THE CONSISTENCY CONDITIONS FOR DENSITY LIMITS OF HYPOPLASTIC CONSTITUTIVE LAW

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Abstract: The so called phase diagram of grain skeletons illustrates the range of possible void ratios between the pressure dependent bounds e_i and e_d. It can be shown that in the framework of the actual hypoplastic model these bounds can be surpassed by particular deformation paths. This inconsistency is particularly acute in recently proposed FE-calculations with density fluctuations. Here we propose a modification to render the formulation consistent.

Keywords: hypoplasticity, intergranular strain, phase diagram, limit void ratio

1. Introduction

The theory of hypoplasticity has been developed by Kolymbas and Gudehus and their associates since the late seventies [7, 8, 10, 13, 9, 3, 1, 12]. Extensions of the hypoplastic model enabled to challenge real engineering problems. In the recent version described in detail in the article by Herle and Gudehus [5] a realistic description of initially dense and initially lose grain skeletons for a wide range of pressure with a single set of parameters related to physical properties is obtained. Therein the range of possible void ratios is bounded by pressure dependent limit void ratios forming a surface in the e–tr$\mathbf{T}$ space. However, using these extensions we are often confronted with the lack of robustness of the numerical calculation. Although for some deformation paths a surpassing of the bounds may be physically accurate [2] the transition requires a change in the material model. However, these kind of problems are not very well understood so far. We propose a small modification to improve the lack of numerical robustness caused by surpassing the density limits. A local violation of the validity range of the constitutive law should not abruptly break the calculation. The error should be clearly reported, but in our opinion, it should be the user who decides whether to interrupt the program or not.

In this paper we assume that the reader is familiar with the recent version of the hypoplastic model [5] and with its extension for small strains [14]. For completeness we repeat briefly all the formulas of the hypoplastic model in the next section.

2. The reference version of hypoplastic model

2.1 The basic equations of the hypoplastic model

The hypoplastic constitutive model is generally described by a single nonlinear tensorial equation that yields the stress rate $\overset{\circ}{\mathbf{T}}$ (objective) with the stretching rate $\mathbf{D}$ [11]:

$$\overset{\circ}{\mathbf{T}} = \mathbf{L} : \mathbf{D} + f_d \mathbf{N} \|\mathbf{D}\| . \tag{1}$$

Multiplication $\mathbf{L} : \mathbf{D}$ corresponds to $L_{ijkl} D_{kl}$ in index notation. The relation between the Cauchy stress rate $\overset{\circ}{\mathbf{T}}$ and the objective (Zaremba–Jaumann) stress rate $\overset{\circ}{\mathbf{T}}$ is:

$$\dot{\mathbf{T}} = \overset{\circ}{\mathbf{T}} + \mathbf{W} \cdot \mathbf{T} - \mathbf{T} \cdot \mathbf{W} , \tag{2}$$

where $\mathbf{W}$ denotes the spin tensor [11] and the multiplication $\mathbf{W} \cdot \mathbf{T}$ corresponds to $W_{ik} T_{kj}$ in index notation. The constitutive tensors $\mathbf{L}(\mathbf{T}, e)$ and $\mathbf{N}(\mathbf{T}, e)$ are functions of stress and void ratio. These functions are of essential importance for the quality of predictions by hypoplastic model. The mathematical representation of $\mathbf{L}(\mathbf{T}, e)$ and $\mathbf{N}(\mathbf{T}, e)$ following Herle and Gudehus [5] reads:

$$\mathbf{L} = f_b f_e \frac{1}{\hat{\mathbf{T}} : \hat{\mathbf{T}}} \left(F^2 \mathcal{I} + a^2 \hat{\mathbf{T}} \hat{\mathbf{T}} \right) , \tag{3}$$

$$\mathbf{N} = f_b f_e \frac{Fa}{\hat{\mathbf{T}} : \hat{\mathbf{T}}} \left(\hat{\mathbf{T}} + \hat{\mathbf{T}}^* \right) , \tag{4}$$

with

$$\hat{\mathbf{T}} = \mathbf{T} / \operatorname{tr} \mathbf{T} , \quad \hat{\mathbf{T}}^* = \hat{\mathbf{T}} - \frac{1}{3} \mathbf{1} , \tag{5}$$

