SOUND WAVES IN A WAVE NOISE

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Abstract: We examine by analytical and numerical means sound waves which propagate in a space- and time-dependent random mass density field in the form of dispersionless wave noise of its spectrum $E(k,\omega) \sim E(k)\delta(\omega - c_r k)$, where c_r is a random speed. Numerical simulations are in agreement with the analytical theory which shows that at $c_r = \omega/k$ resonance occurs and the cyclic frequency ω tends to infinity. For values of c_r which are close to the resonance point, the sound waves are slowed down and attenuated (accelerated and amplified) for $c_r < \omega/k$ ($c_r > \omega/k$).

Keywords: sound waves, random waves, frequency shift, amplitude alteration

1. Introduction

It is known that the dimensionless wave vector $K = kl_x$ and the dimensionless frequency $\Omega = \omega l_x/c_0$ of small-amplitude sound waves propagating in a weak random mass density field $\varrho_r(x,t)$ satisfy the following dispersion relation [1, 2]:

$$\Omega^2 - K^2 = \Omega^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\hat{\Omega}^2 E(\hat{K}-K,\hat{\Omega}-\Omega)}{\hat{\Omega}^2 - \hat{K}^2} \mathrm{d}\hat{K}\mathrm{d}\hat{\Omega}. \tag{1}$$

Here, l_x is the correlation length, $c_0 = \sqrt{\gamma p_0/\varrho_0}$ is the speed of sound at the equilibrium, and E is the Fourier transform of the correlation function $\langle \varrho_r(x,t)\varrho_r(X,T)\rangle/\varrho_0^2$, where $\varrho_r(x,t)$ is a statistically homogeneous random density [3] with a vanishing ensemble average $\langle \varrho_r \rangle$.

It follows from the dispersion relation of Equation (1) that the random sound waves are no longer dispersionless as they experience not only a frequency shift but also an amplitude alteration due to the presence of the random field.

Special limits of Equation (1) have already been considered in the literature. The case of a space-dependent random field $\varrho_r = \varrho_r(x)$ was discussed by [1]. The main conclusion was that sound waves were attenuated by any random field. For Gaussian statistics:

$$E(K) = \frac{\sigma^2}{\pi}\exp(-K^2), \tag{2}$$

sound wave frequencies were increased. Here, $\sigma^2 \ll 1$ is a small parameter which enforces the weakness of random density fluctuations.

The case of a time-dependent random field $\varrho_r = \varrho_r(t)$ was discussed by [2], who showed that such a random field leads to energy transfer from the random field to the sound waves and consequently to wave amplification. Moreover, for a Gaussian random field the frequencies of the sound waves were increased, similarly as in the case of a space-dependent random field.

It is then natural to inquire about the properties of sound waves propagating through the medium of a space- and time-dependent random mass density field. In the following section of this work, we concentrate on the simpler case of *wave noise*.

The paper is organized as follows. In Section 2 we present the dispersion relation for sound waves in wave noise and investigate the influence of random mass densities on frequencies and amplitudes of these waves. Numerical simulations are described in the following part of the paper and their results are presented in Section 4. This paper is concluded with a presentation of the main results in Section 5.

2. Frequency and amplitude alteration by wave noise

We define wave noise through the spectrum:

$$E(K,\Omega) = \frac{\sigma^2}{\pi}E(K)\delta(\Omega - \Omega_r(K)), \tag{3}$$

where Ω_r is the K-dependent frequency of random density fluctuations, and specialize to *dispersionless noise*:

$$\Omega_r(K) = c_r K, \tag{4}$$

where c_r is the phase speed of random noise. For such noise, the initial random mass density profile $\varrho_r(x,t=0)$ is translated in time t by the distance $c_r t$ such that $\varrho_r(x,t) = \varrho_r(x - c_r t, t=0)$. For practical applications, we shall use the Gaussian spectrum of Equation (2) henceforth. Let us now consider a sound wave whose wave number is K. In view of the smallness of σ^2, its frequency, Ω, given by the dispersion relation in Equation (1), can be expanded as:

$$\Omega = K + \sigma^2\Omega_2 + \cdots. \tag{5}$$

Substituting Equations (3)–(5) into Equation (1) and performing an integration, we obtain:

$$\Omega_2 = \frac{1}{8\pi^{3/2}}K(2 - i\sqrt{\pi}K), \quad c_r = -1, \tag{6}$$

$$\Omega_2 = \frac{K}{2\pi^{3/2}}\left[\frac{c_r^2}{c_r^2 - 1} - \frac{c_r - 1}{(c_r+1)^2}KD\left(\frac{2}{c_r+1}K\right)\right] + i\frac{K^2}{4\pi}\left[\frac{1}{c_r - 1} + \left|\frac{c_r - 1}{c_r + 1}\right|\frac{1}{c_r+1}\exp\left(-\frac{4K^2}{(c_r+1)^2}\right)\right], \quad c_r \neq \pm 1. \tag{7}$$

