AN INVERSE MEDIUM PROBLEM FOR THE HEAT EQUATION

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Abstract: In the article we consider the two–dimensional heat equation in a circular domain where the thermal diffusivity is a piecewise constant function in the radial directon and is a constant function in the angular direction. In particular we consider the problem of the computation of a perturbation in this stratified medium having some knowledge about the temperature on the boundary of the domain due to a heat flux applied to the same boundary. A linearized version of this inverse problem is considered and a numerical method to solve the corresponding linear inverse problem is proposed. This numerical method is based on a linear integral equation. Some numerical examples are reported.

Keywords: differential equations, heat transfer equation, inverse problems

1. Introduction

The nondestructive test of an object can be defined as the study of a physical property of some inaccessible parts of the object using only the accessible parts of the same object. These accessible parts are usually used for the measurement of a signal response of the object, that is the signal produced by the interaction of the object and the signal generated by a known source. Depending on the physical context, either acoustic waves, electromagnetic waves or heat fluxes are used to make this nondesctructive test.

The mathematical formulation of a nondestructive test is usually given as an inverse problem for a partial differential equation. In these problems, from some knowledge of the solution of a boundary value problem for the partial differential equation, the computation of some data of the boundary value problem is required. These data, that is the unknown of the inverse problem, are usually a coefficient of the differential equation or a part of the boundary of the domain where the boundary value problem is defined.

In this paper we consider an inverse problem for the two–dimensional heat equation in a circular domain. Inside the circular domain a medium is assumed, having the thermal diffusivity given by a positive function depending only on the radial variable. Moreover, this function is assumed to be a piecewise constant function. In this medium we suppose that there exists a perturbation that we want to calculate using a nondestructive test. In particular, following the idea of [1] we apply a heat flux to the boundary of the domain and from the resulting temperature on the same boundary we want to recover the perturbation on the material inside the domain.

The special form of the domain, i.e. the circular form, arises in a large number of applications, such as for example damage detection of objects like pipes and tanks. In this kind of applications stratified objects in the radial directions are quite natural objects, in fact for example in a simple tank we have two different materials: the wall of the tank and the contents of the tank. Moreover, the wall of the object can be made from a few layers of different materials. Each material in the object usually has a constant thermal diffusivity which is different from the thermal diffusivity of the other materials in the object. Thus the assumption that the thermal diffusivity of the object is given by a piecewise constant function in the radial direction can be seen as a natural assumption in this kind of applications.

In the literature a large number of inverse problems studied from the analytical point of view and from the numerical point of view can be found. This interest in the inverse problems is due to the fact that many important questions, arising in various fields, such as for example damage detections of objects, geological prospecting, noninvasive medical diagnoses and so on can be formulated as inverse problems.

Many authors have considered inverse problems for the heat equation, for example in [1] an inverse problem for the heat equation in a two–dimensional strip defined by two parallel lines is studied. One of these lines is assumed to be the accessible side of the object the other one the inaccessible side. Roughly speaking, the problem of the computation of a perturbation on the line corresponding to the inaccessible side is considered, from some knowledge of the temperature on the accessible side due to a known heat flux applied to the accessible side. In [2] the one–dimensional heat equation in a half–line is considered, where the medium has the thermal diffusivity given by a piecewise constant function. In [3] a two–dimensional version of this problem is examined. In [4] a similar problem for a one–dimensional parabolic equation is considered. Different inverse problems are discusseded in [5–11].

