$$E^{(0)}_{\{n_s\}} = E^s_{\{n_s\}} + 3\hbar\omega_{osc}, \quad (73)$$

Here for convenience we introduce the following notations $\psi^s_{\{n_s\}}$ are eigenfunctions and $E^s_{\{n_s\}}$ and eigenvalues of $\langle H_s\rangle_{ab}$, $\{n_s\}$ are quantum numbers. In the first order of the perturbation theory, the correction reads

$$\Delta E^{(1)} = \langle\psi^s_{\{n_s\}}\psi^a_{0,0,0}\psi^b_{0,0,0}|\Delta H|\psi^s_{\{n_s\}}\psi^a_{0,0,0}\psi^b_{0,0,0}\rangle = = \langle\psi^s_{\{n_s\}}|\langle H_s\rangle_{ab} - \langle H_s\rangle_{ab}|\psi^s_{\{n_s\}}\rangle = 0. \quad (74)$$

Now, let us find corrections of the second order. We can write

$$\Delta E^{(2)} = \sum_{\{n'_s\},\{n^a\},\{n^b\}} \frac{\left|\left\langle \psi^{(0)}_{\{n'_s\},\{n^a\},\{n^b\}}|\Delta H|\psi^{(0)}_{\{n_s\},\{0\},\{0\}}\right\rangle\right|^2}{E^s_{\{n'_s\}} - E^s_{\{n_s\}} - \hbar\omega_{osc}(n^a_1+n^a_2+n^a_3+n^b_1+n^b_2+n^b_3)}. \quad (75)$$

It is important to mention that the set $\{n'_s\}$, $\{n^a\}$, $\{n^b\}$ does not coincide with $\{n_s\}$,$\{0\}$, $\{0\}$. So, in the denominator of all terms in the sum we have oscillator frequency ω_{osc}. Mean values

$$\left\langle \psi^{(0)}_{\{n'_s\},\{n^a\},\{n^b\}}|\Delta H|\psi^{(0)}_{\{n_s\},\{0\},\{0\}}\right\rangle, \quad (76)$$

do not depend on ω_{osc} This follows from the relation (25). In the limit $\omega_{osc}\to\infty$ the second order corrections are equal to zero

$$\lim_{\omega_{osc}\to\infty}\Delta E^{(2)} = 0. \quad (77)$$

This result will be used in our studies of energy levels of different physical systems in the monograph.

So, let us apply the result for finding energy levels of free particle in rotationally-invariant noncommutative phase space.

To find $\langle H_p\rangle_{ab}$ we use representation of noncommutative coordinates and noncommutative momenta by x_i, p_i satisfying the ordinary commutation relations

$$X_i = x_i - \sum_j \frac{1}{2}\theta_{ij}p_j = x_i + \frac{1}{2}[\theta \times \mathbf{p}]_i, \quad (78)$$

$$P_i = p_i + \sum_j \frac{1}{2}\eta_{ij}x_j = p_i - \frac{1}{2}[\eta \times \mathbf{x}]_i, \quad (79)$$

$$\theta_i = \sum_{jk} \varepsilon_{ijk}\frac{\theta_{jk}}{2} = \frac{c_\theta l_P^2}{\hbar}\tilde{a}_i, \quad (80)$$

$$\eta_i = \sum_{jk} \varepsilon_{ijk}\frac{\eta_{jk}}{2} = \frac{c_\eta \hbar}{l_P^2}\tilde{p}_i^b, \quad (81)$$

here $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{p} = (p_1, p_2, p_3)$. For convenience we introduce dimensionless constants c_θ, c_η and dimensionless coordinates and momenta

$$\tilde{a}_i = \frac{a_i}{l_P}, \quad \tilde{b}_i = \frac{b_i}{l_P}, \quad (82)$$

$$\tilde{p}_i^a = \frac{p_i^a l_P}{\hbar}, \quad \tilde{p}_i^b = \frac{p_i^a l_P}{\hbar}. \quad (83)$$

So, the Hamiltonian of a particle reads

$$H_p = \frac{p^2}{2m} - \frac{(\eta \cdot [\mathbf{x} \times \mathbf{p}])}{2m} + \frac{[\eta \times \mathbf{x}]^2}{8m}. \quad (84)$$

Note that H_p does not depend on the a_i, p_i^a. So, we have

$$\langle H_p\rangle_{ab} = \langle \psi_{0,0,0}^b | H_p | \psi_{0,0,0}^b\rangle. \quad (85)$$