It follows from Equation (7) that resonance occurs at a place where the phase speed of the wave noise equals the sound-wave speed. Figure 1 shows the resonance

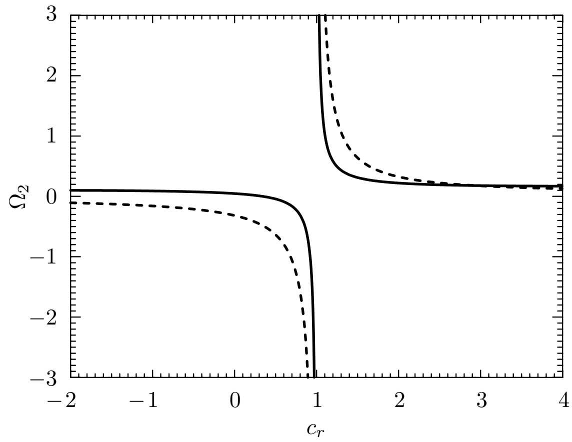

Figure 1. Real (solid line) and imaginary (dashed line) parts of the frequency reduction Ω_2 as functions of c_r for $K=2$

curve for $K=2$. For other values of K the curve is similar. Note that the resonance is of the $1/c_r$-type; for $c_r=1^-$ ($c_r=1^+$) the real and imaginary parts of the frequency shift are negative (positive) and the sound waves are decelerated and attenuated (accelerated and amplified) there. The wave deceleration and attenuation can be explained on physical grounds, as for $c_r=1^-$ the sound wave interacts with the slower-propagating wave noise. This process is accompanied with an energy transfer from the sound wave into the wave noise, leading to sound wave deceleration and attenuation. At the same time, in the $c_r=1^+$ regime the wave noise moves quicker than the sound wave and the energy is transferred into the latter. Consequently, the sound wave is amplified and accelerated.

3. Numerical simulations for hydrodynamic equations

In this section, we present the results of the numerical simulations for the following hydrodynamic equations:

$$\frac{\partial\varrho}{\partial t}+\frac{\partial(\varrho V)}{\partial x}=S_\varrho, \tag{8}$$

$$\varrho\left(\frac{\partial V}{\partial t}+V\frac{\partial V}{\partial x}\right)=-\frac{\partial p}{\partial x}, \tag{9}$$

$$\frac{\partial p}{\partial t}+\frac{\partial(pV)}{\partial x}=(1-\gamma)p\frac{\partial V}{\partial x}, \tag{10}$$

where ϱ is mass density, V is the x-component of flow velocity, p is pressure, γ is the adiabatic index, and S_ϱ is a source term. These simulations are performed with the CLAWPACK code [4], which is a packet of Fortran routines for solving hyperbolic equations. The code utilizes a Godunov-type method [5].

3.1. The initial condition

We solve Euler Equations (8)–(10) numerically by adopting the appropriate code from the CLAWPACK package [4]. This package uses modern shock capturing schemes for solving hyperbolic equations such as the Euler equations [4, 5]. Numerical simulations are performed in the region:

$$0 \le x \le 200. \tag{11}$$

Here, the spatial coordinate x is normalized by $c_0 l_t$, and l_t is the correlation time. At $x=0$ and $x=200$ free boundary conditions are set in, so that a sound wave can freely leave the simulation region. The simulations are carried over time:

$$0 \le t \le 190 \tag{12}$$

normalized by l_t.

Initially, at $t=0$, an equilibrium state is set as follows:

$$\varrho(x,t=0)=\varrho_0=\text{const.},\quad p(x,t=0)=p_0=\text{const.} \tag{13}$$

This equilibrium is perturbed by a small pulse:

$$V(x,t=0)=10^{-3}\cdot\exp\left[-\frac{(x-5)^2}{4}\right], \tag{14}$$

where we normalize velocity, V, by sound speed, c_0. This pulse splits into two counter-propagating pulses. In the non-random medium, for which $\sigma=0$, they are of identical amplitude equal to $5\cdot 10^{-4}$. The left-ward propagating wave reaches the left boundary of the simulation region at $t=5$ and is lost from sight as the boundary is transparent for any outgoing signal.

The above pulse synthesizes from Fourier components whose spectrum is Gaussian. As the shortest waves are most affected by a random field [2], we expect that pulses will be altered by the presence of a random field. For instance, in a random medium a pulse will generally be shifted to a different spatial position and its amplitude will be altered.

3.2. Seeding the random field

The random field is seeded through the term S_ϱ in Equation (8), chosen so that:

$$S_\varrho=\frac{\partial\varrho_r}{\partial t},$$
$$\varrho_r(x_m,t)=\varrho_r(x_m-c_r t,t=0),$$
$$\varrho_r(x_m,t=0)=\sqrt{\frac{2}{N}}\mathrm{Re}\sum_{n=0}^{N-1}\bar{\varrho}(K_n)e^{(-2i\pi m\frac{n}{N}+i\phi_n)}, \tag{15}$$

with the amplitude $\bar{\varrho}(K_n)$:

$$\bar{\varrho}(K_n)=\sqrt{E(K_n)},\qquad K_n=\frac{2\pi n}{N\Delta t}. \tag{16}$$

Here, $0\le\phi_n\le 2\pi$ is a uniformly distributed random phase computed by the `ran1` random number generator [6].

