

# EXPLICIT ESTIMATION OF AN INTEGRAL IN A DOMAIN BY THE MULTIPLE RECIPROCITY METHOD WITH THE USE OF INVERSE OPERATIONS

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**Abstract:** The paper presents two methods of solving the Poisson equation. One is based on the multiple reciprocity method. An analytical form of the basic solution obtained by means of inverse operation technique enabled assessing the method's error. The other is based on source function expansion into a series according to polyharmonic functions. Further polyharmonic functions have been obtained through inverse operations (with the  $\Delta^{-1}$  operator) applied to polyharmonic functions. Numerical results have confirmed perfect efficiency of both methods.

**Keywords:** Poisson equation, multiple reciprocity method, inverse operation, fundamental solution

## 1. Introduction

The use of a fundamental solution to eliminate the domain integral over the adjoint operator is a cornerstone of the boundary element method [1]. Application of the multiple reciprocity method to Poisson's equation enables approximating the source function with the required accuracy. The approximation error for such source function approximation was determined in the present paper. The use of this procedure was a subject of many papers [2-5]. In [5] the multiple reciprocity boundary element method (MRBEM) is applied to modeling Photonic Crystal Fiber. MRBEM converts the Helmholtz equation into an integral equation using a series of higher-order fundamental solutions of the Laplace equation. This method is much more efficient in analyzing the dispersion and non-linear properties of Photonic Crystal Fibers than the conventional direct boundary element method (BEM).

In [2, 3] many cases are considered of solutions of the Poisson equations of various orders with non-linear source functions dependent on the unknown variable. These problems have been effectively solved with the use of the multiple reciprocity method.

The process of solving an inhomogeneous differential equation,  $\Delta u = f$ , gives rise to difficulties related to determining a particular integral  $\Delta^{-1}f$  and estimating the error of its approximate calculation. The work presents two ways of approximate determination of the  $\Delta^{-1}f$  integral. One consists in representing the  $f$  function using the multiple reciprocity method through the values of the  $f$  function and its derivatives at the  $\Gamma$  boundary of the  $m$  range. The other is based on expansion of the  $f$  function into a polyharmonic series [6, 7]. Both methods are meshless.

An analytical form of the fundamental solution of the  $j^{\text{th}}$  order has enabled assessing in advance the error of a particular integral. Another important result of the present work consists in finding a particular integral of the Poisson equation with the use of polyharmonic functions generated as a result of inverse operations (with the  $\Delta^{-1}$  operator) from the harmonic functions. The effectiveness of determining a particular integral with the help of the multiple reciprocity method and polyharmonic functions has been tested on examples of functions with many extreme points.

## 2. Application of the reciprocity principle in representing the $w(x, y, z)$ function within domain $\Omega$

The paper presents an application of the reciprocity principle to meshless function approximation in domain  $\Omega$  delimited with boundary  $\Gamma$ . The Green formula in the following form has been used as a starting point:

$$\int_{\Omega} (u \cdot \Delta v - v \cdot \Delta u) d\Omega = \oint_{\Gamma} \left( \frac{\partial u}{\partial n} v - \frac{\partial v}{\partial n} u \right) d\Gamma, \quad (x, y, z) \in \bar{\Omega}. \quad (1)$$

Substituting  $u = \Delta^k w$ ,  $v = \Delta^{n-k} q$ ,  $k = 0, 1, 2, \dots, n$ ,  $w, q \in C^m(\bar{\Omega})$ , ( $2n < m$ ) yields:

$$\int_{\Omega} (\Delta^k w \cdot \Delta^{n+1-k} q \cdot \Delta^{n-k} q \cdot \Delta^{k+1} w) d\Omega = \oint_{\Gamma} \left( \frac{\partial}{\partial n} \Delta^k w \cdot \Delta^{n-k} q - \frac{\partial}{\partial n} \Delta^{n-k} q \cdot \Delta^k w \right) d\Gamma, \quad (2)$$

while summing (2) for consecutive  $k = 0, 1, \dots, n$  gives us:

$$\int_{\Omega} (w \cdot \Delta^{n+1} q - q \cdot \Delta^{n+1} w) d\Omega = \sum_{k=0}^n \oint_{\Gamma} \left( \frac{\partial}{\partial n} \Delta^k w \cdot \Delta^{n-k} q - \frac{\partial}{\partial n} \Delta^{n-k} q \cdot \Delta^k w \right) d\Gamma. \quad (3)$$