Let us introduce the notation used in the following. Let $\mathbf{R}^N$ be the N–dimensional real euclidean space, let $\mathbf{C}^N$ be the N–dimensional complex euclidean space, we use $\mathbf{R}$, $\mathbf{C}$ in place of $\mathbf{R}^1$, and $\mathbf{C}^1$, respectively. Let $\mathbf{R}^+=\{x \in \mathbf{R}: x \geq 0\}$. Let $\underline{x}$, $\underline{y} \in \mathbf{R}^N$, we denote with $\underline{x}^t \cdot \underline{y}$ the euclidean scalar product in $\mathbf{R}^N$, the superscript t means trans-posed, and we denote with $\|\underline{x}\|$ the euclidean norm in $\mathbf{R}^N$. Let $r > 0$, we denote with $B_r = \left\{ \underline{x} \in \mathbf{R}^2 \ : \ \|\underline{x}\| = r \ \right\}$. Let $A \subset \mathbf{R}^N$, we denote with A^c the closure of A and with ∂A the boundary of A. Let $z \in \mathbf{C}$, we denote with $\bar{z}$ the complex conjugate of z. Let $\underline{z} \in \mathbf{C}^N$, we denote with $\|\underline{z}\|_{\mathbf{C}}$ the euclidean norm of $\underline{z}$ in $\mathbf{C}^N$.

In section 2 we give the mathematical formulation of the inverse problem described above. In section 3 we propose a numerical method to compute an approximation of the solution of the inverse problem considered and some numerical examples are reported.

2. The mathematical formulation

Let $D = \left\{ \underline{x} \in \mathbf{R}^2 : \|\underline{x}\| < 1 \ \right\}$ be the circular disk centered at the origin of the coordinate axes and having unitary radius. We assume D made of a material having a particular thermal diffusivity k, i.e. let v be a positive integer, let $k_1, k_2, \ldots, k_v > 0$ be real constants and let $r_1, r_2, \ldots, r_v \in \mathbf{R}$ be such that $0 < r_1 < r_2 < \ldots < r_v = 1$, then we define:

$$k(\underline{x}) = \begin{cases} k_1, & 0 \le \|\underline{x}\| \le r_1, \\ k_l, & r_{l-1} < \|\underline{x}\| \le r_l, \ l = 2, 3, \ldots, v, \end{cases} \qquad \underline{x} \in D^c. \tag{1}$$

When we apply a heat flux $F(\underline{x}, t)$, $\underline{x} \in \partial D$, $t \in \mathbf{R}^+$, to the boundary of D we have a change in the temperature $U(\underline{x}, t)$, $\underline{x} \in D^c$, $t \in \mathbf{R}^+$, of the disk. We can assume that the function U is the solution of the following problem:

$$\frac{\partial U(\underline{x},t)}{\partial t} - \operatorname{div}(k(\underline{x}) \underline{\nabla} U(\underline{x},t)) = 0, \qquad \underline{x} \in D, \ t > 0, \tag{2}$$

$$\frac{\partial U(\underline{x},t)}{\partial \underline{\hat{n}}(\underline{x})} = F(\underline{x},t), \qquad \underline{x} \in \partial D, \ t > 0, \tag{3}$$

$$U(\underline{x},0) = U_0(\underline{x}), \qquad \underline{x} \in D^c, \tag{4}$$

where "div" is the divergence operator with respect to the $\underline{x}$ variables, $\underline{\nabla}$ is the gradient operator with respect to the $\underline{x}$ variables, $\underline{\hat{n}}(\underline{x})$ is the unit outward normal vector to the boundary of D at the point $\underline{x} \in \partial D$ and $U_0(\underline{x})$, $\underline{x} \in D^c$, is the initial temperature of the disk. We note that U can be considered as the weak solution of problems (2)–(4) when the heat flux F is assumed to be a square integrable function on $\partial D \times \mathbf{R}^+$, see [12] page 116.

Following the approach of [1], let $\omega > 0$ be a given real constant, then we consider $F(\underline{x}, t) = f(\underline{x}) e^{i\omega t}$, $\underline{x} \in \partial D$, $t \in \mathbf{R}^+$, where f is a square integrable function on ∂D and we assume the solution of (2)–(4) to be $U(\underline{x}, t) = u(\underline{x}) e^{i\omega t}$, $\underline{x} \in D^c$, $t \in \mathbf{R}^+$, for a suitable function u. Then at the stationary state, i.e. when $t \to +\infty$, the function U is the solution of problem (2)–(4) when the function u solves the following problem:

$$i\omega u(\underline{x}) - \text{div}(k(\underline{x})\underline{\nabla} u(\underline{x})) = 0, \qquad \underline{x} \in D, \tag{5}$$

$$\frac{\partial u(\underline{x})}{\partial \hat{\underline{n}}(\underline{x})} = f(\underline{x}), \qquad \underline{x} \in \partial D. \tag{6}$$