$$a = \frac{\sqrt{3} (3 - \sin \varphi_c)}{2\sqrt{2} \sin \varphi_c} , \tag{6}$$

$$F = \sqrt{\frac{1}{8} \tan^2 \psi + \frac{2 - \tan^2 \psi}{2 + \sqrt{2} \tan \psi \cos 3\theta}} - \frac{1}{2\sqrt{2}} \tan \psi , \tag{7}$$

$$\tan\psi = \sqrt{3}\left\|\hat{\mathbf{T}}^*\right\|, \quad \cos 3\theta = -\sqrt{6}\frac{\operatorname{tr}\left(\hat{\mathbf{T}}^* \cdot \hat{\mathbf{T}}^* \cdot \hat{\mathbf{T}}^*\right)}{\left[\hat{\mathbf{T}}^* : \hat{\mathbf{T}}^*\right]^{3/2}} . \tag{8}$$

Symbol $\mathcal{I}$ denotes the fourth order unit tensor with components $I_{ijkl} = \delta_{ik}\delta_{jl}$. The multiplication $\hat{\mathbf{T}}\hat{\mathbf{T}}$ denotes the outer product. It reads $\hat{T}_{ij}\hat{T}_{kl}$ in index notation. For $\hat{\mathbf{T}}^* = \mathbf{0}$ is $F = 1$. The parameter φ_c corresponds to the critical friction angle. The scalar factors f_b, f_e and f_d take into account the influence of mean pressure (barotropy) and density (pyknotropy) [3, 1]:

$$f_b = \left(\frac{e_{i0}}{e_{c0}}\right)^{\beta} \frac{h_s}{n}\frac{1+e_i}{e_i}\left(\frac{-\operatorname{tr}\mathbf{T}}{h_s}\right)^{1-n}\left[3 + a^2 - a\sqrt{3}\left(\frac{e_{i0}-e_{d0}}{e_{c0}-e_{d0}}\right)^{\alpha}\right]^{-1}, \tag{9}$$

$$f_e = \left(\frac{e_c}{e}\right)^{\beta}, \tag{10}$$

$$f_d = \left(\frac{e-e_d}{e_c-e_d}\right)^{\alpha} . \tag{11}$$

Three characteristic void ratios — e_i (during isotropic compression at the minimum density), e_c (critical void ratio) and e_d (maximum density) — decrease with mean pressure according to a heuristic relation by Bauer [1]:

$$\frac{e_i}{e_{i0}} = \frac{e_c}{e_{c0}} = \frac{e_d}{e_{d0}} = \exp\left[-\left(\frac{-\operatorname{tr}\mathbf{T}}{h_s}\right)^{n}\right] = h(\operatorname{tr}\mathbf{T}) . \tag{12}$$

The range of admissible void ratios is limited by e_i and e_d. All material properties of the hypoplastic relation can be determined from simple tests [4].

2.2 Intergranular strain

A modified stress-strain relation has been proposed by Niemunis and Herle [14] in order to avoid ratcheting and to improve the small strain behaviour of the model. The general stress-strain relation:

$$\overset{\circ}{\mathbf{T}} = \mathbf{M} : \mathbf{D} , \tag{13}$$

wherein the fourth order tensor $\mathbf{M}$ represents stiffness and is calculated from the hypoplastic tensors $\mathbf{M}(\mathbf{T}, e)$ and $\mathbf{N}(\mathbf{T}, e)$, which may be modified (increased) by scalar multipliers (material constants) m_T and m_R, depending on the so called in tergranular strain $\mathbf{h}$ and the direction of stretching $\mathbf{D}$. Using the definitions $\rho = \|\mathbf{h}\|/R$ and $\hat{\mathbf{h}} = \mathbf{h}/\|\mathbf{h}\|$ we calculate the stiffness $\mathbf{M}$ from the following formula:

$$\mathbf{M} = [\rho^{\chi} m_T + (1-\rho^{\chi}) m_R]\mathbf{L} + \begin{cases} \rho^{\chi}(1-m_T)\mathbf{L}:\hat{\mathbf{h}}\hat{\mathbf{h}} + \rho^{\chi}\mathbf{N}\hat{\mathbf{h}} & \text{for} \quad \hat{\mathbf{h}}:\mathbf{D} > 0 \\ \rho^{\chi}(m_R - m_T)\mathbf{L}:\hat{\mathbf{h}}\hat{\mathbf{h}} & \text{for} \quad \hat{\mathbf{h}}:\mathbf{D} \le 0. \end{cases} \tag{14}$$

wherein χ and R are material constants.

