TASK Quarterly 27 (2) 2023

Edge coloring of small signed graphs

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Abstract

In 2020, Behr [1] introduced the problem of edge coloring of signed graphs and proved that every signed graph (G, σ) can be colored using $\Delta(G)$ or $\Delta(G)+1$ colors, where $\Delta(G)$ denotes the maximum degree of G. Three years later, Janczewski et al. [2] introduced a notion of signed class 1, such that a graph G is of signed class 1 if and only if every signed graph (G, σ) can be colored using $\Delta(G)$ colors.

It is a well-known fact [3] that almost all graphs are of class 1. In this paper, we conjecture that a similar fact is true for signed class 1, that almost all graphs are of signed class 1. To support the hypothesis we implemented an application that colored all the signed graphs with at most 8 vertices. We describe an algorithm behind the application and discuss the results of conducted experiments.

Keywords:

signed graphs, edge coloring of signed graphs

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1. Introduction

This paper exclusively focuses on simple, finite and undirected graphs. Graph G is characterized by a set of vertices $V(G)$ and a set of edges $E(G)$, with their respective cardinalities denoted by $n(G)$ and $m(G)$. The degree of any given vertex v within a graph G is represented by $\deg_G(v)$, while $\Delta(G)$ denotes the highest degree among all the vertices of G. An *incidence* is a pair (v, e), where v denotes a vertex, e denotes an edge and v is one of the endpoints of e. An incidence (v, e) is concisely denoted as $v\colon e$ and the set of all incidences of a graph is denoted by $I(G)$. All remaining definitions and symbols align with those established by Diestel [4].

Signed graphs initially came into existence in the 1950s, introduced by Harary [5] as a generalized form of simple graphs. The primary goal was to more effectively represent social interactions encompassing sentiments of dislike, indifference and liking. A *signed graph* is a pair (G, σ), in which G denotes a graph and $\sigma\colon E(G) \to \{\pm 1\}$ denotes a function. The terms G and σ represent the *underlying* graph and the *signature* of (G, σ), respectively. An edge $e \in E(G)$ is called *positive* (or *negative*) if and only if $\sigma(e) = 1$ (respectively, $\sigma(e) = -1$). A cycle within (G, σ) is called *positive* (or *negative*) when the product of its edge signs is positive (or negative, respectively). A signed graph featuring exclusively positive cycles is called *balanced*, while in all other cases, it is called *unbalanced*.

In a signed graph (G, σ), switching refers to an operation performed on a subset $V' \subseteq V(G)$, leading to a new signed graph (G, σ'), where $\sigma'\colon E(G) \to \{\pm 1\}$ is defined by:

$$\sigma'(uv) = \begin{cases} -\sigma(uv), & \text{if } V' \text{ includes exactly one of } u, v, \\ \sigma(uv), & \text{in all other cases.} \end{cases}$$

As an illustration, when a single vertex is switched—in other words, when a subset of vertices with a cardinality one undergoes switching—the signs of its incident edges are negated. If a signed graph (G, σ') can be derived from (G, σ) through the switching of certain vertices, we state that (G, σ') and (G, σ) are *switching equivalent*. It is well-known that switching equivalence is an equivalence relation within the collection of all signed graphs having a fixed underlying graph.

Theorem 1 (Naserasr et al. [6]). *Let G be a simple graph and c be the number of its connected components. The number of signatures σ such that (G, σ) are not pairwise switching equivalent is $2^{m(G)-n(G)+c}$.* □

In this paper, we examine the problem of edge coloring of signed graphs and make a conjecture about the number of colors required to color almost all signed graphs. We also present the results of running an exponential coloring algorithm on small graphs. In Section 2 we define a problem of edge coloring of signed graphs and present the main conjecture of the paper. In Section 3 we describe an application and the algorithm we implemented in order to test the conjecture. In Section 4 we present results obtained from running the algorithm for small signed graphs.