It is easy to calculate

$$\langle \psi_{0,0,0}^b | \tilde{p}_i^b | \psi_{0,0,0}^b\rangle = 0, \quad (86)$$

$$\langle \psi_{0,0,0}^b | \tilde{p}_i^b \tilde{p}_j^b | \psi_{0,0,0}^b\rangle = \frac{1}{2}\delta_{ij}. \quad (87)$$

So, for $\langle \eta_i\rangle_{ab}$, and $\langle \eta^2\rangle_{ab}$ we obtain

$$\langle \eta_i\rangle_{ab} = 0, \quad (88)$$

$$\langle \eta^2\rangle = \langle \eta^2\rangle_{ab} = \frac{3(\hbar c_\eta)^2}{2l_P^4}. \quad (89)$$

Therefore after averaging H_p over the eigenfunctions of the harmonic oscillators we obtain

$$\langle H_p\rangle_{ab} = \frac{p^2}{2m} + \frac{\langle \eta^2\rangle x^2}{12m}. \quad (90)$$

On the basis of this result (90), we find

$$\Delta H = -\frac{(\eta \cdot [\mathbf{x} \times \mathbf{p}])}{2m} + \frac{[\eta \times \mathbf{x}]^2}{8m} - \frac{\langle \eta^2\rangle x^2}{12m}. \quad (91)$$

Hamiltonian $\langle H_p\rangle_{ab}$ corresponds to the Hamiltonian of harmonic oscillator with mass m and frequency

$$\omega = \sqrt{\frac{\langle \eta^2\rangle}{6m^2}}, \quad (92)$$

in the ordinary space (coordinates and momenta x_i, p_j satisfy the ordinary commutation relations).

Expression for ΔH contains terms of the first and second orders in the parameter of momentum noncommutativity. So, the energy levels of free particle in rotationally-invariant noncommutative phase space up to the second order in the parameter of momentum noncommutativity are as follows

$$E_{n_1,n_2,n_3} = \sqrt{\frac{\hbar^2\langle \eta^2\rangle}{6m^2}}\left(n_1 + n_2 + n_3 + \frac{3}{2}\right), \quad (93)$$

$n_1 = 0, 1, 2...$, $n_2 = 0, 1, 2...$, $n_3 = 0, 1, 2...$.

So, we can conclude that because of the noncommutativity of momenta, the energy levels of free particles are quantized. They correspond to the energy levels of a harmonic oscillator with frequency determined by the parameter of momentum noncommutativity and given by (93)

4. Harmonic oscillator in rotationally-invariant space with noncommutativity of coordinates and noncommutativity of momenta

We consider three-dimensional harmonic oscillator with mass m and frequency ω in the frame of noncommutative algebra (27)-(29)

$$H_{osc} = \sum_i \frac{P_i^2}{2m} + \sum_i \frac{m\omega^2 X_i^2}{2}. \quad (94)$$

Similarly, as in the previous section let us write the total Hamiltonian

$$H = H_0 + \Delta H, \quad (95)$$

$$H_0 = \langle H_{osc}\rangle_{ab} + H_{osc}^a + H_{osc}^b, \quad (96)$$

$$\Delta H = H_{osc} - \langle H_{osc}\rangle_{ab}. \quad (97)$$

To find $\langle H_{osc}\rangle_{ab}$ we use representation (78)-(79) and rewrite the Hamiltonian as follows

$$H_{osc} = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} - \frac{(\eta \cdot [\mathbf{x} \times \mathbf{p}])}{2m} - \frac{m\omega^2(\theta \cdot [\mathbf{x} \times \mathbf{p}])}{2} + \frac{[\eta \times \mathbf{x}]^2}{8m} + \frac{m\omega^2[\theta \times \mathbf{p}]^2}{8}, \quad (98)$$

Using (80), (81), (86), (87) we obtain

$$\langle \psi^a_{0,0,0}|\tilde{a}_i|\psi^a_{0,0,0}\rangle = 0, \tag{99}$$

$$\langle \psi^a_{0,0,0}|\tilde{a}_i\tilde{a}_j|\psi^a_{0,0,0}\rangle = \frac{1}{2}\delta_{ij}. \tag{100}$$

(101)