The evolution equation of the intergranular strain tensor **h** reads:

$$\overset{\circ}{\mathbf{h}} = \begin{cases} (\mathcal{I} - \hat{\mathbf{h}}\hat{\mathbf{h}}\rho^{\beta_r}) : \mathbf{D} & \text{for} \quad \hat{\mathbf{h}} : \mathbf{D} > 0 \\ \mathbf{D} & \text{for} \quad \hat{\mathbf{h}} : \mathbf{D} \le 0, \end{cases} \tag{15}$$

where $\overset{\circ}{\mathbf{h}}$ is the objective rate of intergranular strain to be supplemented by Zaremba-Jaumann terms if necessary. The exponent β_r is a material constant.

3. Barotropy factor f_b

Early versions of hypoplastic constitutive model were first-order homogeneous functions of stress, *i.e.* $\overset{\circ}{\mathbf{T}}(\lambda \mathbf{T}, \mathbf{D}) = \lambda \overset{\circ}{\mathbf{T}}(\mathbf{T}, \mathbf{D})$. This implied that the stiffness vanished for $\mathbf{T} \to 0$ and unrealistically large strains were needed to approach the stress free state. Moreover the calculated void ratio could become negative for extremely high pressures. In order to eliminate these shortcomings Gudehus and Bauer [3, 1] rewrote the hypoplastic equation in terms of dimensionless stress $\hat{\mathbf{T}} = \mathbf{T}/\mathrm{tr}\,\mathbf{T}$ and multiplied it by so called barotropy factor f_b. The factor f_b is a stress function $f_b(\mathrm{tr}\,\mathbf{T})$ that renders the hypoplastic model compatible with the following empirical approach:

$$e \equiv e_i = e_{i0} \exp\left[-\left(\frac{3p}{h_s} \right)^n \right], \tag{16}$$

with $3p = -\mathrm{tr}\,\mathbf{T}$. It describes the special case of isotropic compression that starts at the "percolation limit" $p = 0$ and $e = e_{i0}$ and continue for $p > 0$. Due to this modification the stiffness does not vanish with $\mathrm{tr}\,\mathbf{T} = 0$ and e remains positive for $\mathrm{tr}\,\mathbf{T} \to \infty$. The state $e = e_i$ is assumed to be the loosest possible state of a simple granulate (without macropores), meaning that $e > e_i$ cannot be reached in a process of homogeneous deformation. This will be verified in the following text.

An additional empirical factor $f_e = (e_c / e)^\beta$ called *pyknotropy factor* has been proposed [3, 1] in order to make the stiffness density dependent. The function $e_c(\mathbf{T})$ is a pressure dependent critical void ratio and β is a material constant. In the special case of isotropic compression of a granulate in the loosest state $e = e_i$ we have $f_{ei} = (e_c / e_i)^\beta = (e_{c0} / e_{i0})^\beta = \text{const}$.

Let us derive such function $f_b(\mathrm{tr}\,\mathbf{T})$ that the Equation (16) holds for isotropic compression. We consider monotonic loading only so the intergranular strain can be neglected.

Let us write the isotropic compression/extension in terms of $\dot{p}$ and $\dot{e}$ with:

$$\dot{p} = -\frac{1}{3}\delta_{ij}\overset{\circ}{T}_{ij}, \quad \dot{e} = (1+e)D_{kl}\delta_{kl}, \quad D_{kl} = \frac{\dot{e}}{3(1+e)}\delta_{kl}, \quad \|\mathbf{D}\| = \frac{|\dot{e}|}{3(1+e)}\sqrt{3}. \tag{17}$$

The state of stress **T** remains isotropic so:

$$T_{ij} = -p\delta_{ij},\ \mathrm{tr}\,\mathbf{T} = -3p,\ \hat{T}_{ij} = \frac{1}{\mathrm{tr}\,\mathbf{T}}T_{ij} = \frac{1}{3}\delta_{ij},\ \hat{T}^{*}_{ij} = \hat{T}_{ij} - \frac{1}{3}\delta_{ij} = 0,\ \hat{\mathbf{T}}:\hat{\mathbf{T}} = \frac{1}{3}\frac{1}{3}\delta_{ij}\delta_{ij} = \frac{1}{3}. \quad (18)$$