2. Edge coloring of signed graphs

In the year 2020, Behr [1] proposed the concept of edge coloring in signed graphs as a generalization of standard graph edge coloring. Suppose n is a positive integer, and

$$M_n = \begin{cases} \{0, \pm 1, \ldots, \pm k\}, & \text{if } n = 2k+1, \\ \{\pm 1, \ldots, \pm k\}, & \text{if } n = 2k. \end{cases}$$

An *n-edge-coloring* of a signed graph (G, σ) is a function $c\colon I(G) \to M_n$ which satisfies $c(u\colon uv) = -\sigma(uv)c(v\colon uv)$ for every edge $uv \in E(G)$ and $c(u\colon e_1) \neq c(u\colon e_2)$ for all distinct edges $e_1 \neq e_2$ such that $u\colon e_1,\ u\colon e_2 \in I(G)$. By $\chi'(G, \sigma)$ we denote *the chromatic index* of a signed graph (G, σ), which is the smallest n allowing (G, σ) to have an n-edge-coloring. Since the problem of edge coloring of signed graphs is a generalization of the standard edge coloring problems, all the applications of the standard problem transfer to its signed version.

Behr [1] proved that a signed path can be colored using only two colors, and a signed cycle can similarly be colored with 2 colors if and only if it's balanced. The main result of Behr's article [1] is the generalized Vizing's theorem, also called the Behr's theorem.

Theorem 2 (Behr [1]). $\Delta(G) \leq \chi'(G, \sigma) \leq \Delta(G) + 1$ *for all signed graphs (G, σ).* □

Behr [7] defined the problem of edge coloring in such a way that it works well with the concept of vertex switching. Behr proved that all the switching equivalent signed graphs share the same value of χ'. It follows that in order to get to know χ' of all the switching equivalent signed graphs, it is enough to color only one of these graphs.

A graph G is of *class* 1 if $\chi'(G) = \Delta(G)$ and it is of *class* 2 if $\chi'(G) = \Delta(G) + 1$. Janczewski et al. [2] introduced two new classes of graphs, signed class 1 (denoted by $1^{\pm}$) and signed class 2 (denoted by $2^{\pm}$). A graph G is of class $1^{\pm}$ if $\chi'(G, \sigma) = \Delta(G)$ for any signature σ. Similarly, a graph G is of class $2^{\pm}$ if $\chi'(G, \sigma) = \Delta(G) + 1$ for any signature σ. In other words, a graph is of class $1^{\pm}$ (respectively, $2^{\pm}$) if all the signed graphs having G as an underlying graph have the chromatic index of $\Delta(G)$ (respectively, $\Delta(G) + 1$). Clearly, there are graphs of neither signed class 1 nor signed class 2. An obvious example is a cycle on n vertices—if it is balanced, it can be colored using Δ colors, else it requires $\Delta+1$ colors.

Janczewski et al. [2] proved that some of the well-known classes of graphs belong to signed class 1.

Theorem 3 (Janczewski et al. [2]). *All trees, wheels, necklaces, cacti (other than cycles) and complete bipartite graphs (with parts of different sizes) belong to signed class* 1. □

Behr [7] also defined *class ratio* $\mathscr{C}(G)$ of graph G as the number of possible signatures $\sigma\colon E(G) \to \{\pm 1\}$ such that the signed graph (G, σ) is $\Delta(G)$-colorable, divided by the number of all possible signatures defined on $E(G)$, i.e. $2^{m(G)}$. The class ratio is a rational number satisfying $0 \leq \mathscr{C}(G) \leq 1$ and, by previous definitions, G is of class $1^{\pm}$ (respectively, $2^{\pm}$) if and only if $\mathscr{C}(G) = 1$ (respectively, $\mathscr{C}(G) = 0$).

Theorem 4 (Erdős et al. [3]). *Almost all graphs are of class one.* □

We conjecture that a result similar to Theorem 4 holds for edge coloring signed graphs and class $1^{\pm}$.