So, finally, we find

$$\langle H_{osc}\rangle_{ab} = \left(\frac{1}{2m}+\frac{m\omega^2\langle\theta^2\rangle}{12}\right)p^2 + \left(\frac{m\omega^2}{2}+\frac{\langle\eta^2\rangle}{12m}\right)x^2, \tag{102}$$

where we use the following notation

$$\langle\eta^2\rangle = \langle\eta^2\rangle_{ab} = \frac{3(\hbar c_\eta)^2}{2l_P^4}. \tag{103}$$

Note, that ΔH reads

$$\Delta H = -\frac{(\eta\cdot[\mathbf{x}\times\mathbf{p}])}{2m} - \frac{m\omega^2(\theta\cdot[\mathbf{x}\times\mathbf{p}])}{2} + \frac{[\eta\times\mathbf{x}]^2}{8m} + \frac{m\omega^2[\theta\times\mathbf{x}]^2}{8}\frac{m\omega^2\langle\theta^2\rangle}{12}p^2 - \frac{\langle\eta^2\rangle}{12m}x^2, \tag{104}$$

and it contains terms of the first and second orders in the parameters of noncommutativity. So, up to the second order in the parameters of coordinates and momentum noncommutativity, we obtain the following energy levels of the harmonic oscillator

$$E_{n_1,n_2,n_3} = \hbar\sqrt{\left(m\omega^2+\frac{\langle\eta^2\rangle}{6m}\right)\left(\frac{1}{m}+\frac{m\omega^2\langle\theta^2\rangle}{6}\right)}\left(n_1+n_2+n_3+\frac{3}{2}\right) \tag{105}$$

n_1, n_2, n_3 are quantum numbers, $n_1 = 0,1,2...$, $n_2 = 0,1,2...$, $n_3 = 0,1,2...$.

Note that we have correspondence of the spectrum of harmonic oscillators in the quantum space written up to the second order in the parameters of noncommutativity and spectrum of harmonic oscillator in the ordinary space. Noncommutativity affects the mass and the frequency of the oscillator and does not affect the form of its spectrum. From (102), we have

$$m_{eff} = \frac{6m}{6+m^2\omega^2\langle\theta^2\rangle}, \tag{106}$$

$$\omega_{eff} = \sqrt{\left(m\omega^2+\frac{\langle\eta^2\rangle}{6m}\right)\left(\frac{1}{m}+\frac{m\omega^2\langle\theta^2\rangle}{6}\right)}. \tag{107}$$

Note, that in the limits $\langle\theta^2\rangle\to 0$, $\langle\eta^2\rangle\to 0$ we obtain $m_{eff} = m$, $\omega_{eff} = \omega$. So, the limits expression (105) reduces to the spectrum of the harmonic oscillator in the ordinary space.

Based on the results obtained in this section in the next section we will study the length in quantum space with preserved rotational symmetry. We study squared length operator defined as

$$\mathbf{Q}^2 = \alpha^2\sum_i P_i^2 + \beta^2\sum_i X_i^2, \tag{108}$$

with α and β being constants. Let us find the eigenvalues of the operator in noncommutative phase space with preserved rotational symmetry. So, we consider X_i, P_i satisfying relations of algebra (27)-(29).

Operator $\mathbf{Q}^2$ can be considered as Hamiltonian of a tree-dimensional harmonic oscillator with mass

$$m = \frac{1}{2\alpha^2}, \tag{109}$$

and frequency

$$\omega = 2\alpha\beta. \tag{110}$$

So, we can use results presented in the previous section and write eigenvalues of the operator $\mathbf{Q}^2$ up to the second order in the parameters of noncommutativity as follows

$$q^2_{n_1,n_2,n_3} = \hbar\sqrt{\left(2\beta^2+\frac{\alpha^2\langle\eta^2\rangle}{3}\right)\left(2\alpha^2+\frac{\beta^2\langle\theta^2\rangle}{3}\right)}\left(n_1+n_2+n_3+\frac{3}{2}\right), \tag{111}$$

$n_1 = 0,1,2...$, $n_2 = 0,1,2...$, $n_3 = 0,1,2....$ Let us analyze the minimal length on the basis of result (111). We have

$$q^2_{min} = \sqrt{q^2_{0,0,0}} = \sqrt{\hbar}\sqrt[4]{2\beta^2+\frac{\alpha^2\langle\eta^2\rangle}{3}}\sqrt[4]{2\alpha^2+\frac{\beta^2\langle\theta^2\rangle}{3}} \tag{112}$$

So, the expression for the minimal length depends on the parameters of coordinate and momentum noncommutativity $\langle\theta^2\rangle$, $\langle\eta^2\rangle$.

