AN ANALYSIS OF A PIEZO-ELASTIC RESTRICTOR USING A CAPACITIVE SHUNT

MARINO KEKANA[1] AND JANUSZ BADUR[2]

[1] *Centre for Advanced Materials, Design and Manufacturing Research,*
Durban Institute of Technology,
Durban 4000, South Africa
kekanam@dit.ac.za

[2] *Institute of Fluid Flow Machinery, Polish Academy of Sciences,*
Fiszera 14, 80-952 Gdansk, Poland
jb@imp.gda.pl

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Abstract: The study investigates the control of structural response using collocated piezoelectric elements mounted on both sides of beam. A capacitive shunt is introduced into the circuitry to provide passive control of the beam's configuration. The energy method is used to develop the structure's governing equations. In order to select the appropriate electrostatic relations for the material, *i.e.* conductor or insulator material, the free charge rearrangement time is used. The effects of the capacitive shunt are introduced into the electrostatic relations using additive decomposition of capacitance. As an application of the model, a piezo-elastic beam restrictor using a capacitive shunt is investigated. Numerical results show that the capacitive shunt can effect passive control of the configuration of the beam.

Keywords: passive control, piezo-elastic, deflection

1. Introduction

Piezoelectric materials represent a relatively new approach to vibration suppression and static shape control. By their nature, piezoelectric materials are well suited for use as dampers and shape control elements. When an electric potential is applied to the piezoelectric material, a corresponding strain is observed. Conversely, straining a piece of piezoelectric material induces an electric potential across the element. Due to these properties, piezoelectrics can function as sensors and actuators for active monitoring and control of the structure's response.

Piezoelectric materials can also be used in components of passive damping systems, thus eliminating the need for complex control and feedback systems. They can provide passive damping by using electrical impedance as a dissipating shunt. As the structure deforms, the attached piezoelectric element deforms and an electric field is induced in response. The electrical shunt in turn dissipates the electrical energy, thereby altering the system's response.

We follow closely the work of Pietrzakowski [1] where it has been shown that a capacitive shunt alters the structure's response. This was achieved with the use of a non-homogeneous charge relation. Additionally, brackets were used to distance the piezoelectric elements from the surface of the beam.

In this study we show that electrons cannot move freely within the piezoelectric material (which is typical of all dielectrics) and thus do not contribute to the electric current in the direct and converse piezoelectric effects. Hence, the boundary condition for charge is a homogeneous relation. The effect of a capacitive shunt is investigated along those lines as well.

2. Problem formulation

2.1. Free charge rearrangement time

Since we are interested in what happens to a charge which is placed inside a conducting material, we will consider the continuity equation. The relation between the current density $\mathbf{J}$ and the time rate of change of the free charge density ρ at a point is given by the continuity equation:

$$\nabla\cdot\mathbf{J}+\frac{\partial\rho}{\partial t}=0, \qquad (1)$$

where t is time. Any current that will flow as a result of a charge with a free charge density ρ can be expressed using Ohm's law, $\mathbf{J}=\sigma\mathbf{E}$, where σ is electric conductivity and $\mathbf{E}$ is the electric field. This gives the divergence of the current density in terms of the electric field as $\nabla\cdot\mathbf{J}=\sigma\nabla\cdot\mathbf{E}$. The divergence of this electric field can be expressed in terms of charge density as $\nabla\cdot\mathbf{E}=\rho/\epsilon$, where ϵ is the dielectric constant. By substituting into Equation (1), the following differential equation is obtained for the time variation of charge density $\rho(t)$ at an arbitrary point within the conducting material:

$$\frac{\partial\rho}{\partial t}+\frac{\sigma}{\epsilon}\rho=0. \qquad (2)$$

The solution of this equation is given by:

$$\rho(t)=\rho_0 e^{-t/T}. \qquad (3)$$

We denote the free charge rearrangement time by $T=\epsilon/\sigma$. In the presence of signals that vary with time, the definition of a good conductor will be that the rearrangement time T must be very short in comparison with the signal's period. The converse applies for an insulator. We obtain the following for the time required for an initial distribution ρ_0 to decay to 36.8% of its value:

$$T_{\mathrm{PZT}}=\frac{\epsilon_r\epsilon}{\sigma}=\frac{1200\cdot 8.854\cdot 10^{-12}\ \mathrm{CV^{-1}m^{-1}}}{10^{-13}\ \mathrm{CV^{-1}m^{-1}s^{-1}}}=29.5\mathrm{h}, \qquad (4)$$

where ϵ_r is the relative dielectric constant for the PZT material and $\epsilon=8.854\cdot 10^{-12}\,\mathrm{F/m}$ is the dielectric constant of air. We note here that free electrons in the PZT material cannot move freely within the material and thus do not contribute to the current when an external potential is applied. For practical purposes, we can now conclude that no free charge can be deposited in or extracted from the volume of PZT materials.