Space is sampled over $N=4\,096$ points such that:

$$0\le x_n=n\Delta x\le x_{N-1}=200,\quad n=0,1,2,\cdots,N-1,\quad \Delta x=\frac{x_{N-1}}{N}.$$

Figure 2 shows the initial random mass density obtained with Equation (15) for a particular realization of the random density field at $t=0$.

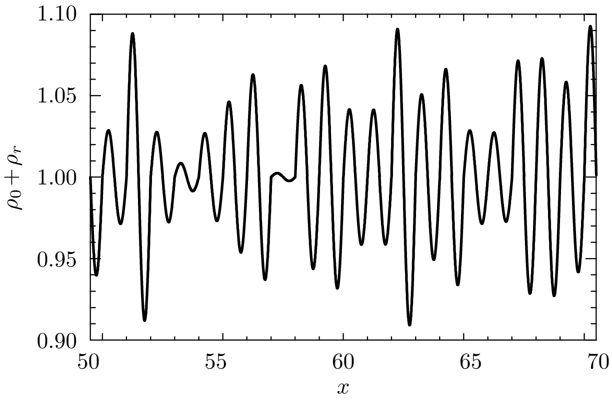

Figure 2. Random mass density as a function of time, x, for a typical realization of a random medium

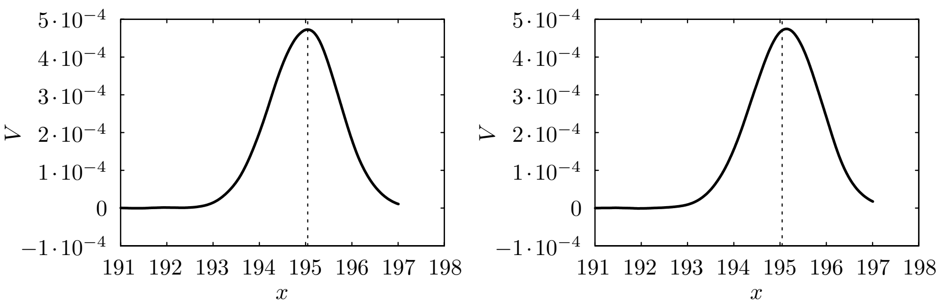

Figure 3. The spatial profile of $V(x,t=190)$ for $c_r=0.95$ (left) and $c_r=1.05$ (right); the dashed line marks the spatial position of the non-random pulse

4. Numerical results

Figure 3 shows spatial profiles of ensemble averaged pulses for $c_r=0.95$ (left panel) and $c_r=1.05$ (right panel). For $c_r=0.95$ the pulse decelerates and for $c_r=1.05$ the pulse accelerates. It is discernible that the pulse amplitudes are reduced, but for $c_r=1.05$ the pulse is wider than in the case of $c_r=0.95$. As these effects are fine, in order to display numerical results we have introduced the coordinate of the geometrical center:

$$x_c \equiv \frac{\int_0^{x_{N-1}} xV\,dx}{\int_{x_0}^{x_{N-1}} V\,dx}, \tag{17}$$

which measures whether a packet of sound waves is accelerated or decelerated by a random field. Figure 4 displays x_c as a function of c_r. It is discernible from this figure that – in an agreement with the analytical findings of Figure 1 for $c_r<1$ ($c_r>1$) – the wave pulses decelerate (accelerate).

5. Summary

In this paper we have studied numerically the propagation of sound impulses in a random mass density field in the form of *wave noise*. The main conclusion of our investigation is that these waves experience resonance when their phase speed,

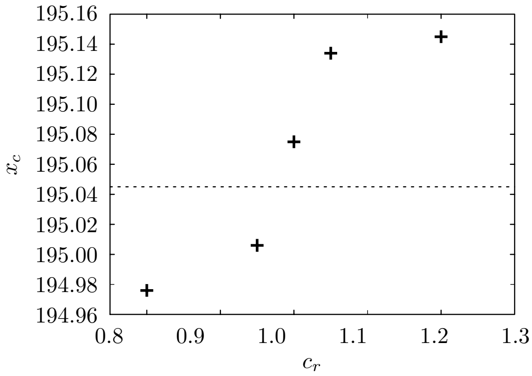

Figure 4. The coordinate of the geometrical center x_c versus the wave noise speed c_r; the horizontal dashed line denotes x_c for the pulse which propagates in a non-random medium with $\sigma = 0$

ω/k, equals the phase speed of the wave noise, c_r. Close to this resonance but for $c_r < \omega/k$ ($c_r > \omega/k$) the sound waves decelerate and attenuate (are accelerated and amplified) as they propagate. This possibility of amplification is akin to the case of a time-dependent random density field discussed by [2].

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