Let us study particular cases. Namely, $\alpha = 0$, $\beta = 1$. In this case one has the squared length operator in the coordinate space

$$\mathbf{R}^2 = \sum_{i=1}^{3} X_i^2. \tag{113}$$

Using (111), the eigenvalues of the operator read

$$r^2_{n_1,n_2,n_3} = \sqrt{\frac{2\hbar^2\langle\theta^2\rangle}{3}}\left(n_1+n_2+n_3+\frac{3}{2}\right), \tag{114}$$

here $n_1 = 0,1,2...$, $n_2 = 0,1,2...$, $n_3 = 0,1,2....$ From (114) expression follows that the squared length is quantized. This is caused by the noncommutativity of coordinates. The minimal length in the coordinate space reads

$$r_{min} = \sqrt{r^2_{0,0,0}} = \sqrt{\frac{3\hbar^2\langle\theta^2\rangle}{2}}. \quad (115)$$

It is determined by the parameter of coordinate noncommutativity $\langle\theta^2\rangle$.

Let us also study another particular case when $\alpha = 1$, $\beta = 0$. In this case we have squared length operator in momentum space. It reads

$$\mathbf{P}^2 = \sum_{i=1}^{3} P_i^2. \quad (116)$$

$$p^2_{n_1,n_2,n_3} = \sqrt{\frac{2\hbar^2\langle\eta^2\rangle}{3}}\left(n_1+n_2+n_3+\frac{3}{2}\right), \quad (117)$$

$n_1 = 0,1,2...$, $n_2 = 0,1,2...$, $n_3 = 0,1,2....$ And the minimal length in the momentum space is determined by parameter of momentum noncommutativity. We have

$$p_{min} = \sqrt{p^2_{0,0,0}} = \sqrt[4]{\frac{3\hbar^2\langle\eta^2\rangle}{2}}. \quad (118)$$

5. Hamiltonian of a system of oscillators in rotationally-invariant noncommutative phase space

We consider a system of N interacting harmonic oscillators of masses m and frequencies ω in uniform field in the frame of rotationally-invariant noncommutatove algebra of canonical type (27)-(29). The system is described with the following Hamiltonian

$$\begin{aligned} H_s &= \sum_n \frac{(\mathbf{P}^{(n)})^2}{2m} + \sum_n \frac{m\omega^2(\mathbf{X}^{(n)})^2}{2} + \\ &+ \frac{k}{2}\sum_{\substack{m,n\\ m\neq n}} (\mathbf{X}^{(n)} - \mathbf{X}^{(m)})^2 + \kappa \sum_n X_1^{(n)}, \end{aligned} \quad (119)$$

where κ, k are constants. For convenience, we choose the direction of the field to coincide with the X_1 axis direction. In the vase of $\kappa = 0$, Hamiltonian (119) describes nondissipative symmetric network of coupled harmonic oscillators [56].

Coordinates and momenta of harmonic oscillators satisfy relations of rotationally-invariant noncommutative algebra

$$\begin{aligned} [X_i^{(n)}, X_j^{(m)}] &= i\hbar\delta_{mn}\theta_{ij}^{(n)}, & (120)\\ [X_i^{(n)}, P_j^{(m)}] &= i\hbar\delta_{mn}\left(\delta_{ij} + \sum_k \frac{\theta_{ik}^{(n)}\eta_{jk}^{(m)}}{4}\right), & (121)\\ [P_i^{(n)}, P_j^{(m)}] &= i\hbar\delta_{mn}\eta_{ij}^{(n)}, & (122)\\ \theta_{ij}^{(n)} &= \frac{c_\theta^{(n)} l_P^2}{\hbar}\sum_k \varepsilon_{ijk}\tilde{a}_k, & (123)\\ \eta_{ij}^{(n)} &= \frac{c_\eta^{(n)}\hbar}{l_P^2}\sum_k \varepsilon_{ijk}\tilde{p}_k^b. & (124) \end{aligned}$$

Here indexes $m,n = (1...N)$ label the oscillators.