2.2. Balance equations

We restrict our considerations to a fairly simple class of piezo-elastic materials which is concerned with laminates that are symmetric in their geometry and material properties about the middle surface. Within this class, the bending and extension coupling stiffness can be ignored. We also conceive of a spatially continuous representation of a piezo-elastic structure in which kinematic behaviour is represented by variables $\mathbf{u}$, a vector displacement field, and ϕ, a scalar electric potential.

The free energy density of the structure is assumed to be a homogeneous, convex and quadratic function F of the kinematic variables. A small change in the kinematic variables, at constant temperature, leads to:

$$\begin{aligned} dF(\mathbf{u},\phi) &= \frac{\partial F}{\partial \mathbf{u}} d\mathbf{u} + \frac{\partial F}{\partial \phi} d\phi \\ &= \mathbf{R} d\mathbf{u} - Q d\phi. \end{aligned} \tag{5}$$

The internal forces, $\mathbf{R}$, are conjugate to the displacement field vector, $\mathbf{u}$. The electric charge, Q, is conjugate to the electric potential, ϕ. The minus sign is chosen so that Q measures the outward flux of the electric charge through the enclosing surface. As a consequence of the assumption made on F, both $\mathbf{R}$ and Q are linear in $\mathbf{u}$ and ϕ, namely:

$$\begin{aligned} \mathbf{R} &= \mathbf{K}\mathbf{u} + \mathbf{G}\phi, \\ -Q &= \mathbf{G}^T \mathbf{u} - H\phi, \end{aligned} \tag{6}$$

where $\mathbf{K}$ is the Laplacian operator of the vector field, $\mathbf{G}$ is the gradient operator of the scalar field, $\mathbf{G}^T$ is the gradient operator of the vector field, H is the Laplacian operator of the scalar field and the T superscript indicates the transpose. For real materials, $\mathbf{K}$ and H are symmetric, and positive definite. Since $\mathbf{R}$ and Q are linear combinations of kinematic variables, it follows that F can be written as:

$$F = \frac{1}{2} \begin{Bmatrix} \mathbf{u} \\ \phi \end{Bmatrix}^T \begin{bmatrix} \mathbf{K} & \mathbf{G} \\ \mathbf{G}^T & -H \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \phi \end{Bmatrix}. \tag{7}$$

The total free charge enclosed in the volume is given by Gauss' law:

$$Q = \int_V \rho dV = \iint_S \rho_s dS dn, \tag{8}$$

where V is the volume enclosing the charge and ρ is the measure of free charge per unit volume. In view of Equation (4), charge deposited in the piezo volume will not be distributed on the surface immediately. Therefore, for practical purposes, we take $\rho_s = 0$, which means that the initial charge-free surface will remain charge free. Consequently, the electric flux vector per unit surface becomes zero.

Besides, piezoelectric materials (like all dielectrics) do not contain any free charge inside the bounding surface and no net free charge can be enclosed in the volume occupied by the material (in the context of current application). It then follows that the electric flux density vector $\mathbf{D}$ for the piezo-elastic structure is divergence-free, *i.e.* $\nabla \cdot \mathbf{D} = 0$. Hence, the total charge is a homogeneous function, *i.e.* $Q = 0$.

Let the external forces be denoted by $\mathbf{P}(t)$. In static problems, time simply measures the order of events rather than their real time. For static analysis, equilibrium is attained when

$$\mathbf{R}(t)=\mathbf{P}(t). \tag{9}$$

From Equations (6), (8) and (9), the balance equations of the piezo-elastic structure may be written as follows:

$$\begin{bmatrix} \mathbf{K} & \mathbf{G} \\ \mathbf{G}^T & -H \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \phi \end{Bmatrix} = \begin{Bmatrix} \mathbf{P} \\ 0 \end{Bmatrix}. \tag{10}$$

A similar result has been obtained in a study conducted by Kekana *et al.* [2] using variational principles to investigate the behaviour of piezo-elastic composite beams.