Let $W^{1,2}(D)$ be the space of square integrable functions in D having square integrable generalized derivatives of first order in D. We note that the function $u \in W^{1,2}(D)$ can be considered as the weak solution of problem (5), (6), see [12] page 61, i.e. u satisfies the following relation:

$$\int_D (k(\underline{x})\underline{\nabla}' u(\underline{x}) \cdot \underline{\nabla}\phi(\underline{x}) + i\omega u(\underline{x})\phi(\underline{x}))d\underline{x} = \int_{\partial D} f(\underline{x})\phi(\underline{x})ds(\underline{x}), \qquad \forall \phi \in W^{1,2}(D). \tag{7}$$

Integral relation (7) gives imediately the uniqueness of the solution u of problem (5), (6), in fact taking advantage of the linearity of problem (5), (6) and choosing $f = 0$, $\phi = \bar{u}$ in (7), we have that $u = 0$ is the unique function that satisfies (7).

For the particular choice of the thermal diffusivity k we have that the function u is a smooth function in $D \setminus \{\underline{x} \in D\colon \|\underline{x}\| = r_l, l = 1, 2, \ldots, \nu-1\}$, where $r_1, r_2, \ldots, r_{\nu-1}$ are the parameters that define the function k and we have:

$$u \text{ is continuous in } D, \tag{8}$$

$$k_l \frac{\partial^- u}{\partial \rho}(\underline{x}) = k_{l+1}\frac{\partial^+ u}{\partial \rho}(\underline{x}), \ \underline{x} \in D \text{ such that } \|\underline{x}\| = r_l, l = 1, 2, \ldots, \nu - 1, \tag{9}$$

$$\frac{\partial u(\underline{x})}{\partial \hat{\underline{n}}(\underline{x})} = f(\underline{x}), \qquad \underline{x} \in \partial D, \tag{10}$$

where $\dfrac{\partial^- u}{\partial \rho}$, $\dfrac{\partial^+ u}{\partial \rho}$ denote the right derivative and the left derivative along the radial direction, respectively (see [13] page 329 for details). From the physical point of view equations (8), (9) state the well known property of the temperature in the heat conduction theory, that is the temperature u and the flux $k\underline{\nabla}\, u$ are continuous functions on the surface separation of two media, see for example [14] page 25.

Let (ρ,θ), $\rho \in \mathbf{R}^+$, $\theta \in [0,2\pi)$, be the polar coordinates associated to $\underline{x} \in \mathbf{R}^2$, i.e. $\underline{x} = \rho(\cos\theta, \sin\theta)' \in \mathbf{R}^2$, $\rho \in \mathbf{R}^+$, $\theta \in [0,2\pi)$. The particular expression of the function k and relations (8)–(10) allow to compute explicitely the solution of problems (5), (6). We show this computation in the particular case that $f(\underline{x}) = \cos(n(\theta-\alpha))$, where θ is such that $\underline{x} = (\cos\theta, \sin\theta)' \in \partial D$, n is an integer number and $\alpha \in [0,2\pi)$ is a real number. Using the separation variables technique we obtain the following expression for the solution u of problems (5), (6):

$$u(\underline{x}) = R_n(\rho)\cos(n(\theta-\alpha)), \qquad \underline{x} = \rho(\cos\theta, \sin\theta)', \tag{11}$$

$$R_n(\rho) = \begin{cases} A_1 \mathrm{be}_n(\sqrt{\omega/k_1}\ \rho), & 0 \le \rho \le r_1, \\ A_l \mathrm{be}_n(\sqrt{\omega/k_l}\ \rho) + B_l \mathrm{ke}_n(\sqrt{\omega/k_l}\ \rho), & r_{l-1} < \rho \le r_l,\ l = 2,3,\dots,\nu, \end{cases} \tag{12}$$