If masses of harmonic oscillators are equal m. Using (123), (124), we can write

$$\begin{aligned} \theta_{ij}^{(n)} &= \theta_{ij} = \frac{c_\theta l_P^2}{\hbar}\sum_k \varepsilon_{ijk}\tilde{a}_k, & (125)\\ \eta_{ij}^{(n)} &= \eta_{ij} = \frac{c_\eta\hbar}{l_P^2}\sum_k \varepsilon_{ijk}\tilde{p}_k^b, & (126)\\ c_\theta &= \frac{\tilde{\gamma}}{m}, & (127)\\ c_\eta &= \tilde{\alpha}m. & (128) \end{aligned}$$

Using representation of noncommutative coordinates and noncommutative momenta over coordinates and momenta satisfying the ordinary commutation relations, one has

$$\begin{aligned} H_s = &\sum_n \left(\frac{(\mathbf{p}^{(n)})^2}{2m} + \frac{m\omega^2(\mathbf{x}^{(n)})^2}{2} + \kappa x_1^{(n)}\right) + \\ &+ \frac{k}{2}\sum_{\substack{m,n\\ m\neq n}} (\mathbf{x}^{(n)} - \mathbf{x}^{(m)})^2 + \\ &+ \sum_n \left(-\frac{(\eta\cdot\mathbf{L}^{(n)})}{2m} - \frac{m\omega^2(\theta\cdot\mathbf{L}^{(n)})}{2} + \frac{\kappa}{2}[\theta\times\mathbf{p}^{(n)}]_1 + \right.\\ &\left. + \frac{m\omega^2}{8}[\theta\times\mathbf{p}^{(n)}]^2 + \frac{[\eta\times\mathbf{x}^{(n)}]^2}{8m}\right) - \frac{k}{2}\sum_{\substack{m,n\\ m\neq n}} \theta\cdot \\ &\cdot[(\mathbf{x}^{(n)} - \mathbf{x}^{(m)})\times(\mathbf{p}^{(n)} - \mathbf{p}^{(m)})] + \sum_{\substack{m,n\\ m\neq n}} \frac{k}{8}[\theta\times(\mathbf{p}^{(n)} - \mathbf{p}^{(m)})]^2, \end{aligned} \quad (129)$$

The total Hamiltonian reads

$$H = H_s + H^a_{osc} + H^b_{osc} = H_0 + \Delta H. \quad (130)$$

We have

$$\langle[\eta\times\mathbf{x}^{(n)}]^2\rangle_{ab}=\frac{2}{3}\langle\eta^2\rangle(\mathbf{x}^{(n)})^2, \tag{131}$$

$$\langle[\theta\times\mathbf{p}^{(n)}]^2\rangle_{ab}=\frac{2}{3}\langle\theta^2\rangle(\mathbf{p}^{(n)})^2, \tag{132}$$

$$\langle[\theta\times(\mathbf{p}^{(n)}-\mathbf{p}^{(m)})]^2\rangle_{ab}=\frac{2}{3}\langle\theta^2\rangle(\mathbf{p}^{(n)}-\mathbf{p}^{(m)})^2. \tag{133}$$

So, for ΔH we can write

$$\begin{aligned}\Delta H=\sum_n\Bigg(&-\frac{(\eta\cdot\mathbf{L}^{(n)})}{2m}-\frac{m\omega^2(\theta\cdot\mathbf{L}^{(n)})}{2}+\frac{\kappa}{2}[\theta\times\mathbf{p}^{(n)}]_1+\\&+\frac{m\omega^2}{8}[\theta\times\mathbf{p}^{(n)}]^2+\frac{[\eta\times\mathbf{x}^{(n)}]^2}{8m}\Bigg)-\\&-\frac{k}{2}\sum_{\substack{m,n\\m\neq n}}\theta\cdot[(\mathbf{x}^{(n)}-\mathbf{x}^{(m)})\times(\mathbf{p}^{(n)}-\mathbf{p}^{(m)})]+\\&+\sum_{\substack{m,n\\m\neq n}}\frac{k}{8}[\theta\times(\mathbf{p}^{(n)}-\mathbf{p}^{(m)})]^2-\sum_n\Bigg(\frac{\langle\eta^2\rangle(\mathbf{x}^{(n)})^2}{12m}+\\&+\frac{\langle\theta^2\rangle m\omega^2(\mathbf{p}^{(n)})^2}{12}\Bigg)-\frac{k}{12}\sum_{\substack{m,n\\m\neq n}}\langle\theta^2\rangle(\mathbf{p}^{(n)}-\mathbf{p}^{(m)})^2.\end{aligned} \tag{134}$$