3. Application to beam analysis

3.1. Internal forces

A rectangular beam is considered of length L, width b and thickness h under a transverse bending load q with piezoelectric elements attached (see Figure 1). The beam is defined in the Cartesian co-ordinates x, y and z with axes x and y lying on the middle surface of the beam. The displacement field employed to analyse the problem is $u(x,y,z)=-zw_{,x}$, where $w=w(x)$ is the deflection of the reference surface in the z direction and the subscript after the comma denotes differentiation with respect to the variable following the comma. The associated strains are given as $\varepsilon=-zw_{,xx}$.

The free energy per unit volume of the piezo-elastic beam can now be written as:

$$F_V(w,\phi)=\frac{1}{2}z^2E(w_{,xx})^2+ze_{31}w_{,xx}\phi_{,z}-\frac{1}{2}g_{33}(\phi_{,z})^2, \tag{11}$$

where E is the elastic modulus, e_{31} – the piezoelectric constant, g_{33} – the dielectric constant. The internal forces per unit length and the electric charge density are:

$$R=EI\ w_{,xxxx}+\int ze_{31}\phi_{,zxx}dz, \tag{12}$$

$$\rho=ze_{31}w_{,xx}-g_{33}\phi_{,z}, \tag{13}$$

respectively, where $EI=\sum_{k=1}^{N}E_kI_k$ is the bending stiffness, $k=1,2,\ldots,N$, and N is the number of layers. For the portion of the beam bonded with the collocated piezo-elements on the top and bottom surfaces $N=3$, while $N=1$ for the rest of the beam. The balance equation now becomes:

$$P=EIw_{,xxxx}+\int ze_{31}\phi_{,zxx}dz, \tag{14}$$

$$0=ze_{31}w_{,xx}-g_{33}\phi_{,z}, \tag{15}$$

where Equations (4) and (8) have been enforced.

3.2. Capacitive shunt

Pietrzakowski [1] has investigated the design of semi-active absorbers in beams undergoing transverse vibration. The configuration comprises rectangular piezoelectric patches mounted on the beam's upper and lower surfaces. Brackets are used to distance the collocated piezoelectric elements from the beam's upper and lower surfaces. A capacitive shunt is also introduced with the purpose of altering the vibration

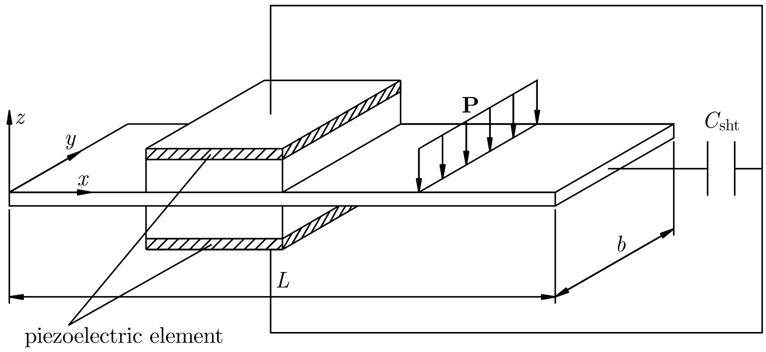

Figure 1. Piezo-elastic structure

characteristic. In [1], the Gaussian surface enclosed one side of the electrode so that $\epsilon\int \mathbf{E}\cdot d\mathbf{A}=\rho$, and the effect of the shunt was introduced by relating the enclosed charge ρ to the capacitance C_{sh}. In this study, the Gaussian surface encloses the entire dielectric so that $\epsilon\int \mathbf{E}\cdot d\mathbf{A}=0$. The capacitive shunt is introduced using additive decomposition of capacitance. It turns out that both approaches yield the same effect of the capacitive shunt in controlling the structure's response.