where $\mathrm{be}_n(s) = \mathrm{ber}_n(s) + i\mathrm{bei}_n(s)$, $s \ge 0$, $\mathrm{ke}_n(s) = \mathrm{ker}_n(s) + i\mathrm{kei}_n(s)$, $s > 0$ are the Kelvin functions of order n, see [15] page 379, and the coefficients A_1, A_l, B_l, $l = 2, 3, \dots, \nu$ are the solution of the following linear system:

$$A_1 \mathrm{be}_n\left(\sqrt{\tfrac{\omega}{k_1}} r_1\right) - A_2 \mathrm{be}_n\left(\sqrt{\tfrac{\omega}{k_1}} r_1\right) - B_2 \mathrm{ke}_n\left(\sqrt{\tfrac{\omega}{k_1}} r_1\right) = 0, \tag{13}$$

$$A_1\sqrt{k_1}\,\mathrm{be}_n'\left(\sqrt{\tfrac{\omega}{k_1}} r_1\right) - A_2\sqrt{k_2}\,\mathrm{be}_n'\left(\sqrt{\tfrac{\omega}{k_2}} r_1\right) - B_2\sqrt{k_2}\,\mathrm{ke}_n'\left(\sqrt{\tfrac{\omega}{k_2}} r_1\right) = 0, \tag{14}$$

$$\begin{aligned} &A_l \mathrm{be}_n\left(\sqrt{\tfrac{\omega}{k_l}} r_l\right) + B_l \mathrm{ke}_n\left(\sqrt{\tfrac{\omega}{k_l}} r_l\right) - A_{l+1} \mathrm{be}_n\left(\sqrt{\tfrac{\omega}{k_{l+1}}} r_l\right) - \\ &- B_{l+1} \mathrm{ke}_n\left(\sqrt{\tfrac{\omega}{k_{l+1}}} r_l\right) = 0, \qquad l = 2,3,\dots,\nu-1, \end{aligned} \tag{15}$$

$$\begin{aligned} &A_l\sqrt{k_l}\,\mathrm{be}_n'\left(\sqrt{\tfrac{\omega}{k_l}} r_l\right) - B_l\sqrt{k_l}\,\mathrm{ke}_n'\left(\sqrt{\tfrac{\omega}{k_l}} r_l\right) - A_{l+1}\sqrt{k_{l+1}}\,\mathrm{be}_n'\left(\sqrt{\tfrac{\omega}{k_{l+1}}} r_l\right) - \\ &- B_{l+1}\sqrt{k_{l+1}}\,\mathrm{ke}_n'\left(\sqrt{\tfrac{\omega}{k_{l+1}}} r_l\right) = 0, \qquad l = 2,3,\dots,\nu-1, \end{aligned} \tag{16}$$

$$A_\nu \mathrm{be}_\nu'\left(\sqrt{\tfrac{\omega}{k_\nu}}\right) - B_\nu \mathrm{ke}_n'\left(\sqrt{\tfrac{\omega}{k_\nu}}\right) = \tfrac{k_\nu}{\omega}, \tag{17}$$

where $\mathrm{be}_n'(s)$ and $\mathrm{ke}_n'(s)$ denote the derivatives with respect to s of $\mathrm{be}_n(s)$, and $\mathrm{ke}_n(s)$, respectively. Note that equations (13), (15) are consequences of (8), equations (14), (16) are consequences of (9) and equation (17) is a consequence of (10). Function u given by (11)–(17) is the solution of problem (5), (6) when the heat flux f has the particular expression considered above. For a more general form of function f we can compute easily the Fourier series expansion of the function u solution of (5), (6) using the Fourier series expansion of f and the corresponding solutions given by (11)–(17).