So, up to the second order in ΔH (or up to the second order in the parameters of noncommutativity) the Hamiltonian of a system of interacting harmonic oscillators in uniform field reads

$$\begin{aligned}H_0 = &\sum_n\left(\frac{(\mathbf{p}^{(n)})^2}{2m}+\frac{m\omega^2(\mathbf{x}^{(n)})^2}{2}+\kappa x_1^{(n)}\right)+\\&+\frac{k}{2}\sum_{\substack{m,n\\m\neq n}}(\mathbf{x}^{(n)}-\mathbf{x}^{(m)})^2+\\&+\sum_n\left(\frac{\langle\eta^2\rangle(\mathbf{x}^{(n)})^2}{12m}+\frac{\langle\theta^2\rangle m\omega^2(\mathbf{p}^{(n)})^2}{12}\right)+\\&+\frac{k}{12}\sum_{\substack{m,n\\m\neq n}}\langle\theta^2\rangle(\mathbf{p}^{(n)}-\mathbf{p}^{(m)})^2+H^a_{osc}+H^b_{osc}.\end{aligned} \tag{135}$$

6. Effect of noncommutativity on spectrum of interacting oscillators

For convenience, let us introduce

$$m_{eff}=m\left(1+\frac{m^2\omega^2\langle\theta^2\rangle}{6}\right)^{-1}, \tag{136}$$

$$\omega_{eff}=\left(\omega^2+\frac{\langle\eta^2\rangle}{6m^2}\right)^{\frac{1}{2}}\left(1+\frac{m^2\omega^2\langle\theta^2\rangle}{6}\right)^{\frac{1}{2}}. \tag{137}$$

So, Hamiltonian (135) can be rewritten as

$$\begin{aligned}H_0 = &\sum_n\left(\frac{(\mathbf{p}^{(n)})^2}{2m_{eff}}+\frac{m_{eff}\omega^2_{eff}(\tilde{\mathbf{x}}^{(n)})^2}{2}\right)-\frac{N\kappa^2}{2m_{eff}\omega^2_{eff}}+\\&+\frac{k}{2}\sum_{\substack{m,n\\m\neq n}}(\tilde{\mathbf{x}}^{(n)}-\tilde{\mathbf{x}}^{(m)})^2+\frac{k}{12}\sum_{\substack{m,n\\m\neq n}}\langle\theta^2\rangle(\mathbf{p}^{(n)}-\mathbf{p}^{(m)})^2+\\&+H^a_{osc}+H^b_{osc}.\end{aligned} \tag{138}$$

Here $\tilde{\mathbf{x}}^{(n)}$ is defined as

$$\tilde{\mathbf{x}}^{(n)}=\left(x_1^{(n)}+\frac{\kappa}{m_{eff}\omega^2_{eff}},x_2^{(n)},x_3^{(n)}\right). \tag{139}$$

For operators $\tilde{\mathbf{x}}^{(n)}$, $\mathbf{p}^{(n)}$ we have the ordinary commutation relations

$$[\tilde{x}_i^{(n)},\tilde{x}_j^{(m)}]=0, \tag{140}$$

$$[\tilde{x}_i^{(n)},p_j^{(m)}]=i\hbar\delta_{nm}\delta_{ij}, \tag{141}$$

$$[p_i^{(n)},p_j^{(m)}]=0. \tag{142}$$

It is also important to mention that

$$[H_0,H^a_{osc}]=[H_0,H^b_{osc}]=0. \tag{143}$$

So, the energy levels of H_0 are

$$E_{\{n_1\},\{n_2\},\{n_3\}}=\sum_{a=1}^N\hbar\omega_a\left(n_1^{(a)}+n_2^{(a)}+n_3^{(a)}+\frac{3}{2}\right)-\frac{N\kappa^2}{2m_{eff}\omega^2_{eff}}+3\hbar\omega_{osc}. \tag{144}$$

Here $n_i^{(a)}$ are quantum numbers ($n_i^{(a)}=0,1,2...$) and

$$\omega_1=\omega_{eff}, \tag{145}$$

$$\omega_2=\omega_3=...=\omega_N=\left(\omega^2_{eff}+\frac{2kN}{m_{eff}}+\frac{kN\langle\theta^2\rangle m_{eff}\omega^2_{eff}}{3}+\frac{2k^2\langle\theta^2\rangle N^2}{3}\right)^{\frac{1}{2}} \tag{146}$$