Equation (15) can be rewritten as:

$$0=e_{31}\frac{h_k+h_{k-1}}{2}w_{,xx}-C_k\frac{(h_k-h_{k-1})}{b_k(x_{k2}-x_{k1})}\phi_{,z}, \tag{16}$$

assuming a rectangular cross-section of the k^{th} layer. At the same time, C_k is the capacitance of layer k, *i.e.* $C_k=g_{33}^{(k)}b_k(x_{k2}-x_{k1})/(h_k-h_{k-1})$. This representation is required to facilitate additive decomposition of capacitance. The parallel coupling of the piezo-shunt units correspond to a state in which capacitance can be decomposed into the piezoelectric part, C_s, and the shunt part, C_{sh}, so that:

$$0=e_{31}\frac{h_s+h_{s-1}}{2}w_{,xx}-(C_s+C_{sh})\frac{(h_s-h_{s-1})}{b_s(x_{s2}-x_{s1})}\phi_{,z}. \tag{17}$$

This yields the electric field as follows:

$$\phi_{,z}=\frac{b_s(x_{s2}-x_{s1})e_{31}(h_s+h_{s-1})}{2(C_s+C_{sh})(h_s-h_{s-1})}w_{,xx}. \tag{18}$$

Now, the aim is to introduce the effect of varying the position of the piezoelectric elements with respect to the beam's neutral axis. The sensor potential is monitored as a function of the offset from the beam's neutral axis using:

$$\begin{aligned}\phi&=\alpha\frac{b_s(x_{s2}-x_{s1})e_{31}(h_s^2-h_{s-1}^2)}{2(C_s+C_{sh})(h_s-h_{s-1})}w_{,xx}\\&=\alpha\frac{b_s\beta l e_{31}(h_s^2-h_{s-1}^2)}{2C_s(1+\text{ratio})(h_s-h_{s-1})}w_{,xx},\end{aligned} \tag{19}$$

where $\alpha\geq 1$ is the offset parameter, $0\leq\beta\leq 1$, l is the length of the beam and $\text{ratio}=C_{sh}/C_s$. The electric field (18) may be used to decouple Equation (14) so that it yields the response in terms of displacements only. Under the thin beam assumptions adopted here, validity of the results will be governed by the beam's aspect ratio [3].

4. Numerical results

We consider a simply supported beam of unit length (*i.e.* $l = 1$m) subjected to a uniform load $q = 1$N/m. Piezoelectric patches are attached to the top and bottom surfaces of the beam. The symmetry conditions in geometry and material are observed. Continuity of the generalised forces and kinematic variables has also been enforced.

Figure 2 shows the rotation of the beam fibres for various lengths of the piezoelectric patches, βl, where $\beta = 0.125$, 0.250, 0.375 and 0.5. Figure 3 shows the deflection of the middle surface of the beam for the same lengths. It is noticeable that as the length of the piezoelectric elements is increased, deflections and slopes are reduced. This indicates that the weaker elastic part of the beam is reduced while the stiffer piezo-elastic part is increased. As a result, the entire beam becomes stiffer.

Figure 4 shows the maximum deflection at $0 \leq \text{ratio} \leq 1000$ for the β values given above. The results indicate that the influence of capacitive is bounded. At low shunt capacitance, *i.e.* $\beta = 0$, the stiffness of the piezo-elastic part of the beam is at its minimum. This is depicted by the maximum deflection at this state. As the shunt capacitance is increased, stiffness approaches its maximum. At this point, the slope of the deflection curve approaches zero.

Figure 5 shows the maximum deflection at $0 \leq \text{ratio} \leq 200$ and $0.1 \leq \beta \leq 0.5$. It can be observed here that the maximum influence on deflections occurs at low shunt capacitance. For higher shunt capacitance, an enormous increase in the capacitive shunt results in a minute change in deflection. Further studies need to be conducted in order to find the optimum capacitive shunt range.

According to these results, the most effective passive control of the capacitive shunt occurs at low shunt capacitance relative to the piezoelectric capacitance. As the shunt capacitance is increased to higher values relative to the piezo-capacitance, the effect of passive control becomes relatively weaker. It can also be noticed that the span of the piezoelectric elements across the host beam has a pronounced effect on controlling beam's configuration.

5. Conclusion

In this study we have investigated the control of a beam using collocated piezoelectric elements mounted on both sides of the beam. A capacitive shunt was introduced into the circuitry using additive decomposition. The energy method was used to develop the governing equations of the structure. The free charge rearrangement time was used to determine the charge boundary conditions. The effects of the shunt capacitor were introduced into the electrostatic relations using additive decomposition of capacitance. As an application of the model, a piezo-elastic beam restrictor was investigated.