Conjecture 5. *Almost all graphs*[4] *are of signed class one.*

3. Coloring algorithm

To test Conjecture 5 we implemented an application capable of finding the chromatic index of a given signed graph. The application uses an exponential backtracking algorithm that exhaustingly tries to create a proper edge coloring. We used it to color all non-switching-equivalent non-isomorphic connected signed graphs on up to eight vertices. We focused only on connected graphs since the chromatic index of a disconnected graph is always the maximum of the chromatic indices of its connected components.

In order to color all those graphs, we needed a practical way to generate them. We used the “geng” tool, part of the “nauty” [8] package, to generate all non-isomorphic non-empty connected graphs with up to eight vertices. These graphs were generated once and preserved on the drive in the g6 format—a standard format for representing graphs as strings. Our application was capable of reading such a file and constructing a graph based on its corresponding g6 string. For computation purposes, we represented a graph by using a combination of an adjacency matrix and an adjacency list. This approach combines the advantages of both solutions—it enables us to verify the existence of a given edge in $O(1)$ time and enumerate all the neighbours of a given vertex in $O(\Delta)$ time. While this method slows the graph creation process, the graph is created once and remains unchanged during the exponential coloring algorithm, so the creation time is negligible. We store signs of edges within the adjacency matrix, which also allows us to check the sign of a given edge in $O(1)$ time.

Clearly, graphs created from g6 strings are not signed graphs, so for each graph, we needed a way to generate signed graphs having this graph as an underlying graph. Since all the switching equivalent signed graphs possess the same χ', it was sufficient to only generate all non-switching-equivalent signed graphs having a given graph as an underlying graph. A simple algorithm for generating all non-switching-equivalent graphs follows from the proof of Theorem 1. For a given graph G, it finds an arbitrary spanning tree T on this graph. To produce all non-switching-equivalent signed graphs on graph G, it assigns a positive sign to each edge from the set $E(T)$ and considers all possible assignment of signs to the remaining edges of graph G. Since $m(T) = n(G) - 1$, the algorithm generates $2^{m(G)-m(T)} = 2^{m(G)-n(G)+1}$ non-switching-equivalent signed graphs—all possible non-switching-equivalent signed graphs with a shared underlying graph.

For every signed graph, our program attempted to color it using Δ colors. If successful, this indicated that the graph is Δ-edge-colorable. Otherwise, it is not and requires $\Delta+1$ colors to be colored properly. The procedure for coloring a single signed graph using Δ colors is a simple exponential backtracking algorithm that examines all possible coloring of incidences of the input graph. Initially, it selects a single incidence for every edge since coloring one of the edge's incidences dictates the color that must be used for the other incidence. Following this, it orders all the selected incidences. Then, it considers two possible colorings of the first incidence—either using color 0 or 1. It does not have to check all the possible Δ colors. This is because if there is a Δ-coloring that assigns a color $c \notin \{0, 1\}$ to that incidence, it is possible to create another Δ-coloring of that graph by swapping colors $\pm c$ and ± 1 in such a way that the first incidence gets color 1. When coloring the ith incidence ($i > 1$), the algorithm branches into Δ cases, assigning all possible colors to that incidence. If a given color cannot be used to color the incidence (due to the presence of another incidence already utilizing that color and both incidences sharing a common vertex), the entire branch is discarded. If the algorithm successfully colors all the incidences without causing any conflicts, the input graph can be colored using Δ colors. If no branches are left and a full coloring has not been produced, a Δ-coloring of an input graph does not exist.

It would be possible to optimize the algorithm even further. One could safely remove all the leaves from the

[4] *Almost all graphs* means that the proportion of graphs with n vertices having a property tends to 1 as n tends to infinity.

input graph as long as that does not change the maximum degree of the graph. It surely does not change the result since all the removed leaves in such a case could be easily colored using available colors. Another possible optimization would be checking if an input graph is of any of the classes listed in Theorem 3. If it was, it would be clear that it can be colored using Δ colors.