The spectrum of the center-of-mass of the system of the harmonic oscillators is represented by the first term in (144). The spectrum of the relative motion is described by the terms with $a=2..N$. To show this let us introduce coordinates and moments of the center of mass

$$\mathbf{x}^c = \frac{\sum_n \mathbf{x}^{(n)}}{N}, \tag{147}$$

$$\mathbf{p}^c = \sum_n \mathbf{p}^{(n)}, \tag{148}$$

coordinates and momenta of he relative motion

$$\Delta\mathbf{x}^{(n)} = \mathbf{x}^{(n)} - \mathbf{x}^c, \tag{149}$$

$$\Delta\mathbf{p}^{(n)} = \frac{\mathbf{p}^{(n)} - \mathbf{p}^c}{N}. \tag{150}$$

Taking into account (138), we have

$$H_0 = H^c + H_{rel} + H^a_{osc} + H^b_{osc}, \tag{151}$$

$$H^c = \frac{(\mathbf{p}^c)^2}{2Nm_{eff}} + \frac{Nm_{eff}\omega^2_{eff}(\tilde{\mathbf{x}}^c)^2}{2} - \frac{N\kappa^2}{2m_{eff}\omega^2_{eff}}, \tag{152}$$

$$H_{rel} = \sum_n \left(\frac{(\Delta\mathbf{p}^{(n)})^2}{2m_{eff}} + \frac{m_{eff}\omega^2_{eff}(\Delta\mathbf{x}^{(n)})^2}{2} \right) + \frac{k}{2}\sum_{\substack{m,n\\ m\neq n}} (\Delta\mathbf{x}^{(n)} - \Delta\mathbf{x}^{(m)})^2 + \frac{k}{12}\sum_{\substack{m,n\\ m\neq n}} \langle\theta^2\rangle(\Delta\mathbf{p}^{(n)} - \Delta\mathbf{p}^{(m)})^2, \tag{153}$$

$$[H^c, H_{rel}] = [H^c, H^a_{osc} + H^b_{osc}] = [H_{rel}, H^a_{osc} + H^b_{osc}] = 0. \tag{154}$$

Here $\tilde{\mathbf{x}}^c$ reads

$$\tilde{\mathbf{x}}^c = \left(x^c_1 + \kappa/(m_{eff}\omega^2_{eff}), x^c_2, x^c_3\right). \tag{155}$$

Let us analyze the obtained result. From (144) follows that frequencies in the spectra of the center-of-mass and relative motion of the system of interacting oscillators are affected by the noncommutativity of coordinates and noncommutativity of momenta. The uniform field causes to the shift of the spectrum of the system on a constant.

Considering limit $\langle\theta^2\rangle \to 0$, $\langle\eta^2\rangle \to 0$ form $E_{\{n_1\},\{n_2\},\{n_3\}}$ one obtains well known expression

$$E_{\{n_1\},\{n_2\},\{n_3\}} = \hbar\omega\left(n^{(1)}_1 + n^{(1)}_2 + n^{(1)}_3 + \frac{3}{2}\right) + \sum_{a=2}^{N}\hbar\left(\omega^2 + \frac{2Nk}{m}\right)^{\frac{1}{2}} \left(n^{(a)}_1 + n^{(a)}_2 + n^{(a)}_3 + \frac{3}{2}\right) - \frac{N\kappa^2}{2m\omega^2}. \tag{156}$$

On the basis of (144) we can write the spectrum of a system of N particles of mass m with harmonic oscillator interaction. Considering $\omega=0$, we have

$$E_{\{n_1\},\{n_2\},\{n_3\}} = \frac{\hbar\langle\eta^2\rangle}{6m^2}\left(n^{(1)}_1 + n^{(1)}_2 + n^{(1)}_3 + \frac{3}{2}\right) + \hbar\left(\frac{2kN}{m} + \frac{\langle\eta^2\rangle}{6m^2} + \frac{2k^2\langle\theta^2\rangle N^2}{3}\right)^{\frac{1}{2}} \sum_{a=2}^{N}\left(n^{(a)}_1 + n^{(a)}_2 + n^{(a)}_3 + \frac{3}{2}\right) - \frac{3N\kappa^2 m}{\langle\eta^2\rangle} + 3\hbar\omega_{osc}. \tag{157}$$

The spectrum of the center-of-mass of the system is described by (157). It is important to note that this spectrum is discreet, that is caused by momentum noncommutativity. The spectrum of the center-of-mass of the system if the spectrum of harmonic oscillator with a frequency determined by the parameter of momentum noncommutativity $\hbar\langle\eta^2\rangle/6m^2$. The spectrum of the relative motion of the system is affected by noncommutativity of coordinates and noncommutativity of momenta (see second term in (157)).