We consider now a perturbation of the medium in D, in particular we assume that the thermal diffusivity of this medium is given by $k_c(\underline{x}) = k(\underline{x})(1+c(\underline{x}))$, $\underline{x} \in D^c$, where k is the function given in (1) and $c : D^c \to \mathbf{R}$ is a Lipschitz continuous function such that $c(\underline{x}) > -1$, $\underline{x} \in D^c$, and $c(\underline{x}) = 0$, $\underline{x} \in \partial D$. As above, applying a heat flux f on the boundary of D we have that the temperature u_c in the disk D is the solution of the following problem:

$$\left(\sqrt{\tfrac{\omega}{k_1}}\rho\right) i\omega u_c(\underline{x}) - \mathrm{div}(k_c(\underline{x})\underline{\nabla} u_c(\underline{x})) = 0, \qquad \underline{x} \in D, \tag{18}$$

$$\frac{\partial u_c(\underline{x})}{\partial \underline{\hat{n}}(x)} = f(\underline{x}), \qquad \underline{x} \in \partial D, \tag{19}$$

where $u_c = W^{1,2}(D)$ can be seen as the weak solution of problem (18), (19). We set $u_c(\underline{x}) = u(\underline{x}) + w_c(\underline{x})$, $\underline{x} \in D^c$, where u is the temperature due to the flux f for the unperturbed medium and w_c is a suitable function depending on c. Function w_c can be seen as the perturbation on the temperature u due to the perturbation c in the medium. In the inverse problem we want to recover the perturbation c in the medium from some knowledge of the perturbation w_c. In particular we consider the following inverse problem:

Problem 1 Given $\omega_j > 0$, $j=1, 2, \ldots, N_\omega$, the heat flux $f(\underline{x})$, $\underline{x} \in \partial D$, the thermal diffusivity of the unperturbed medium $k(\underline{x})$, $\underline{x} \in D^c$, and given $u_{c,j}(\underline{x})$, $\underline{x} \in \partial D$, $j=1, 2, \ldots, N_\omega$, where $u_{c,j}$ is the solution of problems (18), (19) corresponding to $\omega=\omega_j$, $j=1, 2, \ldots, N_\omega$, compute the perturbation $c(\underline{x})$, $\underline{x} \in D$.

This inverse problem is a nonlinear problem and we propose a numerical solution which is based on a linearized version of problems (18), (19). More precisely we consider the linearization of w_c with respect to c using as base point $c = 0$, i.e. we approximate the function w_c with the function v_c solution of the following problem:

$$i\omega \upsilon_c(\underline{x}) - \mathrm{div}(k(\underline{x})\underline{\nabla}\upsilon_c(\underline{x})) = \mathrm{div}(k(\underline{x})\, c(\underline{x})\underline{\nabla} u(\underline{x})), \quad \underline{x} \in D, \tag{20}$$

$$\frac{\partial \upsilon_c}{\partial \underline{\hat{n}}}(\underline{x}) = 0, \quad \underline{x} \in \partial D, \tag{21}$$

where $\upsilon_c = W^{1,2}(D)$ can be seen as the weak solution of problem (20), (21). Note that, when $c(\underline{x}) \geq c_0$, $\underline{x} \in D$, for a suitable constant $c_0 > -1$, υ_c can be seen as the second term of the regular perturbation series for the function u_c solution of (18), (19), see [13] page 619 for details, so that we expect that υ_c is an accurate approximation of w_c when the perturbation c is small with respect to the unity.

The mathematical formulation of Problem 1 is based on an integral equation. Let $\underline{\xi} \in \partial D$ and $g_{\underline{\xi}}$ be the solution of the following problem:

$$i\omega g_{\underline{\xi}} - div(k(\underline{x})\underline{\nabla} g_{\underline{\xi}}(\underline{x})) = 0, \quad \underline{x} \in D, \tag{22}$$

$$\frac{\partial g_{\underline{\xi}}(\underline{x})}{\partial \underline{\hat{n}}(\underline{x})} = \delta_{\underline{\xi}}(\underline{x}), \quad \underline{x} \in \partial D, \tag{23}$$

where $\delta_{\underline{\xi}}(\underline{x}) = \delta(\underline{x} - \underline{\xi})$, $\underline{x} \in D$ and δ is the Dirac delta.

