THE INFLUENCE OF WAVE-NOISE ON WAVE SPEEDS AND AMPLITUDES OF SURFACE-GRAVITY WAVES

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Abstract: We have analytically examined surface-gravity waves which propagate in space- and time-dependent random velocity fields. Using a perturbative method, we have derived a dispersion relation which is solved for the case of wave-noise whose spectrum $E(k,\omega) \sim E(k)\delta(\omega - c_r k)$, where δ is Dirac's delta-function and c_r is the random phase speed. We have found that for a dispersionless noise resonance occurs when c_r is equal to the group velocity c_g of the surface-gravity wave. In this resonance the real part of the wave frequency is finite, but its imaginary part exhibits the characteristic $1/x$ singularity. The wave-noise interacts with a packet of the surface-gravity waves in such a way that the waves are attenuated for $c_r < c_g$ and are amplified for $c_r > c_g$. As the real part is positive for high values of k, the surface-gravity waves are accelerated by the wave-noise.

Keywords: random waves, dispersion relation, wave-noise

1. Introduction

Random waves have been the subject of intensive studies. For instance, random sound waves in a weakly stratified atmosphere has been discussed by Murawski (2002) [1]. Wave propagation in random media, with oceanic applications, has been reviewed by Mysak (1978) [2]. In another context, the theory of sound propagation in media of random sound speed, density, and flow speed in the atmosphere and ocean has been discussed by Ostashev (1994) [3]. The author has considered the Born-approximation, ray, Rytov and parabolic-equation methods and the theory of multiple-scattering. These techniques can be used to develop new remote-sensing methods for the atmosphere and ocean seismology (Ostashev 1997 [4]). As a particular application we can mention surface-gravity waves which propagate in a basin of random bottom (Pelinovsky, Razin and Sasorova 1998 [5]). The analytical findings for the exponential correlation function have revealed that these waves are attenuated and their wave speed is altered. The attenuation reaches its maximum value at an intermediate value of the normalized horizontal wavenumber kh_0, but it is negligibly small for high values of kh_0. Here, h_0 is the mean depth of the basin. The wave speed is lower for shallow water and higher for deep water. This has direct implications for explaining the increase of tsunami travel time variations which are lower than 1–2 seconds even for transoceanic paths (Pelinovsky, Razin and Sasorova 1998 [5]). At the same time, Kawahara (1996) [6] has found that the effect of a random bottom on surface gravity waves is attenuation and wave speed variations. These results have been presented in the form of a non-linear Schrödinger equation which shows that the wave speed increases for long waves and decreases for short waves.

In another approach, it has been shown that a space-dependent Gaussian random flow produces a reduction of the surface-gravity wave frequency and its amplitude, while a time-dependent flow produces an increase (Murawski and Roberts 1993 [7]). Thus, the surface-gravity wave is attenuated and decelerated by a space-dependent Gaussian random, and amplified and accelerated by a time-dependent flow (Murawski 2000 [8]). Nevertheless, the treatment of surface-gravity waves is still unsatisfactory as no space- and time-dependent random fields have been discussed so far. The simplest realization of such a field is wave-noise, which has been studied in the context of sound waves by Murawski, Nocera and Pelinovsky (2001) [9]. It has been found that when the wave speed of wave-noise (which represents a random mass density field) c_r, is equal to the phase speed of sound waves, ω/k, resonance occurs at which the cyclic frequency tends to infinity. For values of c_r which are close to the resonance point, the frequency shift may be negative or positive; the imaginary part of the frequency attains the negative sign for $c_r < \omega/k$ and the positive sign for $c_r > \omega/k$.

The main goal of this paper is to examine the influence of a space- and time-dependent random flow in the form of non-dispersive wave-noise in frequencies and amplitudes of the surface-gravity wave. This goal is additionally motivated by the fact that the surface-gravity wave is dispersive, while sound waves are non-dispersive. Thus, it is interesting to see how these waves behave in wave-noise. This research is also motivated by the fact that it has been shown that flows can affect waves, modifying dispersion relations and changing line widths (Nakariakov and Roberts 1995 [10], Nakariakov, Roberts and Murawski 1998 [11], Pintér, Erdélyi and New 2001 [12]).

We start by introducing wave-noise in the next section. In Section 3, we present the dispersion relation for a random surface-gravity wave which propagates along the interface between the incompressible bottom medium and the evacuated atmosphere. In the following section, we investigate the influence of a random flow on wave frequencies and amplitudes. We conclude the paper with a summary of the main results in Section 5.

2. Wave-noise

We define wave-noise by introducing the correlation function of a random field f_r, *viz.*:

$$R(|x_2-x_1|,|t_2-t_1|)=\langle f_r(x_2,t_2)f_r(x_1,t_1)\rangle, \qquad (1)$$

where x_i and t_i, $i=1,2$, denote, for simplicity, normalized spatial and temporal points, and $\langle\rangle$ stands for the ensemble average (Ostashev 1997 [4]). The Fourier transform of the correlation function is:

$$E(K,\Omega)=\int_{-\infty}^{\infty} e^{-i(Kx-\Omega t)}R(x,t)dx\,dt \tag{2}$$

with K as normalized wavenumber and Ω as dimensionless frequency.