It is important to stress that from (144) and (157) follows that the influence of noncommutativity on the spectrum increases with increasing of the number of particles N.

In the case of $k=0$ we obtain energy levels of a system of N free particles in uniform field in a space with noncommutativity of coordinates and noncommutativity of momenta

$$E_{\{n_1\},\{n_2\},\{n_3\}} = \sum_{a=1}^{N}\frac{\hbar\langle\eta^2\rangle}{6m^2}\left(n^{(a)}_1 + n^{(a)}_2 + n^{(a)}_3 + \frac{3}{2}\right) - \frac{3N\kappa^2 m}{\langle\eta^2\rangle} + 3\hbar\omega_{osc}. \tag{158}$$

The expression corresponds to the spectrum of N oscillators with frequencies $\hbar\langle\eta^2\rangle/6m^2$. Noncommutativity of coordinates does not affect on the energy levels of free particle system

7. Conclusions

A way to construct algebra with noncommutativity of coordinates and noncommutativity of momenta which is rotationally-invariant and equivalent to noncommutative algebra of canonical type has been proposed. The idea of generalization of the parameters of noncommutativity to tensors has been used to construct algebra (7)-(9). The tensors have been defined with the help of additional coordinates and conjugate momenta of them that are governed by harmonic oscillators. The length of the oscillators has been considered to be the Planck length. The frequency of the oscillators is assumed to be very large. Therefore harmonic oscillators that are in the ground state remains in them.

The spectrum of free particle has been studied in the frame of rotationally-invariant noncommutative algebra. Up to the second order in the parameters of noncommutativity it is shown that the energy levels of a free particle in noncommutative phase space correspond to the energy levels of harmonic oscillator (93) with frequency defined by the parameter of momentum noncommutativity (92).

Also, harmonic oscillator has been examined in rotationally-invariant noncommutative phase space. We have found energy levels of the oscillator up to the second order in the parameters of noncommutativity. We have concluded that noncommutativity of coordinates and noncommutativity of momenta affect on the mass and the frequency of the oscillator. The expression for the energy levels of the harmonic oscillator in noncommutative phase space corresponds to that in the ordinary space.

Based on the obtained results the minimal length has been studied in the frame of rotationally-invariant noncommutative algebra. Squared length operator has been considered in coordinate, momentum space, and phase space. The eigenvalues of the operators (111), (114), (117) have been obtained up to the second order in the parameters of coordinate and momentum noncommutativity. Based on the results the minimal lengths in coordinate space, momentum space, and phase space have been obtained (115), (118), (112).

We have also examined energy levels of a system of N harmonic oscillators with harmonic oscillator interaction in uniform field in rotationally-invariant noncommutative phase space of canonical type. Up to the second order in the parameters of noncommutativity we have obtained influence of noncommutativity of coordinates and noncommutativity of momenta on the energy levels of the system. We have concluded that space quantization affects on the frequencies of the system (144). Uniform field shifts of the spectrum of the system on a constant (144). Particular case of a system of two interacting oscillators and a system of three interacting oscillators have been examined. We have found energy levels of the systems in rotationally-invariant noncommutative phase space.

On the basis of the obtained results a system of particles with harmonic oscillator interaction and a system of free particles in uniform field have been examined. We have concluded that up to the second orders in the parameters of noncommutativity, the noncommutativity of coordinates does not affect the spectrum of free particle system in uniform field. The spectrum of free particles in uniform field has the form of the spectrum of a system of N harmonic oscillators with frequencies determined by parameters of momentum noncommutativity a s $\hbar\langle\eta^2\rangle/6m^2$ (158). We have also shown that a spectrum of the center-of-mass of a system of particles with harmonic oscillator interaction in uniform field corresponds to the spectrum of harmonic oscillator (see first term in (157)) and is affected only by noncommutativity of momenta.

We have also found that the spectrum of the relative motion of the system of interacting harmonic oscillators corresponds to the spectrum of harmonic oscillators with frequencies that depends on the parameters of noncommutativity (see second term in (157)). We have also showed that effect of coordinates noncommutativity on the spectra of systems with harmonic oscillator interaction increases with increasing of the number of particles (144), (157).

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