The assumption of

$$\Delta^{n+1} q(x - \xi, y - \eta, z - \zeta) = \delta(x - \xi, y - \eta, z - \zeta), \quad \xi, \eta, \zeta \in \bar{\Omega}, \quad (4)$$

leads to the following particular integral:

$$\Delta^{n-k} q = \Delta^{-(k+1)} \delta, \quad q = \Delta^{-(n+1)} \delta, \quad (5)$$

while formula (3) takes the following form:

$$\begin{aligned} c \cdot w(\xi, \eta, \zeta) = & \sum_{k=0}^n \oint_{\Gamma} \left[ \frac{\partial}{\partial n} \Delta^k w \cdot \Delta^{-(k+1)} \delta - \frac{\partial}{\partial n} \Delta^{-(k+1)} \delta \cdot \Delta^k w \right] d\Gamma + \\ & + \int_{\Omega} \Delta^{n+1} w(x, y, z) \cdot \Delta^{-(n+1)} \delta(x - \xi, y - \eta, z - \zeta) d\Omega. \end{aligned} \quad (6)$$

An analytical form of the particular integral of Equation (5) will be determined below.

### 3. Representation of the $w(x, y, z)$ function within domain $\Omega$ with polyharmonic functions

An important advantage of representing the function in the form of polyharmonic functions consists in the ease of determination of the  $\Delta^{-k}w$ ,  $k \geq 1$ , functions [8, 7]. Let us consider a finite expansion of the  $w$  function into a Taylor series up to the  $N^{\text{th}}$  order. This gives us:

$$\Delta^{M+1}\tilde{w}(x, y, z) = 0, \quad M = \left\lfloor \frac{N}{2} \right\rfloor. \tag{7}$$

The application of consecutive inverse operations to the above equation yields:

$$\begin{aligned} \Delta^{-1}(\Delta^{M+1}\tilde{w}) &= \Delta^{-1}(0) = H_0 \\ \Delta^{-1}(\Delta^M\tilde{w}) &= \Delta^{-1}H_0 + H_1 \\ \Delta^{-1}(\Delta^{M-1}\tilde{w}) &= \Delta^{-2}H_0 + \Delta^{-1}H_1 + H_2 \\ \Delta^{-1}(\Delta^{M-2}\tilde{w}) &= \Delta^{-3}H_0 + \Delta^{-2}H_1 + \Delta^{-1}H_2 + H_3 \\ &\dots\dots\dots \\ \Delta^{M-n}\tilde{w} &= \sum_{k=0}^n \Delta^{-k}H_{n-k} \end{aligned} \tag{8}$$

While for  $n = M$

$$\tilde{w}(x, y, z) = \sum_{k=0}^M \Delta^{-k}H_{M-k}(x, y, z) = \sum_{k=0}^M \sum_{j=0}^{\infty} a_{M-k,i} \cdot \Delta^{-k}h_j(x, y, z), \tag{9}$$

where functions  $h_j(x, y, z)$ ,  $j = 0, 1, 2, \dots$ , are consecutive base harmonic functions. Further considerations will be carried out for functions of two variables. In this case, the base harmonic functions are the elements of expansion of the  $e^{p(x+iy)}$  function into a power series with respect to  $p$ , *i.e.*:

$$e^{p(x+iy)} = \sum_{n=0}^{\infty} p^n \frac{(x+iy)^n}{n!} = \sum_{n=0}^{\infty} p^n [F_n(x, y) + iG_n(x, y)], \quad F_0 = 1, \quad G_0 = 0, \tag{10}$$

$$\{h\} = \{F_0, G_1, F_1, G_2, F_2, \dots\}. \tag{11}$$

Substituting (11) into Equation (9) yields:

$$\tilde{w}(x, y) = \sum_{k=0}^M \sum_j^{\infty} [A_{M-k,j} \cdot \Delta^{-k}F_j(x, y) + B_{M-k,j} \cdot \Delta^{-k}G_j(x, y)]. \tag{12}$$