Theorem 6 (Behr [7])**.** *Let (G, σ) be a signed graph with an even value of $\Delta(G)$. Let $M(G)$ be a subgraph induced by all vertices of maximum degree. If $M(G)$ is an independent set, then G can be colored using $\Delta(G)$ colors.* □

Another optimization follows from the above Theorem 6. One could check if an input graph meets all the conditions—it has an even Δ and the subgraph induced by all its vertices of maximum degree is an independent set, then it is clear that it can be colored using Δ colors.

We did not implement such optimizations since the number of graphs we considered was low enough that additional optimizations were not needed. Since the exponential nature of the algorithm they also probably would not allow us to color all signed graphs with 9 vertices.

4. Results

There are exactly 12112 non-empty non-isomorphic connected graphs with up to 8 vertices and more than 19 million non-empty non-isomorphic non-switching-equivalent connected signed graphs with up to 8 vertices. We colored all these signed graphs using the algorithm described in Section 3. For every graph, we also calculated its class ratio to find out how common graphs of signed classes 1 and 2 are.

We did not consider larger graphs since there are more than 4 billion non-empty non-isomorphic non-switching-equivalent connected signed graphs with 9 vertices and more than 3 trillion of such graphs with 10 vertices. We stopped at graphs with 8 vertices since they represent a meaningful part of small graphs and can all be colored in a reasonable amount of time.

Table 1 presents the numbers of n-vertex non-empty non-isomorphic connected graphs that belong to signed class 1 or signed class 2. The results show that there are no graphs with 8 or less vertices that belong to signed class 2. Clearly, signed class 2 is not empty—Janczewski et al. [2] proved that there exists a signed class 2 graph for every odd Δ value. Results presented in Table 1 support Conjecture 5. They show that more than 90% of all graphs with 6, 7 and 8 vertices belong to signed class 1. More than 99% of all graphs with 8 vertices belong to that class. We expect that the trend continues and almost all larger graphs are of class $1^{\pm}$.

Proof of Theorem 4 follows directly from the fact proved by Vizing [9] that every graph of class 2 has at least 3 vertices of maximum degree. We analyzed the structure of all the 119 graphs with at most 8 vertices that are outside signed class 1 and observed that all of them have at least 3 such vertices. We propose the following conjecture:

Conjecture 7. *Every graph outside signed class 1 has at least 3 vertices of maximum degree.*

One could think that it follows directly from the Vizing results but it does not. Indeed, it is true that if graph G is of class 2 it cannot be of signed class 1, since a signed graph with G as an underlying graph and all the signs negative requires $\Delta(G)+1$ colors. But there are also graphs outside signed class 1 that are not of class 2 (meaning they are of class 1). An obvious example is an even cycle that is of class 1 and is not of signed class 1. So potentially there could be graphs outside signed class 1 that do have no more than 2 vertices of maximum degree but we think there are no such graphs.

We observe that if Conjecture 7 is true, then Conjecture 5 is also true. It follows directly from the fact that almost all graphs have a unique vertex of maximum degree [3].

Table 2 presents a distribution of all signed graphs with n vertices into graphs with $\chi' = \Delta$ and $\chi' = \Delta + 1$. Again, for all signed graphs with 6, 7 and 8 vertices, over 90% of them can be colored using Δ colors. More than 99% of all signed graphs with 8 vertices are Δ-colorable.

We have also analyzed some classes of graphs for which the problem of edge coloring of signed graphs has not yet been solved. Table 3 presents calculated class ratio of complete graphs. Based on these results we propose another conjecture:

Conjecture 8. *All complete graphs K_{2n} are of class $1^{\pm}$.*

Based on Table 2, one could observe that signed graphs with an even number of vertices can be colored using Δ colors more often than signed graphs with an odd number of vertices. For example, from all signed graphs with 6 and 7 vertices, as many as 97.66% and 99.39% of them, respectively, can be colored using Δ colors but only 91.23% of all signed graphs with 7 vertices can be colored using Δ colors. It seems to be true and is mostly caused by complete graphs. We observe that complete graphs K_n are the graphs with the most number of switching equivalence classes among all n vertex graphs and it seems that complete signed graphs with an odd number of vertices do not belong to signed class 1, and their class ratio is not larger than 0.5. It follows that most of the n-vertex signed graphs that require $\Delta + 1$ colors are indeed complete graphs. We can observe that there are 16901 graphs with 7 vertices that require $\Delta + 1$ colors to