For wave-noise we have:

$$E(K,\Omega)=\frac{\sigma^2}{\pi}E(K)\delta(\Omega-\Omega_r(K)), \tag{3}$$

where Ω_r is the K-dependent frequency of the random fluctuations. We specialize to *non-dispersive noise*:

$$\Omega_r(K)=c_rK, \tag{4}$$

where c_r is the normalized phase speed of random noise. For such noise, the initial random profile $f_r(x,t=0)$ is translated in time t by the distance c_rt such that $f_r(x,t)=f_r(x-c_rt,t=0)$. For practical applications, we shall use henceforth the normalized Gaussian spectrum:

$$E(K)=\frac{1}{\pi}e^{-K^2}. \tag{5}$$

Although this spectrum may not be completely physically justifiable for each medium, it is quite convenient for an analytical exercise.

3. Surface-gravity waves in a random velocity field

We consider a surface-gravity wave that propagates along an interface between two semi-infinite layers of perfect gas under constant gravity, g, which is assumed to be pointing in the z-direction. The simplest case considered is a closed medium at $z=0$, *i.e.* all the atmospheric effects are excluded. As the surface-gravity wave is incompressible ($\nabla\cdot\vec{V}=0$) and the plasma is assumed to be magnetic field-free, the motions in the bottom medium are described by incompressible hydrodynamic equations (Lighthill 1978 [13]).

The normalized wavenumber, $K=kl_x$, and frequency, $\Omega=\sqrt{l_x/g}\,\omega$, of small amplitude surface-gravity waves which propagate in a weak random velocity field, $V_r(x,t)$, satisfy the following dispersion relation (Murawski 2000 [8]):

$$gl_x(\Omega^2-K)=4\Omega K\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\hat{K}\hat{\Omega}E(\hat{\Omega}-\Omega,\hat{K}-K)}{\hat{\Omega}^2-\hat{K}}d\hat{K}d\hat{\Omega}, \tag{6}$$

where l_x is the correlation length, and g – surface gravity.

We now consider a surface-gravity wave whose wavenumber is K. In view of the smallness of σ^2, its frequency, Ω, can be expanded as:

$$\Omega=\sqrt{K}+\sigma^2\Omega_2+\cdots. \tag{7}$$

Substituting Equations (3)–(7) into Equation (6) we obtain:

$$\bar{\Omega}_2\equiv gl_x\Omega_2=\frac{2K}{\pi^{3/2}}\left[c_r-2\left(c_r(K+w)+\left(1+\frac{K}{w}\right)\sqrt{K}\right)D(w)\right]-$$

$$i\frac{2}{\pi}K\left[\frac{K^{3/2}}{w}+\mathrm{sign}(c_r)\left(c_r(K+w)+\left(1+\frac{K}{w}\right)\sqrt{K}\right)e^{-w^2}\right], \tag{8}$$

where

$$w=\frac{1-2c_r\sqrt{K}}{c_r^2} \tag{9}$$

and

$$D(\xi)=e^{-\xi^2}\int_0^{\xi}e^{\zeta^2}d\zeta \tag{10}$$

is the Dawson integral (Press *et al.* 1992 [14]).

It is noteworthy that when $w=0$ or

$$c_r=\frac{1}{2\sqrt{K}}=c_g, \tag{11}$$

resonance occurs. Here, c_g is the group velocity of the surface-gravity wave. At this resonance, the real part of $\bar{\Omega}_2$ is finite, but the imaginary part of $\bar{\Omega}_2$ attains an infinite value.

4. Numerical results

Figure 1 exhibits the normalized frequency correction, $\bar{\Omega}_2$, for a given value of wave-noise speed, c_r. It is interesting that in the case of $c_r=-1$ (top panel) the