It has been shown in [8, 9] that:

$$\begin{aligned} \Delta^{-k}F_j(x, y) &= \frac{1}{2^{2k}} \left[ 2G_{j+k} \cdot G_k + \binom{j+2k}{k} \cdot F_{j+2k} \right] \\ &= \frac{1}{2^{2k}} \left[ 2F_{j+k} \cdot F_k + \binom{j+2k}{k} \cdot F_{j+2k} \right], \\ \Delta^{-k}G_j(x, y) &= \frac{1}{2^{2k}} \left[ 2G_{j+k} \cdot F_k - \binom{j+2k}{k} \cdot G_{j+2k} \right] \\ &= \frac{1}{2^{2k}} \left[ -2G_{j+k} \cdot F_k + \binom{j+2k}{k} \cdot G_{j+2k} \right], \end{aligned} \tag{13}$$

$$\text{rank } \Delta^{-k} F_j = \text{rank } \Delta^{-k} G_j = 2k + j. \quad (14)$$

Taking into account that function (7) is a polynomial of the  $N^{\text{th}}$  order and the order of inverse operations determines relationship (14), the summation is carried out only for those elements for which  $2k + j \leq N$ , i. e.  $j \leq N - 2k$ . Hence, substitution of (13) to Equation (12) gives us:

$$\tilde{w}(x, y) = \sum_{k=0}^M \sum_{j=0}^{N-2k} [A_{kj}^* \cdot F_{k+j}(x, y) + B_{kj}^* \cdot G_{k+j}(x, y)] F_k(x, y). \quad (15)$$

Harmonic functions  $F_l, G_l$  will be determined as follows on the basis of Equation (10):

$$F_k + i \cdot G_k = \frac{(x + iy)^k}{k!} = \frac{(x + iy)^{k-1}}{(k-1)!} \frac{x + iy}{k} = (F_{k-1} + iG_{k-1}) \frac{x + iy}{k}, \quad (16)$$

$$F_k = \frac{1}{k} (xF_{k-1} - yG_{k-1}), \quad G_k = \frac{1}{k} (yF_{k-1} - xG_{k-1}), \quad k \geq 1. \quad (17)$$

#### 4. Estimating the integral in the domain according to the multiple principle of reciprocity

In order to present an approximate function  $w(x, y, z)$  in domain  $\bar{\Omega}$  by means of a series of integrals at the  $\Gamma$  boundary, the error caused by the omission of the following integral:

$$I_n = \int_{\Omega} \Delta^{n+1} w(x, y, z) \cdot \Delta^{-(n+1)} \delta(x - \xi, y - \eta, z - \zeta) d\Omega, \quad (18)$$

occurring in formula (6), should be estimated. As a result of inverse operations [8, 9] for finding a particular integral of Equation (9), we obtain:

$$\Delta^{-j} \delta = \begin{cases} \frac{1}{2} \frac{r^{2j-1}}{(2j-1)!}, & r = |x - \xi| \quad \text{for the 1D case,} \\ \frac{1}{2\pi} \frac{r^{2(j-1)}}{(2^{j-1}(j-1)!)^2} \left[ \sum_{k=1}^{j-1} \frac{1}{k} - \ln r \right], & \text{for the 2D case,} \\ \frac{1}{4\pi} \frac{r^{2j-1}}{(2j)!}, & \text{for the 3D case.} \end{cases} \quad (19)$$

where  $r = \sqrt{(x - \xi)^2 + (y - \eta)^2}$  and  $r = \sqrt{(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2}$  for the 2D and 3D case, respectively. Therefore, substituting result (19<sub>2</sub>) to (18) gives us (for  $r \leq 1$ ):

$$\begin{aligned} |I_n| &= \left| \int_{\Omega} \Delta^{n+1} w(x, y) \frac{1}{2\pi} \frac{r^{2n}}{2^{2n}(n!)^2} \left( \sum_{k=1}^n \frac{1}{k} - \ln r \right) d\Omega \right| \leq \\ &\leq \max |\Delta^{n+1} w| \frac{1}{\pi 2^{2n+1} (n!)^2} \int_0^1 r^{2n+1} \left( \sum_{k=1}^n \frac{1}{k} - \ln r \right) dr = \\ &= \frac{\max(\Delta^{n+1} w)}{\pi 2^{2n+1} (n!)^2} \left( \frac{1}{2n+2} \sum_{k=1}^n \frac{1}{k} + \frac{1}{(2n+2)^2} \right) = \\ &= \frac{\max(\Delta^{n+1} w)}{\pi 2^{2(n+1)} (n+1) (n!)^2} \left( \frac{1}{2(n+1)} + \sum_{k=1}^n \frac{1}{k} \right). \end{aligned} \quad (20)$$

Hence, for a confined function  $\Delta^{n+1}w$  values of the  $|I_n|$  integral decidedly decrease with the growing number  $n$  of applications of the reciprocity principle.