Table 1: The numbers of n-vertex non-empty non-isomorphic connected graphs of signed classes 1 and 2.

n	all graphs	graphs in $1^{\pm}$	%	graphs in $2^{\pm}$	%	graphs outside of $1^{\pm}$ and $2^{\pm}$	%
2	1	1	100	0	0	0	0
3	2	1	50	0	0	1	50
4	6	5	83.33	0	0	1	16.67
5	21	17	80.95	0	0	4	19.05
6	112	107	95.54	0	0	5	4.46
7	853	821	96.25	0	0	32	3.75
8	11117	11041	99.32	0	0	76	0.68

Table 2: The numbers of n-vertex non-empty non-isomorphic non-switching-equivalent connected signed graphs requiring either Δ or $\Delta+1$ colors.

n	number of graphs	number of graphs with $\chi' = \Delta$	%	number of graphs with $\chi' = \Delta+1$	%
2	1	1	100	0	0
3	3	2	66.67	1	33.33
4	18	17	94.44	1	5.56
5	193	156	80.83	37	19.17
6	4316	4215	97.66	101	2.34
7	192817	175916	91.23	16901	8.77
8	19402921	19285367	99.39	117554	0.61

be properly colored and as many as 16405 of them are complete graphs.

We present all graphs with up to 8 vertices that are not of signed class 1 in Appendix A.

5. Conclusions

In the paper, we conjectured that almost all graphs are of signed class 1. We implemented an application that uses an exponential algorithm for coloring signed graphs. We colored a lot of small graphs and the results we received support the hypothesis—it seems like indeed almost all graphs are of signed class 1. We also noticed that graphs of signed class 2 are extremely rare—there is no such graph within all the graphs with up to eight vertices. We observed that all graphs outside signed class 1 have at least 3 vertices of maximum degree, and it is well-known that almost all graphs have a unique vertex of maximum degree. All these results suggest that Conjecture 5 is true.

We will continue computer experiments to verify the hypothesis for larger graphs and examine the problem for more classes of graphs.

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A. Graphs outside signed class 1

Using the application described in Section 3 we found all graphs with up to 8 vertices that are not of signed class 1. There is no such graph with 2 vertices, since the only connected graph with 2 vertices is the path P_2. Clearly, all signed graphs with underlying graph P_2 can be colored using only color 0 and $\Delta(P_2) = 1$. Below we present all the graphs with at least 3 and at most 8 vertices which are not of signed class 1.

For graphs with 3 and 4 vertices only cycles C_3 and C_4, respectively, are not of signed class 1.

There are 4 graphs with 5 vertices that are not of signed class 1. They are presented in Figure 1.

There are 5 graphs with 6 vertices that are not of signed class 1. They are presented in Figure 2.

There are 32 graphs with 7 vertices that are not of signed class 1. We present some of them in Figure 3.

There are 76 graphs with 8 vertices that are not of signed class 1. We present some of them in Figure 4.

Table 3: The numbers of non-empty non-isomorphic non-switching-equivalent signed graphs having a given K_n as an underlying graph requiring either Δ or $\Delta+1$ colors.

underlying graph	signed graphs with $\chi' = \Delta$	signed graphs with $\chi' = \Delta+1$	class ratio
K_2	1	0	1
K_3	1	1	$1/2$
K_4	8	0	1
K_5	31	33	$31/64$
K_6	1024	0	1
K_7	16363	16405	$16363/32768$
K_8	2097152	0	1

Table 4: All graphs with 3 vertices which are not of signed class 1.