The constitutive equation for isotropic compression can be written in the following equivalent forms:

$$\overset{\circ}{\mathbf{T}} = f_{ei}f_b\left(\mathbf{L}:\mathbf{D} + f_{di}\mathbf{N}\|\mathbf{D}\|\right),$$

$$\dot{p} = -\frac{1}{3}f_{ei}f_b\left(L_{iikl}:D_{kl} + f_{di}N_{ii}\|\mathbf{D}\|\right) = -\frac{1}{9(1+e)}f_{ei}f_b\left(L_{iikl}\delta_{kl}\dot{e} + f_{di}N_{ii}\sqrt{3}|\dot{e}|\right),$$

$$\dot{p} = -\frac{1}{9(1+e)}f_{ei}f_b\left(L_{iikk}\dot{e} + f_{di}N_{ii}\sqrt{3}|\dot{e}|\right), \quad (19)$$

wherein the value of f_{di} denotes the factor $f_d(e, \mathrm{tr}\,\mathbf{T})$ function calculated at $e = e_i$, *i.e.*:

$$f_{di} = \left(\frac{e_{i0} - e_{d0}}{e_{c0} - e_{d0}}\right)^{\alpha}. \quad (20)$$

We calculate now scalar $L_{iikk} = \delta_{ij}L_{ijkl}\delta_{kl}$ for the hypoplastic model by von Wolffersdorff:

$$L_{iikk} = \mathbf{1}:\mathbf{L}:\mathbf{1} = \frac{1}{\hat{\mathbf{T}}:\hat{\mathbf{T}}}\mathbf{1}:\left(F^2\mathcal{I} + a^2\hat{\mathbf{T}}\hat{\mathbf{T}}\right):\mathbf{1} = \frac{1}{1/3}\delta_{ij}\left(\delta_{ik}\delta_{jl} + a^2\frac{1}{3}\frac{1}{3}\delta_{ij}\delta_{kl}\right)\delta_{kl},$$

$$L_{iikk} = 3\left(3 + a^2\right). \quad (21)$$

Note that due to the isotropic stress state we may substitute $F = 1$. Now we proceed similarly with $N_{ii} = \delta_{ij}N_{ij}$ keeping $F = 1$:

$$N_{ii} = \mathbf{1}:\mathbf{N} = \frac{Fa}{\hat{\mathbf{T}}:\hat{\mathbf{T}}}\mathbf{1}:\left(\hat{\mathbf{T}} + \hat{\mathbf{T}}^{*}\right) = \frac{a}{1/3}\delta_{ij}\hat{T}_{ij}3a\delta_{ij}\frac{1}{3}\delta_{ij} = 3a. \quad (22)$$

Finally we set the expressions for L_{iikk} and N_{ii} in to the constitutive Equation (19):

$$\dot{p} = -\frac{1}{9(1+e)}f_{ei}f_b\left(3\left(3+a^2\right)\dot{e} + f_{di}3a\sqrt{3}|\dot{e}|\right). \quad (23)$$

For isotropic compression with $\dot{e} < 0$ holds, and:

$$\dot{p} = -\left[\frac{1}{3(1+e)}f_b f_{ei}\left(\left(3+a^2\right) - f_{di}a\sqrt{3}\right)\right]\dot{e}. \quad (24)$$

Time differentiation of the compression curve (16) results in:

$$\dot{e} = e_{i0}\exp\left[-\left(\frac{3p}{h_s}\right)^n\right](-n)\left(\frac{3p}{h_s}\right)^{n-1}\frac{3}{h_s}\dot{p} = -\left(\frac{3p}{h_s}\right)^{n-1}\frac{3ne_i}{h_s}\dot{p}$$

or

$$\dot{p} = -\left(\frac{3p}{h_s}\right)^{1-n} \frac{h_s}{3ne_i}\dot{e}, \tag{25}$$

which can be compared with the final form (24) of the constitutive equation yielding:

$$f_b = \frac{1}{f_{ei}}\left(\frac{3p}{h_s}\right)^{1-n} \frac{3(1+e)h_s}{3ne_i}\left(3+a^2-f_{di}a\sqrt{3}\right)^{-1}. \tag{26}$$

Finally, substituting $e = e_i$ one obtains:

$$f_b = \frac{1}{f_{ei}}\left(\frac{3p}{h_s}\right)^{1-n} \frac{3(1+e_i)h_s}{3ne_i}\left(3+a^2-f_{di}a\sqrt{3}\right)^{-1}, \tag{27}$$

which is identical with the expression given in the first section.