From the Green's formula, problems (20), (21) and problems (22), (23) we have:

$$\upsilon_c(\underline{\xi}) = \frac{1}{k_v} \int_D g_{\underline{\xi}}(\underline{x}) \mathrm{div}(k(\underline{x})\, c(\underline{x}) \underline{\nabla} u(\underline{x})) d\underline{x}, \quad \underline{\xi} \in \partial D, \tag{24}$$

moreover, from the Green's formula, problems (5), (6) and the assumption that $c(\underline{x}) = 0$, $\underline{x} \in \partial D$, we can rewrite (24) as follows:

$$\upsilon_c(\underline{\xi}) = \frac{1}{k_v} \int_D k(\underline{x})\, c(\underline{x}) \underline{\nabla}' u(\underline{x}) \cdot \underline{\nabla} g_{\underline{\xi}}(\underline{x}) d\underline{x}, \quad \underline{\xi} \in \partial D. \tag{25}$$

We note that integral equation (25) is the mathematical formulation of the linearized version of Problem 1, in fact this integral equation gives a straightforward relation beetwen v_c, the approximation of the data of Problem 1, and c, the unknown of Problem 1.

3. The numerical solution

In the numerical experience we must consider two different questions: the generation of the data of Problem 1, the numerical solution of Problem 1.

The generation of the data of Problem 1, i.e. given k, c, f, ω generates u_c, can be done either with experimental temperature measurements or with synthetic temperature measurements using a numerical simulation. In particular in the numerical examples considered later in this section we have generated the synthetic temperature measurements solving numerically problems (18), (19). We refer to the problem of the computation of the data of Problem 1 as the *direct problem*. An approximation of u_c is computed using a finite difference discretization of equations (18), (19) in polar coordinates (ρ, θ). More precisely we consider the following problem:

$$i\omega U_c - \frac{\partial}{\partial \rho}\left(k_c \frac{\partial U_c}{\partial \rho}\right) - \frac{k_c}{\rho}\frac{\partial U_c}{\partial \rho} - \frac{\partial}{\partial \theta}\left(k_c \frac{\partial U_c}{\partial \theta}\right) = 0, \quad \rho \in (0,1), \ \theta \in (0, 2\pi), \tag{26}$$

$$\frac{\partial U_c}{\rho}(1,\theta) = \Phi(\theta), \quad \theta \in [0, 2\pi), \tag{27}$$

$$\lim_{\theta \to 2\pi^-} U_c(\rho,\theta) = U_c(\rho,0), \quad \rho \in (0,1], \tag{28}$$

$$\lim_{\theta \to 2\pi^-} \frac{\partial U_c}{\partial \theta}(\rho,\theta) = \frac{\partial^+ U_c}{\partial \theta}(\rho,0), \quad \rho \in (0,1], \tag{29}$$

$$U_c(0,\theta) = \frac{1}{\pi r^2} \lim_{r \to 0^+} \int_{B_r} U_c(\underline{x}) d\underline{x}, \quad \theta \in [0, 2\pi), \tag{30}$$