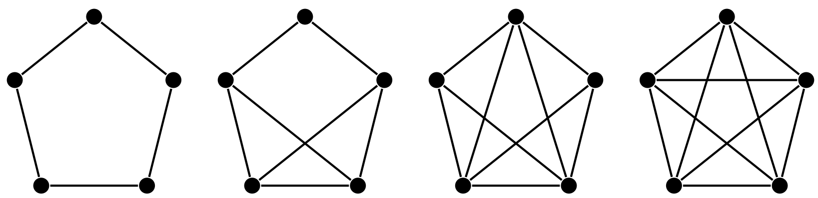
g6 string of G	$n(G)$	$m(G)$	no. of signed graphs with $\chi' = \Delta$	no. of signed graphs with $\chi' = \Delta+1$	class ratio
Bw	3	3	1	1	$1/2$

Table 5: All graphs with 4 vertices which are not of signed class 1.

g6 string of G	$n(G)$	$m(G)$	no. of signed graphs with $\chi' = \Delta$	no. of signed graphs with $\chi' = \Delta+1$	class ratio
C]	4	4	1	1	$1/2$

Table 6: All graphs with 5 vertices which are not of signed class 1.

g6 string of G	$n(G)$	$m(G)$	no. of signed graphs with $\chi' = \Delta$	no. of signed graphs with $\chi' = \Delta+1$	class ratio
DUW	5	5	1	1	$1/2$
D]w	5	7	6	2	$3/4$
D^{	5	9	31	1	$31/32$
D~{	5	10	31	33	$31/64$

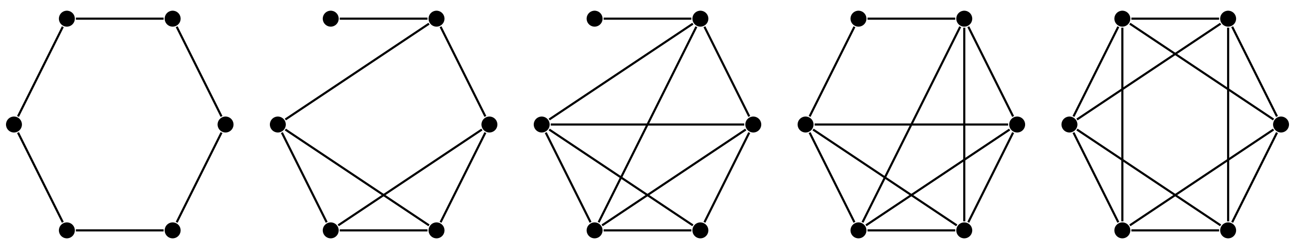


DUW D]w D^{ D~{

Figure 1: All four non-isomorphic connected graphs on 5 vertices that do not belong to signed class 1.

Table 7: All graphs with 6 vertices which are not of signed class 1.

g6 string of G	$n(G)$	$m(G)$	no. of signed graphs with $\chi' = \Delta$	no. of signed graphs with $\chi' = \Delta+1$	class ratio
EEh_	6	6	1	1	$1/2$
EEzO	6	8	6	2	$3/4$
ETno	6	10	31	1	$31/32$
E]zg	6	11	31	33	$31/64$
E]~o	6	12	64	64	$1/2$

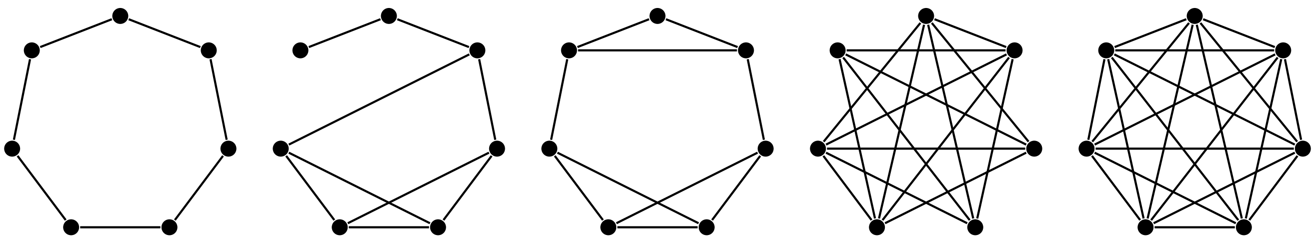


EEh_ EEzO ETno E]zg E]~o

Figure 2: All five non-isomorphic connected graphs on 6 vertices that do not belong to signed class 1.