4. Consistency of the upper bound $e = e_i$

Now we examine if there is a direction of strain **D** for which the void ratio becomes larger than the upper bound e_i. Let us rewrite the (hypothetical as yet) upper bound surface in $e - \operatorname{tr}\mathbf{T}$ space in the following form:

$$F_i(e, \operatorname{tr}\mathbf{T}) \equiv e - e_{i0} b(\operatorname{tr}\mathbf{T}) = 0. \tag{28}$$

The vector

$$\mathbf{M}^{(i)}(\operatorname{tr}\mathbf{T}, e) = \left[M_e^{(i)}, M_T^{(i)}\right] = \left[\frac{\partial F_i}{\partial e}, \frac{\partial F_i}{\partial \operatorname{tr}\mathbf{T}}\right], \tag{29}$$

normal to this surface, see Figure 1, has two components:

$$M_e^{(i)} = \frac{\partial F_i}{\partial e} = 1, \tag{30}$$

$$M_T^{(i)} = \frac{\partial F_i}{\partial \operatorname{tr}\mathbf{T}} = -\frac{e_i}{h_s} n \left(\frac{-\operatorname{tr}\mathbf{T}}{h_s}\right)^{n-1}. \tag{31}$$

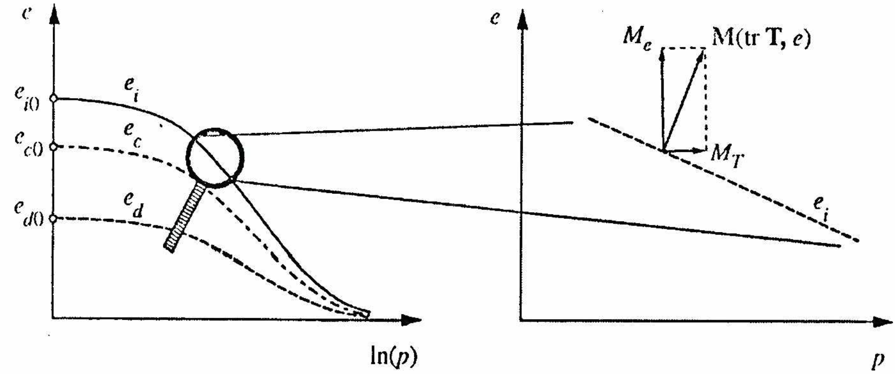

Figure 1. *Definition of vector M in the e – trT space on* e_i

Any process of deformation described in terms of $\dot{e}$ and $\mathrm{tr}\dot{\mathbf{T}}$ must satisfy the following condition:

$$A^{(i)} = \frac{\partial F_i}{\partial e}\dot{e} + \frac{\partial F_i}{\partial \,\mathrm{tr}\,\mathbf{T}}\mathrm{tr}(\dot{\mathbf{T}}) \le 0 , \tag{32}$$

$$A^{(i)} = M_e^{(i)}\,\mathrm{tr}\,\mathbf{D}(1+e) + M_T^{(i)}\,\mathrm{tr}\left(\mathbf{L}:\mathbf{D} + f_d \mathbf{N}\|\mathbf{D}\|\right) \le 0 , \tag{33}$$

for all strain rates $\mathbf{D}$. The scalar product $A^{(i)}$ must not be positive at $e = e_i$, otherwise the bound surface can be surpassed.

First we calculate the strain rated $\mathbf{D} = \bar{\mathbf{D}}$ for which the product $A^{(i)}$ reaches the maximum under condition $\|\mathbf{D}\| = 1$. With other words, among all strain rates such that $\|\mathbf{D}\| = 1$ we seek the most dilatant one keeping in mind that the density limit is pressure-dependent. From the system of 7 equations (disregarding the symmetric components of $\mathbf{D}$):

$$\begin{cases} \dfrac{\partial\left[A^{(i)} + \lambda\left(\|\mathbf{D}\| - 1\right)\right]}{\partial D_{rs}} & = \ 0 \\ \|\mathbf{D}\| & = \ 1, \end{cases} \tag{34}$$

we find the most dilatant direction:

$$D_{rs} \sim M_e^{(i)}(1+e)\delta_{rs} + M_T^{(i)}\delta_{ij}L_{ijrs} , \tag{35}$$

in which:

$$L_{iirs} = \frac{f_b f_e}{1/3}\delta_{ij}\left(\delta_{ir}\delta_{js} + a^2\frac{1}{3}\frac{1}{3}\delta_{ij}\delta_{rs}\right) = 3 f_b f_e\left(\delta_{rs} + a^2\frac{1}{3}\delta_{rs}\right), \tag{36}$$

so $\bar{D}_{rs}$ is an *isotropic* deformation:

$$\bar{D}_{rs} = -\frac{1}{\sqrt{3}}\delta_{rs} . \tag{37}$$

Since the function f_b has been derived for such isotropic deformation, it automatically constitutes the upper bound. Note that this result holds for the particular form of $\mathbf{L}$ only and disregards the intergranular strain. In Section 7 we demonstrate a special zig-zag strain path that goes beyond the e_i–limit due to the effects of the intergranular strain.