### 5. Numerical calculations

Numerical properties related to expressing the source function in terms of the multiple principle of reciprocity and polyharmonic functions may be demonstrated by comparing the solutions of Poisson's equation of the following form:

$$\Delta T = q, \tag{21}$$

in a domain  $\Omega = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$  with a boundary condition of the first  $T|_{\Gamma} = f$ .

In order to estimate the solution error, let us assume two relative norms, a maximal  $\varepsilon_{\max}$  and a mean square  $\varepsilon_s$ :

$$\varepsilon_{\max} = \frac{\max_{(x_i, y_i)} |\bar{T} - T_d|}{\max_{(x_i, y_i)} |T_d|}, \quad \varepsilon_s = \frac{\sqrt{\sum_{i=1}^N \sum_{j=1}^N (\bar{T} - T_d)^2}}{N \max_{(x_i, y_i)} |T_d|}. \tag{22}$$

**Example 1**

The source function is defined by formula [10]:

$$q(x, y) = -2(x + y - x^2 - y^2),$$

with the following boundary conditions:

$$f(x, 0) = 0, \quad f(1, y) = 0, \quad f(x, 1) = 0, \quad f(0, y) = 0.$$

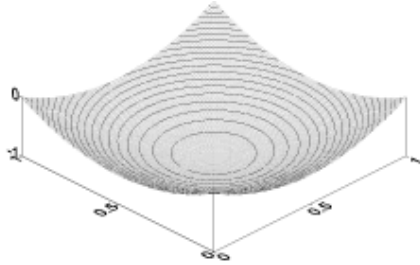
The following function is the solution of the Poisson equation:

$$T(x, y) = xy(x - 1)(y - 1).$$

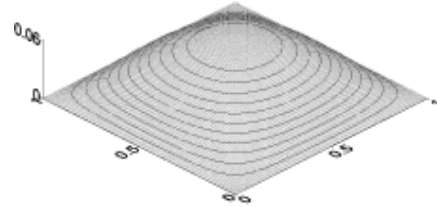
Figure 1 shows the form of the source function in the imposed domain, while Figure 2 presents the solution of the Poisson equation for the source function represented by repeated application of the reciprocity principle (6). Numerical results are given in Table 1.  $N$  is the number of rows,  $M$  – the number of columns. When using the multiple principle of reciprocity, the rank of the equations system matrix is equal to  $N$  (rank  $A = N$ ), while with the use of polyharmonic functions the rank is  $A = M$ .

**Table 1.** Numerical results for Example 1

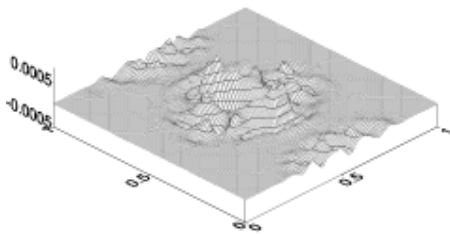
$N = 400$	$T$		$q$		$M$
	$\varepsilon_{\max}$	$\varepsilon_s$	$\varepsilon_{\max}$	$\varepsilon_s$	
MRM	$0.682337 \cdot 10^{-07}$	$0.155272 \cdot 10^{-07}$	$0.914075 \cdot 10^{-03}$	$0.163878 \cdot 10^{-03}$	400
Polyhar. Func.	$0.644623 \cdot 10^{-17}$	$0.306654 \cdot 10^{-18}$	$0.759455 \cdot 10^{-19}$	$0.121920 \cdot 10^{-19}$	75



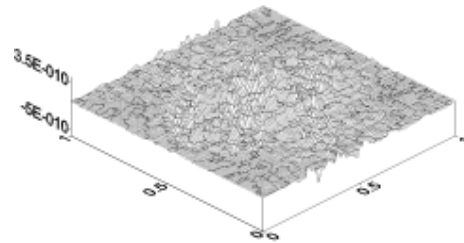
**Figure 1.** Source function for Poisson's equation