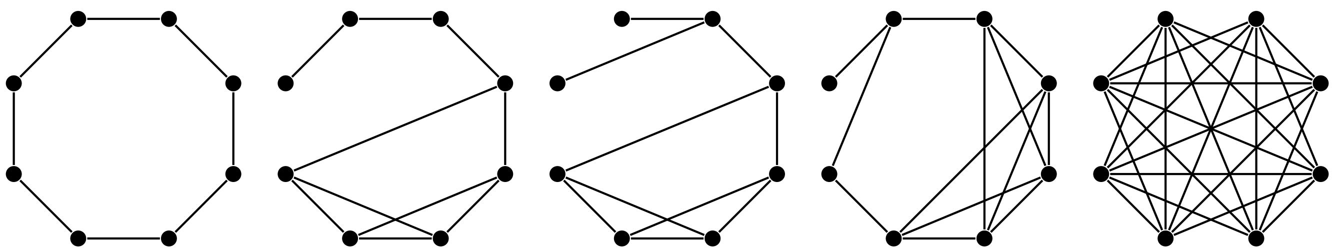
Table 8: All graphs with 7 vertices which are not of signed class 1.

g6 string of G	$n(G)$	$m(G)$	no. of signed graphs with $\chi' = \Delta$	no. of signed graphs with $\chi' = \Delta + 1$	class ratio
FCp`_	7	7	1	1	1/2
FCZbO	7	9	6	2	3/4
FCxv?	7	10	15	1	15/16
FCusw	7	11	31	1	31/32
FEhv?	7	10	13	3	13/16
FEhuO	7	10	12	4	3/4
FEzUg	7	12	31	33	31/64
FEzvO	7	13	127	1	127/128
FFzf_	7	13	127	1	127/128
FFzeo	7	13	127	1	127/128
FFzvO	7	14	127	129	127/256
FFz~o	7	16	1023	1	1023/1024
FQjR_	7	10	14	2	7/8
FQinO	7	11	31	1	31/32
FQytW	7	12	62	2	31/32
FQzvO	7	13	64	64	1/2
FQzuo	7	13	127	1	127/128
FUZv_	7	13	64	64	1/2
FUxv_	7	13	127	1	127/128
FUxvO	7	13	127	1	127/128
FUxuo	7	13	126	2	63/64
FUzro	7	14	127	129	127/256
FU~vo	7	16	1023	1	1023/1024
FU~vW	7	16	1023	1	1023/1024
FV~~w	7	19	8191	1	8191/8192
F]zno	7	16	1023	1	1023/1024
F]znW	7	16	1022	2	511/512
F]zlw	7	16	1023	1	1023/1024
F]~vo	7	17	2026	22	1013/1024
F]~~w	7	19	8191	1	8191/8192
F^~~w	7	20	16363	21	16363/16384
F~~~w	7	21	16363	16405	16363/32768



FCp`_ FCZbO FQjR_ FU~vo F~~~w

Figure 3: Some of non-isomorphic connected graphs on 7 vertices that do not belong to signed class one.



G?qa`_ G?ovEO G?oppg GCrJeW G]~v~w

Figure 4: Some of non-isomorphic connected graphs on 8 vertices that do not belong to signed class one.