5. Consistency of the lower bound $e = e_d$

We proceed now to examine the complementary lower bound $e = e_d$ in a similar way. The nonlinear part $\mathbf{N}$ of the hypoplastic model is multiplied by the factor:

$$f_d = \left(\frac{e - e_d}{e_c - e_d}\right)^{\alpha} , \tag{38}$$

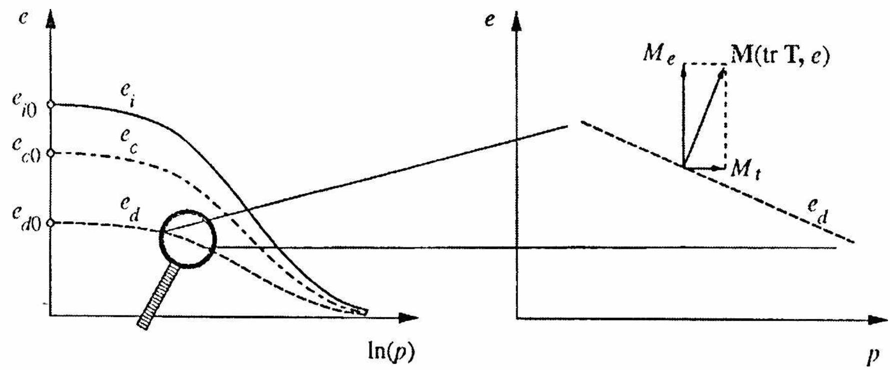

Figure 2. *Definition of vector M in the e — trT space on* e_d

so this part must vanish at the vicinity of $e = e_d$ (because of $f_d \approx 0$) and instead of (1) we obtain:

$$\overset{\circ}{\mathbf{T}} \approx \mathbf{L} : \mathbf{D}. \tag{39}$$

In order to keep the void ratio above the lower bound $e = e_d(\mathrm{tr}\,\mathbf{T})$:

$$e = e_d = e_{d0}\, h(\mathrm{tr}\,\mathbf{T}), \tag{40}$$

wherein:

$$h(\mathrm{tr}\,\mathbf{T}) = \exp\left[-\left(\frac{-\mathrm{tr}\,\mathbf{T}}{h_s}\right)^n\right], \tag{41}$$

we require that for any rate of deformation **D** the increment of the void ratio must be suficiently large. Let us rewrite the lower bound surface in $e - \mathrm{tr}\,\mathbf{T}$ space in the following form:

$$F_d(e, \mathrm{tr}\,\mathbf{T}) \equiv e - e_{d0}\, h(\mathrm{tr}\,\mathbf{T}) = 0. \tag{42}$$

The vector:

$$\mathbf{M}(\mathrm{tr}\,\mathbf{T}, e)^{(d)} = \left[M_e^{(d)}, M_T^{(d)}\right] = \left[\frac{\partial F_d}{\partial e}, \frac{\partial F_d}{\partial\, \mathrm{tr}\,\mathbf{T}}\right], \tag{43}$$

normal to this surface, see Figure 2, has two components:

$$M_e^{(d)} = \frac{\partial F_d}{\partial e} = 1, \tag{44}$$

$$M_T^{(d)} = \frac{\partial F_d}{\partial\, \mathrm{tr}\,\mathbf{T}} = -\frac{e_d}{h_s} n \left(\frac{-\mathrm{tr}\,\mathbf{T}}{h_s}\right)^{n-1}. \tag{45}$$

Any process of deformation described in terms of $\dot{e}$ and $\mathrm{tr}\,\dot{\mathbf{T}}$ must satisfy the following condition:

$$A^{(d)} = \frac{\partial f}{\partial e}\dot{e} + \frac{\partial f}{\partial \operatorname{tr}\mathbf{T}}\operatorname{tr}(\dot{\mathbf{T}}) \geq 0, \tag{46}$$

for all strain rates **D**. The scalar product A must be positive at $e = e_d$, otherwise the bound surface can be surpassed. This condition is not fulfilled by the current hypoplastic model. Usage of (39) is inconsistent and may result in numerical problems. Computation is simply halted whenever expression (38) is evaluated with $e < e_d$.