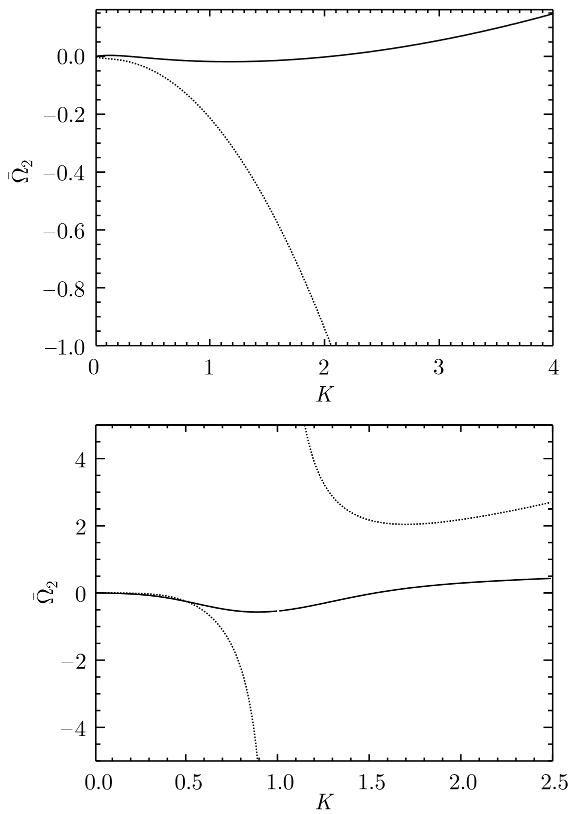

Figure 1. Real (solid line) and imaginary (dashed line) parts of the frequency correction $gl_x\Omega_2$ versus the normalized wavenumber $K=kl_x$ for $c_r=-1$ (top panel) and $c_r=1/2$ (bottom panel)

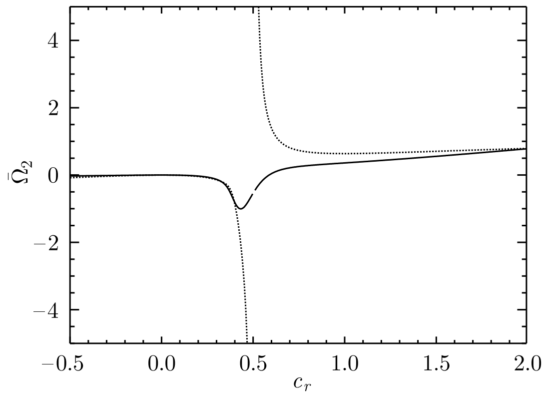

Figure 2. Real (solid line) and imaginary (dashed line) parts of the frequency correction $gl_x\Omega_2$ versus c_r for the normalized wavenumber $K = kl_x = 1$

imaginary part of $\bar{\Omega}_2$ is negative over the whole range of K. This results in wave attenuation, the process which leads to an amplitude decrease in time. The real part of $\bar{\Omega}_2$ is positive and, consequently, frequencies of the surface-gravity wave increase and the random effect grows with K. Thus, wave-noise, while propagating in the direction opposite to the direction of the surface-gravity wave, transfers some energy into the wave, whose speed increases.

The bottom panel of Figure 1 shows the normalized frequency correction $\bar{\Omega}_2$ for a down-wave propagating wave-noise with a speed of $c_r = 1/2$. According to formula (11), resonance occurs at $K = 1$. This resonance should be present in the imaginary part of $\bar{\Omega}_2$, while Equation (8) shows that the real part of $\bar{\Omega}_2$ attains a finite value. Indeed, this kind of resonance is discernible in the bottom panel of Figure 1. On its left side the wave is attenuated, while it is amplified for $K > 1$. The closer K approaches the unity, the more pronounced the process. As the real part of the frequency correction is negative for low values of K, the surface-gravity wave is decelerated by a down-wave propagating wave-noise.

Figure 2 displays $\bar{\Omega}_2$ versus c_r for $K = 1$. The imaginary part of $\bar{\Omega}_2$ exhibits the $1/x$-type singularity at $c_r = 1/2$. For $c_r = 1/2^-$ the imaginary part of $\bar{\Omega}_2$ grows to minus infinity, the process which leads to wave attenuation. For $c_r > 1/2$ the imaginary part of $\bar{\Omega}_2$ is positive, leading to wave amplification. The real part of $\bar{\Omega}_2$ continues at this point. As it is negative at the resonant point, the surface-gravity wave is decelerated there.

5. Summary and discussion

In this paper we have studied the propagation of surface-gravity waves in a random velocity field given in the form of wave-noise. The main result of our research is that these waves experience resonance when the speed of the wave-noise, c_r, is equal to the group velocity of the surface-gravity wave. At this resonance, the real part of the wave frequency continues, but its imaginary part exhibits the characteristic $1/x$ singularity. Surface-gravity waves may transfer their energy into the random flow as the imaginary part of the frequency is negative for values of c_r lower than the wave group velocity. For higher values of c_r, surface-gravity waves may gain energy from the wave-noise, as the imaginary part is positive. The real part of the frequency of a surface-gravity wave attains positive values for high K and c_r, but it is negative for long waves and close to the resonance point.

There is a notable difference between the behavior of surface-gravity waves and sound waves. For sound waves, resonance occurs when their phase speed ω/k is equal to the phase speed of the wave-noise, c_r (Murawski, Nocera and Pelinovsky 2001 [9]). At this resonance, both the real and imaginary parts of the cyclic frequency, ω, of a sound wave tend to infinity. For values of c_r close to the resonance point, the waves are decelerated and attenuated for $c_r < \omega/k$ and accelerated and amplified for $c_r > \omega/k$. The possibility of amplification is akin to the the case of a time-dependent random density field, which has been discussed by Murawski, Nocera and Mędrek (2001) [15]. As non-random sound waves are non-dispersive and a surface-gravity wave is dispersive, we conclude that this property is important as far as wave propagation in non-dispersive wave-noise is concerned.

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