Table 9: All graphs with 8 vertices which are not of signed class 1.

g6 string of G	$n(G)$	$m(G)$	no. of signed graphs with $\chi' = \Delta$	no. of signed graphs with $\chi' = \Delta + 1$	class ratio
G?qa`_	8	8	1	1	$1/2$
G?ovEO	8	10	6	2	$3/4$
G?oppg	8	10	6	2	$3/4$
G?rLeW	8	12	31	1	$31/32$
G?rNeW	8	13	31	33	$31/64$
G?qjf?	8	11	13	3	$13/16$
G?o~F?	8	11	15	1	$15/16$
G?zffO	8	14	127	1	$127/128$
G?zfeW	8	14	127	1	$127/128$
G?zTb_	8	12	31	1	$31/32$
G?~vf_	8	16	256	256	$1/2$
GCRcqo	8	11	12	4	$3/4$
GCQuck	8	12	31	1	$31/32$
GCQuQo	8	11	14	2	$7/8$
GCrRug	8	14	127	1	$127/128$
GCrRuW	8	14	126	2	$63/64$
GCpuuo	8	14	64	64	$1/2$
GCrJeW	8	13	62	2	$31/32$
GCXe`W	8	11	12	4	$3/4$
GCZfVG	8	14	127	1	$127/128$
GCZfUW	8	14	127	1	$127/128$
GCZTck	8	13	62	2	$31/32$
GCZUuo	8	14	127	1	$127/128$
GCZUug	8	14	64	64	$1/2$
GCZUuW	8	14	127	1	$127/128$
GCXnBW	8	13	31	33	$31/64$
GCXnbW	8	14	62	66	$31/64$
GCZvf_	8	15	127	129	$127/256$
GCZnfO	8	15	127	129	$127/256$
GCdcuo	8	12	31	1	$31/32$
GCxvVG	8	15	127	129	$127/256$
GCzvbo	8	16	256	256	$1/2$

Table 10: All graphs with 8 vertices which are not of signed class 1. (cont.)

g6 string of G	$n(G)$	$m(G)$	no. of signed graphs with $\chi' = \Delta$	no. of signed graphs with $\chi' = \Delta + 1$	class ratio
GCv^ew	8	17	1023	1	1023/1024
GCv^es	8	17	1023	1	1023/1024
GCv^e[	8	17	1023	1	1023/1024
GCv^c{	8	17	1023	1	1023/1024
GEr^es	8	17	1023	1	1023/1024
GEr^ek	8	17	1023	1	1023/1024
GEr^e[	8	17	1023	1	1023/1024
GEr]vo	8	17	1023	1	1023/1024
GEr^vo	8	18	2046	2	1023/1024
GEr^uw	8	18	2026	22	1013/1024
GEjeqw	8	15	127	129	127/256
GEhuuW	8	15	127	129	127/256
GEhttW	8	15	126	130	63/128
GEzU~o	8	18	2046	2	1023/1024
GEv]~w	8	20	8191	1	8191/8192
GEv]|{	8	20	8191	1	8191/8192
GEv~u{	8	21	16382	2	8191/8192
GEnfbW	8	16	256	256	1/2
GEnbvG	8	16	256	256	1/2
GFzfvo	8	19	4092	4	1023/1024
GQhTVO	8	13	62	2	31/32
GQjRak	8	14	124	4	31/32
GQjVRW	8	15	127	129	127/256
GQjnVK	8	17	1023	1	1023/1024
GQjnT[	8	17	1022	2	511/512
GQzTrg	8	16	256	256	1/2
GQyurg	8	16	256	256	1/2
GQzvVW	8	18	2046	2	1023/1024
GQzvVS	8	18	2046	2	1023/1024
GQzn^[	8	20	8191	1	8191/8192
GUZvfg	8	18	2046	2	1023/1024
GUZvfK	8	18	2046	2	1023/1024
GUZvek	8	18	2046	2	1023/1024
GUZvvg	8	19	4092	4	1023/1024
GUxvvo	8	19	4092	4	1023/1024
GUz]}{	8	21	16382	2	8191/8192
GTm~vw	8	21	16382	2	8191/8192
GTm~vs	8	21	16382	2	8191/8192
GTm|~w	8	21	16363	21	16363/16384
G]zn^w	8	22	32764	4	8191/8192
G]zn^[	8	22	16363	16405	16363/32768
G]zn\{	8	22	32764	4	8191/8192
G]~vvs	8	23	32764	32772	8191/16384
G]~v~w	8	24	65536	65536	1/2