In order to establish the consistency condition at the maximum density limit we modify the expression (38) for f_d. We seek such f_d that the inequality:

$$A^{(d)} = M_e^{(d)}\operatorname{tr}\mathbf{D}(1+e) + M_T^{(d)}\operatorname{tr}\left(\mathbf{L}:\mathbf{D} + f_d\mathbf{N}\|\mathbf{D}\|\right) \geq 0 \tag{47}$$

holds for all **D** at $e = e_d$. For simplicity we have assumed that the tensors **L** and **N** contain the barotropy and pyknotropy multipliers already. First we calculate the strain rated $\mathbf{D} = \bar{\mathbf{D}}$, for which the product A reaches the minimum under condition $\|\mathbf{D}\| = 1$:

$$\begin{cases} \dfrac{\partial\left[A^{(d)} + \lambda\left(\|\mathbf{D}\| - 1\right)\right]}{\partial D_{rs}} & = & 0 \\ \|\mathbf{D}\| & = & 1, \end{cases} \tag{48}$$

we find the most contractant direction:

$$\bar{D}_{rs} \sim M_e^{(d)}(1+e)\delta_{rs} + M_T^{(d)}\delta_{ij}L_{ijrs}, \tag{49}$$

in which:

$$\delta_{ij}L_{ijrs} = L_{iirs} = \frac{f_b f_e}{1/3}\delta_{ij}\left(\delta_{ir}\delta_{js} + a^2\frac{1}{3}\frac{1}{3}\delta_{ij}\delta_{rs}\right) = 3f_b f_e\left(\delta_{rs} + a^2\frac{1}{3}\delta_{rs}\right), \tag{50}$$

so D_{rs} is an *isotropic* deformation:

$$\bar{D}_{rs} = \frac{1}{\sqrt{3}}\delta_{rs}. \tag{51}$$

6. Modified expression for f_d to establish consistency

We modify the stiffness of the hypoplastic constitutive model according to the lower bound (42) for $e = e_d$. In our proposition we follow the principle "first, do no harm" (primum non nocere) and we modify as little as possible, namely we change merely the expression for the density factor f_d. The required value of $\bar{f}_d = f_d(e_d)$ follows directly from the condition $A^{(d)}(\bar{\mathbf{D}}) = 0$ *i.e.*:

$$\bar{f}_d = -\frac{M_e^{(d)}\operatorname{tr}\bar{\mathbf{D}}(1+e) + M_T^{(d)}\operatorname{tr}\left(\mathbf{L}:\bar{\mathbf{D}}\right)}{M_T^{(d)}\operatorname{tr}\mathbf{N}\|\bar{\mathbf{D}}\|} = -\frac{M_e^{(d)}\sqrt{3}(1+e) + M_T^{(d)}f_b f_e\frac{3}{\sqrt{3}}\left(3+a^2\right)}{M_T^{(d)}f_b f_e 3a}. \tag{52}$$

This value of f_d–factor (usually less than zero) applies at $e = e_d$ only. In order to keep the rest of the hypoplastic model intact we replace the expression (38) for f_d with the following interpolation rule:

$$f_d = \left(\frac{e-e_d}{e_c-e_d}\right)^{\alpha} + \left[1-\left(\frac{e-e_d}{e_c-e_d}\right)^{\alpha}\right]\bar{f}_d\,. \tag{53}$$

The remaining elements of hypoplastic model need not be changed. Despite this precaution the void ratio may *numerically* become smaller than e_d due to large strain increments. Therefore, for a better numerical stability we prefer to calculate:

$$f_d = \mathrm{sign}(e-e_d)\left(\frac{|e-e_d|}{e_c-e_d}\right)^{\alpha} + \left[1-\mathrm{sign}(e-e_d)\left(\frac{|e-e_d|}{e_c-e_d}\right)^{\alpha}\right]\bar{f}_d\,. \tag{54}$$

7. Numerical examples

7.1 A path that goes below the lower bound

Neglecting intergranular strain we consider isotropic deformation in the vicinity of the lower bound e_i. In the range of all allowed constitutive constants the lower bound cannot be surpassed by deformations with increasing pressure (Herle [6]). However, a change in the deformation path from isotropic compression to isotropic extension can lead to a violation of the condition $e \geq e_d$ (see Figure 3a). Including the proposed modification of scalar f_d the lower bound cannot be surpassed anymore as illustrated in Figure 3b.