**Figure 2.** Accurate solution



**Figure 3.** Multiple Reciprocity Method (MRM): distribution of source function errors



**Figure 4.** Multiple Reciprocity Method (MRM): distribution of solution errors

### Example 2

Source function [10]:

$$q = -6(x+y) - 4 \left[ 1 - \frac{(x-b)^2 + (y-b)^2}{a^2} \right] e^{-((x-b)^2 + (y-b)^2)/a^2}.$$

Boundary conditions:

$$\begin{aligned} f(x,0) &= -x^3 + e^{-((x-b)^2 + b^2)/a^2}, & f(1,y) &= -1 - y^3 + e^{-((1-b)^2 + (y-b)^2)/a^2}, \\ f(x,1) &= -x^3 - 1 + e^{-((x-b)^2 + (1-b)^2)/a^2}, & f(0,y) &= -y^3 + e^{-((b^2 + (y-b)^2)/a^2)}. \end{aligned}$$

The following function is the solution of the Poisson equation:

$$T(x,y) = -x^3 - y^3 + e^{-((x-b)^2 + (y-b)^2)/a^2}.$$

**Table 2.** Numerical results for Example 2

$N = 400$	$T$		$q$		$M$
	$\varepsilon_{\max}$	$\varepsilon_s$	$\varepsilon_{\max}$	$\varepsilon_s$	
MRM	$0.508103 \cdot 10^{-03}$	$0.809722 \cdot 10^{-04}$	$0.795474 \cdot 10^{-03}$	$0.123899 \cdot 10^{-03}$	400
Polyhar. Func.	$0.107744 \cdot 10^{-13}$	$0.410255 \cdot 10^{-14}$	$0.211782 \cdot 10^{-11}$	$0.191817 \cdot 10^{-12}$	75

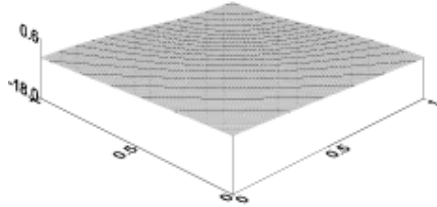


Figure 5. Source function for Poisson's equation

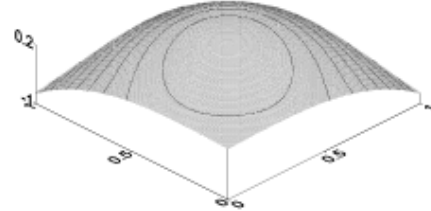


Figure 6. Accurate solution

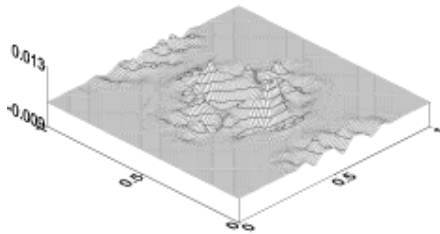


Figure 7. Multiple Reciprocity Method (MRM): distribution of source function errors

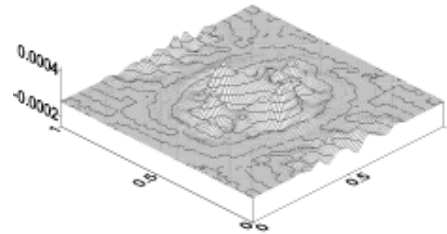


Figure 8. Multiple Reciprocity Method (MRM): distribution of solution errors

**Example 3**

Source function [11]:

$$q = -\frac{751\pi^2}{144} \sin \frac{\pi x}{6} \sin \frac{7\pi x}{4} \sin \frac{3\pi y}{4} \sin \frac{5\pi y}{4} +$$

$$+ \frac{7\pi^2}{12} \cos \frac{\pi x}{6} \cos \frac{7\pi x}{4} \sin \frac{3\pi y}{4} \sin \frac{5\pi y}{4} +$$

$$+ \frac{15\pi^2}{8} \sin \frac{\pi x}{6} \sin \frac{7\pi x}{4} \cos \frac{3\pi y}{4} \cos \frac{5\pi y}{4}.$$

Boundary conditions:

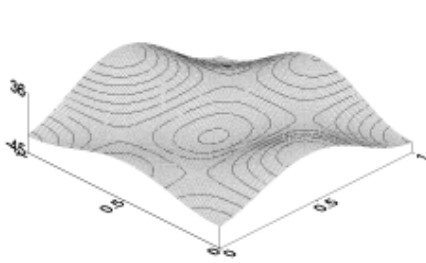
$$f(x,0)=0, \quad f(1,y)=-\frac{1}{2} \sin \frac{3\pi y}{4} \sin \frac{5\pi y}{4}, \quad f(x,1)=-\frac{\sqrt{2}}{4} \sin \frac{\pi x}{6} \sin \frac{7\pi x}{4}, \quad f(0,y)=0.$$

The following function is the solution of the Poisson equation:

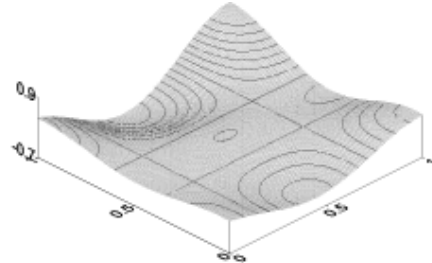
$$T(x,y) = \sin \frac{\pi x}{6} \sin \frac{7\pi x}{4} \sin \frac{3\pi y}{4} \sin \frac{5\pi y}{4}.$$

**Table 3.** Numerical results for Example 3

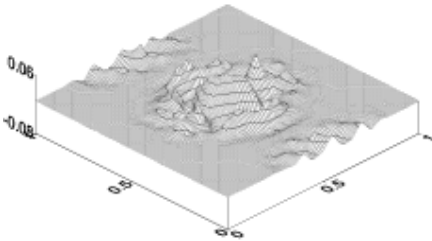
$N = 400$	$T$		$q$		$M$
	$\epsilon_{\max}$	$\epsilon_s$	$\epsilon_{\max}$	$\epsilon_s$	
MRM	$0.152527 \cdot 10^{-02}$	$0.290300 \cdot 10^{-03}$	$0.187807 \cdot 10^{-02}$	$0.372177 \cdot 10^{-03}$	400
Polyhar. Func.	$0.276120 \cdot 10^{-10}$	$0.124352 \cdot 10^{-10}$	$0.137696 \cdot 10^{-08}$	$0.199642 \cdot 10^{-09}$	75



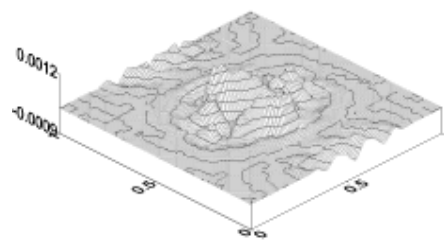
**Figure 9.** Source function for Poisson's equation



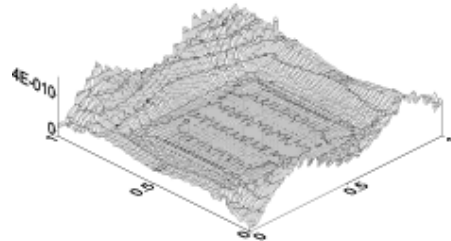
**Figure 10.** Accurate solution



**Figure 11.** Multiple Reciprocity Method (MRM): distribution of source function errors



**Figure 12.** Multiple Reciprocity Method (MRM): distribution of solution errors



**Figure 13.** Polyharmonic functions: distribution of source function errors

#### **Example 4**

Source function:

$$q = \sin px \sin sy, \quad p = 4\pi, \quad s = 4\pi.$$

Boundary conditions:

$$f(x,0) = 0, \quad f(1,y) = 0, \quad f(x,1) = 0, \quad f(0,y) = 0.$$

The following function is the solution of the Poisson equation:

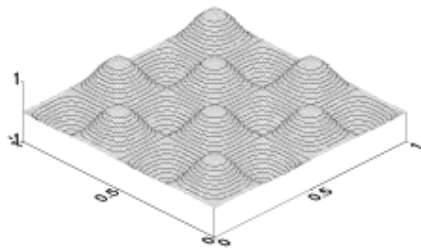
$$T(x,y) = -\frac{1}{p^2 + s^2} \sin px \sin sy.$$

The examples presented above indicate that MRM and polyharmonic functions are highly useful in approximation of continuous and differentiable functions. This, together with the use of inverse operations, provides an effective method for solving