The results have shown that the free charge rearrangement time is relatively high, so that no charge can be deposited in or removed from the volume of the PZT material in the current context. This leads to a homogeneous charge relation for the electrostatic equation. It is apparent from the numerics that the capacitive shunt does affect the configuration of the beam. At low shunt capacitance the controlling effect is pronounced, whereas at high shunt capacitance the effect of control decreases.

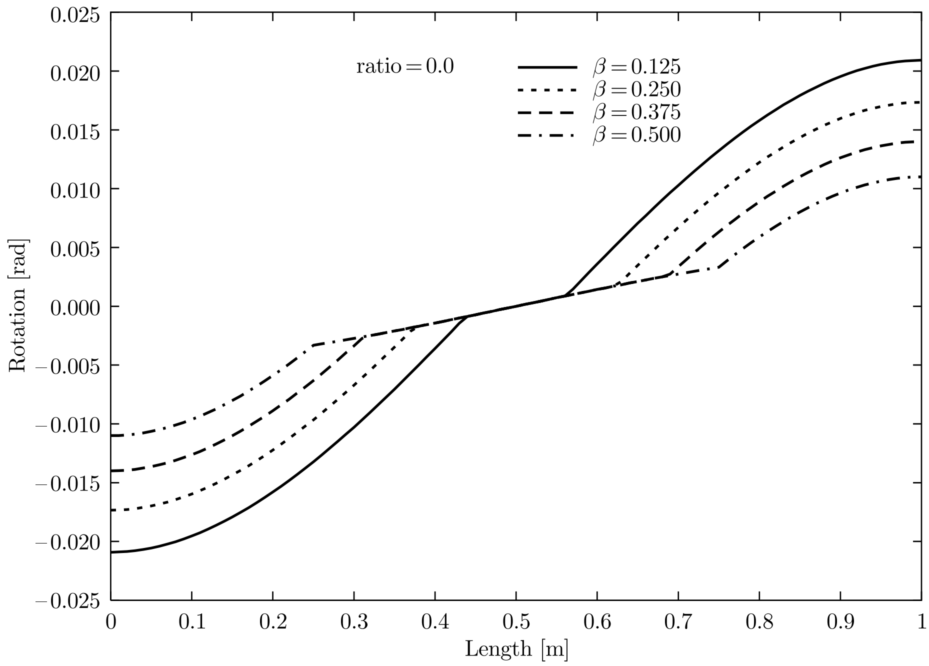

Figure 2. The slope of the beam at $\beta = 0.125$, 0.250, 0.375, 0.500

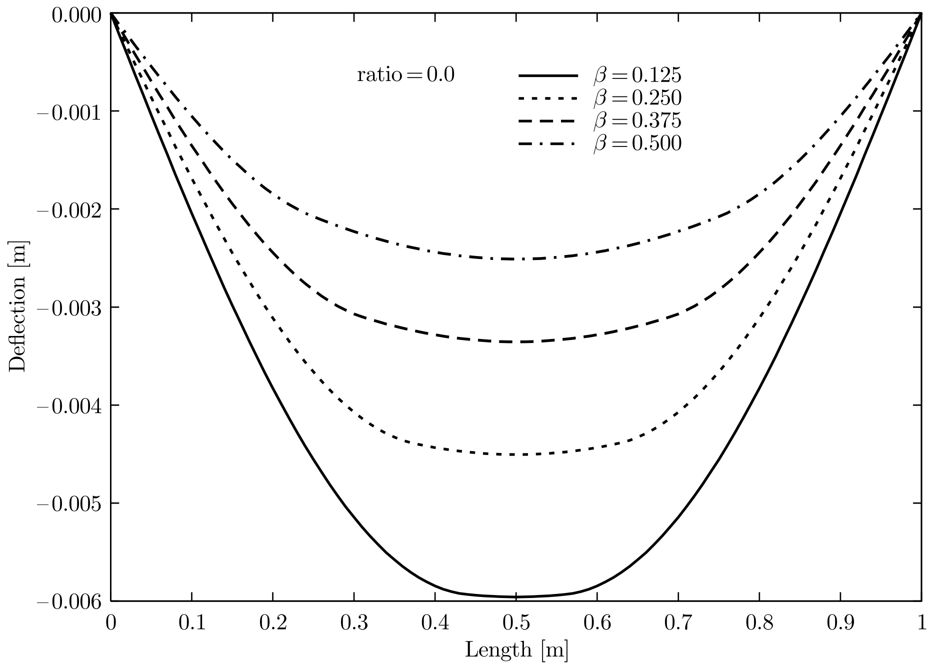

Figure 3. Deflection of the beam at $\beta = 0.125$, 0.250, 0.375, 0.500

Therefore, it can be claimed that introducing the capacitive shunt into the piezoelectric circuitry improves the passive control of the piezo-elastic structure. However, the validity of these numerics depend on results of experiments which remain to be conducted.