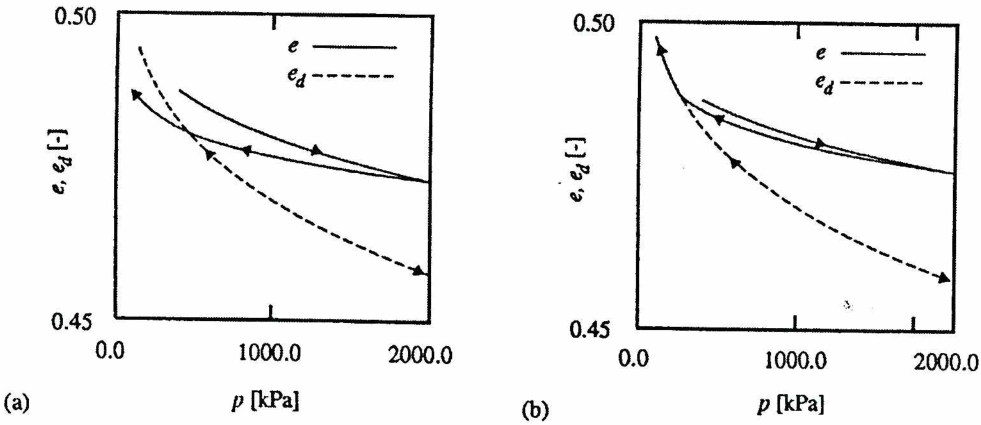

Figure 3. *(a) Violation of the condition $e \geq e_d$ while decreasing p_s in reference version, (b) the lower bound cannot be surpassed in modified formulation of f_d*

7.2 A path that goes beyond the upper bound

The initial state of single homogeneous soil element with uniquely defined **T**, e, **D** and intergranular strain **h** was assumed to be in an isotropic stress state. The intergranular strain was initialized with $\mathbf{h}_{ij} = 0$. Stress-controlled cycles along the p-axes were applied to the soil sample with typical parameters. The amplitude of the zig-zag stresses were increased proportionally to the number of cycles. Due to the intergranular strain extension of the hypoplastic model the upper bound is surpassed as illustrated in Figure 4 p-axes.

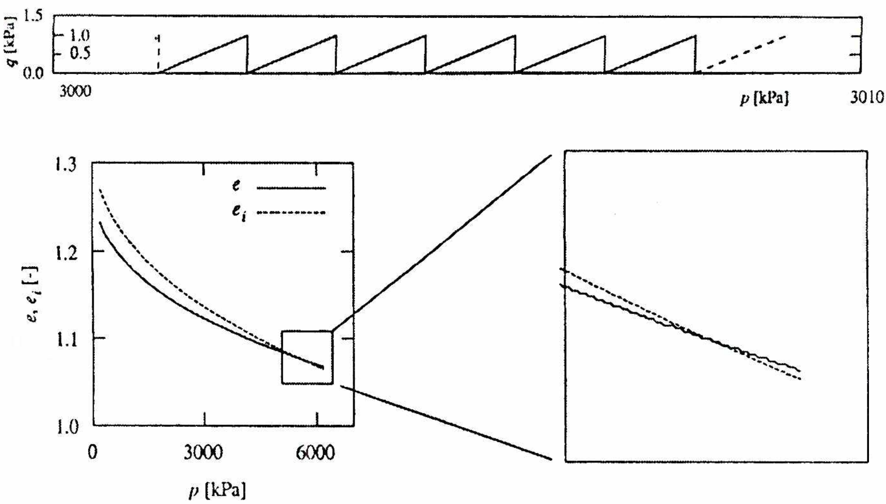

Figure 4. *Violation of the condition* $e \leq e_i$ *(left) by a zig-zag stress path along the p-axes (right)*

8. Conclusion

In the framwork of hypoplasticity limiting void ratios were proposed. Since particular deformation paths violate these boundaries, a consistency conditions was derived. Therefore the factor f_d in the constitutive model was slightly modified. The entire mathematical representation was given. A verification shows the behaviour of the modified model.

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