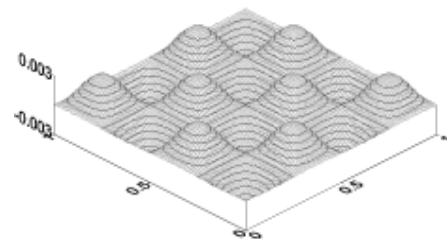


**Table 4.** Numerical results for Example 4

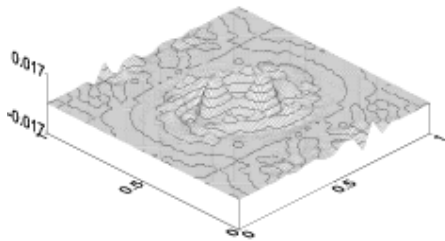
$N = 400$	$T$		$q$		$M$
	$\epsilon_{\max}$	$\epsilon_s$	$\epsilon_{\max}$	$\epsilon_s$	
MRM	$0.219959 \cdot 10^{-01}$	$0.393477 \cdot 10^{-02}$	$0.219959 \cdot 10^{-01}$	$0.393477 \cdot 10^{-02}$	400
Polyhar. Func.	$0.499032 \cdot 10^{-04}$	$0.457495 \cdot 10^{-05}$	$0.966855 \cdot 10^{-03}$	$0.542302 \cdot 10^{-04}$	75



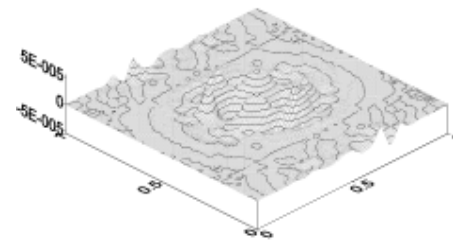
**Figure 14.** Source function for Poisson's equation



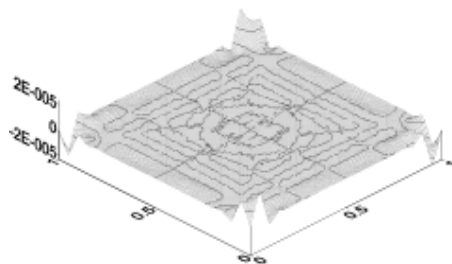
**Figure 15.** Accurate solution



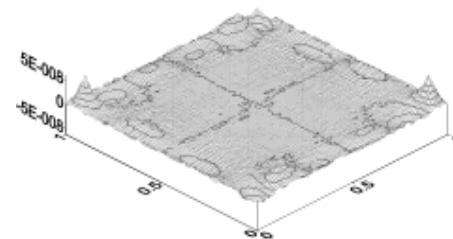
**Figure 16.** Multiple Reciprocity Method (MRM): distribution of source function errors



**Figure 17.** Multiple Reciprocity Method (MRM): distribution of solution errors



**Figure 18.** Polyharmonic functions: distribution of source function errors



**Figure 19.** Polyharmonic functions: distribution of solution errors

differential equations (such an approach is presented on the example of Poisson's equation).

The examples have been chosen so that the obtained results can be compared with those of [2]. The computation presented in the paper is related to the approxi-

mation of any analytical function of two real variables by means of radial and polyharmonic functions.

## 6. Conclusions

The solutions to Poisson's equation obtained with the MRM method have been compared with the approach where the source function is approximated by a linear combination of polyharmonic functions in the sense of relative norms, *viz.* maximal,  $\varepsilon_{\max}$ , and average,  $\varepsilon_s$ , ones.

The comparison has shown that for smooth source functions the solution may be better approximated with a polyharmonic function, while for a source function of considerable variability the Poisson equation may be better solved with MRM.

This due to the fact that solution of the problem consists in computation of an inverse matrix in order to express the coefficients of the source function's expansion as dependent on the function's node values. For this purpose, an SVD (Singular Value Decomposition) algorithm was applied in the calculation. Taking into account that polyharmonic functions become numerically linearly-dependent as their number increases, the algorithm enables obtaining a pseudo-inverse matrix rather than an inverse one. Hence, the problem was converted into an approximation of the source function. For a  $400 \times 400$  matrix the rank of the matrix was equal to 75. A similar matrix was generated for the MRM method. An application of the SVD algorithm caused no change in the matrix rank, which was equal to 400 in this instance. This is equivalent to an interpolation of the source function. In this case, the results were not smoothed, which gave us worse properties of the norms but better mapping of the source function.

### Acknowledgements

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