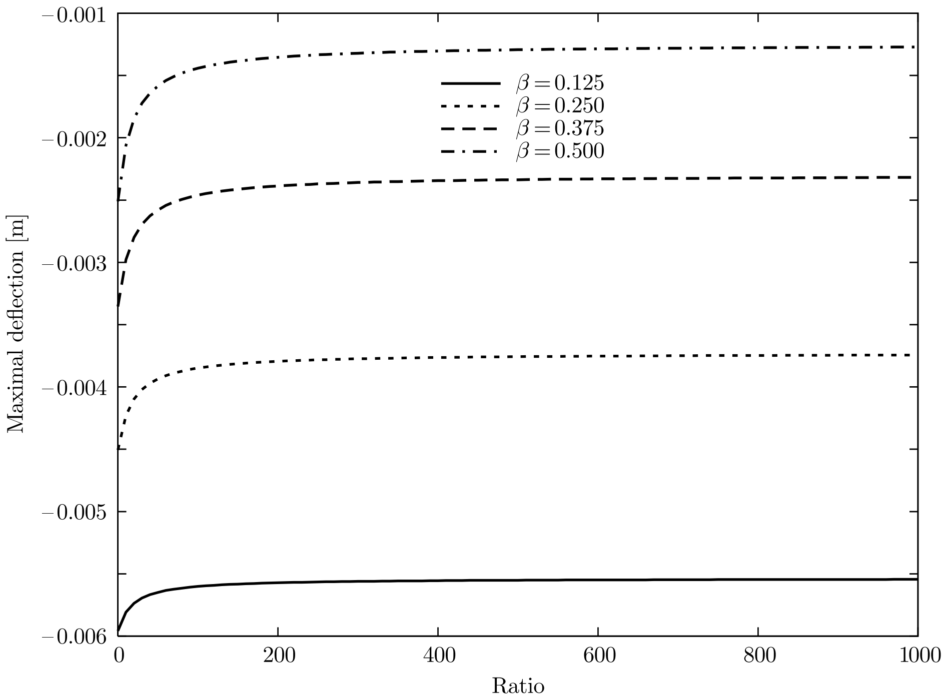

Figure 4. Maximum deflection versus ratio at $\beta = 0.125$, 0.250, 0.375, 0.500

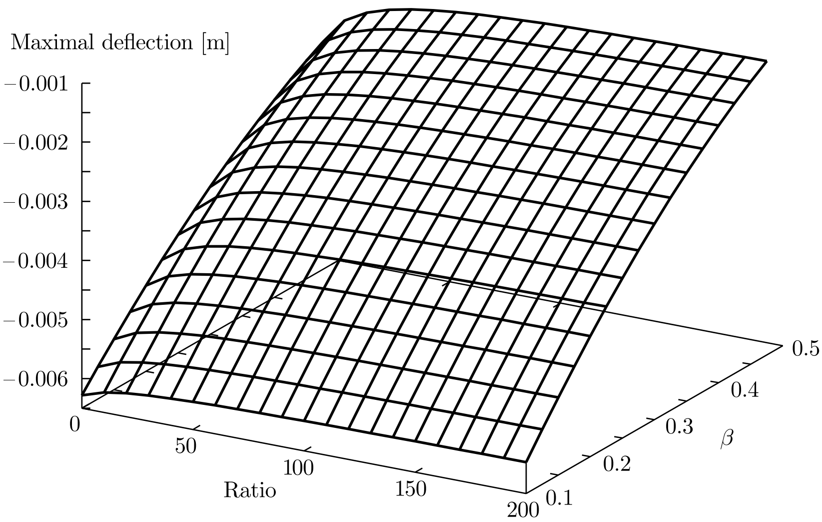

Figure 5. Maximum deflection at $0 \leq \text{ratio} \leq 200$ and $0.1 \leq \beta \leq 0.5$

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