where $U_c(\rho,\theta) = u_c(\rho\cos\theta, \rho\sin\theta)$, $\rho \in [0,1]$, $\theta \in [0,2\pi)$, $\Phi(\theta) = f(\cos\theta, \sin\theta)$, $\theta \in [0,2\pi)$. Note that in condition (30) it is required that the solution of (18), (19) is given by its average on the ball B_r when $r \to 0^+$. This requirement is not restrictive due to the regularity properties of the solution of problem (18), (19). Problems (26)–(30) have been discretized using a classical finite difference scheme where we have considered N_ρ discretization points uniformly distributed in (0,1] along the radial direction, and we have considered N_θ discretization points uniformly distributed in $[0,2\pi)$ along the angular direction. In this way the finite difference scheme is obtained approximating the derivatives of problems (26)–(30) with the classical finite difference formulas on the mesh grid (ρ_l,θ_j), (26)–(30), $l = 1, 2, \ldots, N_\rho$, $j = 1, 2, \ldots, N_\theta$, where $\rho_l = l/N_\rho$, $l = 1, 2, \ldots, N_\rho$, and $\theta_j = 2\pi(j-1)/N_\theta$, $j = 1, 2, \ldots, N_\theta$. This discretization of (26)–(30) gives a linear system where the unknowns are $U_{l,j} = U_c(\rho_l,\theta_j)$ $l = 1, 2, \ldots, N_\rho$, $j = 1, 2, \ldots, N_\theta$. This linear system has usually a large number of unknowns, so it cannot be solved using a direct factorization method. In the numerical results given below this system has been solved using the routine **LINBCG** contained in [17], this routine is based on the biconjugate gradient method. Finally let $\tilde{U}_{l,j}$, $l = 1, 2, \ldots, N_\rho$, $j = 1, 2, \ldots, N_\theta$, be the solution of this linear system, then the data of Problem 1 are $\tilde{U}_{N_\rho,j}$, $j = 1, 2, \ldots, N_\theta$.

Problem 1 is solved computing a numerical solution of integral equation (25). In particular for $j = 1, 2, \ldots, N_\theta$ the approximation V_j of $\upsilon_c(\cos\theta_j, \sin\theta_j)$, are obtained as the difference of the function $\tilde{U}_{N_\rho,j}$ and the function $u(\cos\theta_j, \sin\theta_j)$ solution of problem (5), (6). This function, as described in section 2, can be computed in terms of the solution of linear system (13)–(17). This linear system, depending on the choice of ω and the function k, can be ill conditioned, so it must be solved with special care. We solve the linear systems (13)–(17) using Tikhonov regularization, see [16] pages 215, 243 for details. More precisely let M be the square complex matrix of order $2\nu-1$ with entries given by the coefficients of the unknowns of the linear system (13)–(17), let $\underline{b} \in \mathbf{C}^{2\nu-1}$ be the vector given by the terms on the right hand side of linear system (13)–(17), then the solution $\underline{\alpha} = (A_1, A_2, B_2, \ldots, A_\nu, B_\nu)' \in \mathbf{C}^{2\nu-1}$ of (13)–(17) is computed as the unique minimizer of the following problem:

$$\min_{\underline{\alpha} \in \mathbf{C}^{2\nu-1}} \left\{ \left\| M\underline{\alpha} - \underline{b} \right\|_{\mathbf{C}}^2 + \lambda \left\| \underline{\alpha} \right\|_{\mathbf{C}}^2 \right\}, \tag{31}$$

where $\lambda > 0$ is a parameter.

Moreover, the integral equation (25) is based on a numerical approximation of the function $g_{\underline{\xi}}$ solution of problems (22), (23). The solution of problems (22), (23) can be expressed as a series of functions where each term is given by formulas (11)–(17), that is we have:

$$\begin{gathered} g_{\underline{\xi}}(\underline{x}) = \frac{R_0(\rho)}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty} R_n(\rho)\cos(n(\theta-\alpha)), \\ \underline{x} = \rho(\cos\theta, \sin\theta)' \in D,\ \xi = (\cos\alpha, \sin\alpha)' \in \partial D, \end{gathered} \tag{32}$$

where for $n = 0, 1, \ldots$ the function R_n is given by (12)–(17). Note that $R_0(\rho)$, $R_n(\rho)\cos(n(\theta-\alpha))$, $n = 1, 2, \ldots$, $\rho \in (0,1)$, $\theta \in (0,2\pi)$, solve the differential equation (22) and $(\partial R_0/\partial\rho)(1) = 1$, $(\partial R_n/\partial\rho)(1) = 1$, $n = 1, 2, \ldots$, $\rho \in (0,1)$, so that (32) holds as a consequence of the well known formula that gives the Fourier expansion of the Dirac delta, that is:

$$\delta(\theta-\alpha) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\cos(n(\theta-\alpha)) \qquad \theta, \alpha \in [0, 2\pi), \tag{33}$$

see [18] page 31. Note that relation (33) holds in the sense of distributions, so also (32) holds in the sense of distributions, see [18] page 13. In the numerical computation we consider a truncation of the series appearing in (32), i.e. we replace the series with a finite sum corresponding to indices from $n = 1$ to $n = N_g$ and we compute the sum of these terms using the classical Cesaro summation rule, see [19] page 411 for a detailed discussion.

In the numerical solution of the integral equation (25) we consider a piecewise constant approximation for the function c, i.e. let $\rho_l = l/L$, $l = 0, 1, \ldots, L$, $\vartheta_j = 2\pi j/J$, $j = 0, 1, \ldots, J$, then we assume that $c(\underline{x}) = c_{l,j}$ for $\underline{x} = (\rho\cos\vartheta, \rho\sin\vartheta)'$ and $\rho \in (\rho_{l-1}, \rho_l]$, $l = 1, 2, \ldots, L$, $\vartheta \in [\vartheta_{j-1}, \vartheta_j)$, $j = 0, 1, \ldots, J$. From (25), (32) we obtain:

$$V_m = -\frac{1}{\pi k_v}\sum_{l=1}^{L-1}\sum_{j=1}^{J} c_{l,j}\left[(\vartheta_j - \vartheta_{j-1})\int_{\rho_{l-1}}^{\rho_l} d\rho\, \rho k(\rho)\frac{\partial R_0(\rho)}{\partial\rho}\,\frac{\partial R_0(\rho)}{\partial\rho} + \right.$$

$$\left. +2\sum_{n=1}^{N_g}\frac{N_g - n + 1}{N_g + 1}\,\frac{\sin(n(\vartheta_j - \alpha)) - \sin(n(\vartheta_{j-1} - \alpha))}{n}\int_{\rho_{l-1}}^{\rho_l} d\rho\, \rho k(\rho)\frac{\partial R_0(\rho)}{\partial\rho}\,\frac{\partial R_0(\rho)}{\partial\rho}\right]$$

$$m = 1, 2, \ldots, N_\theta, \qquad (34)$$

where the functions R_0, R_1, $\ldots$, R_{N_g} are defined in (12)–(17) and the factor $2(N_g - n + 1)/(N_g + 1)$ is due to the Cesaro summation rule. Formula (34) defines a linear system for the unknowns $c_{l,j}$, $l = 1, 2, \ldots, L-1$, $j = 1, 2, \ldots, J-1$. Note that $c_{L,j} = 0$, $j = 1, 2, \ldots, J$ as consequence of the assumption $c(\underline{x}) = 0$, $\underline{x} \in \partial D$. The solution $\tilde{\underline{c}}_{l,j}$, $l = 1, 2, \ldots, L-1$, $j = 1, 2, \ldots, J$ of the linear system (34) is the approximation of the solution of Problem 1. In the numerical results given below the integrals appearing in (34) are computed using the routine **D01AKF** contained in [20], this routine is an adaptive integrator based on a particular Gaussian quadrature rule. We note that the integral equation (25) is a Fredholm integral equation of the first kind. It is a well known fact that the solution of such an integral equation is an ill posed problem, see [16] page 1 for details, so we expect that the linear system (34), obtained from the discretization of (25), is ill conditioned. We solve (34) using the Tikhonov regularization. More precisely let $\underline{V} = (V_1, V_2, \ldots, V_{N_\theta})' \in \mathbf{C}^{N_\theta}$ and let G be the complex matrix having N_θ rows and $(L-1)J$ columns such that the linear system (34) can be rewritten in the following form:

$$G\underline{c} = \underline{V}, \qquad (35)$$

where $\underline{c} \in \mathbf{R}^{(L-1)J}$ is the real vector containing the unknowns $c_{l,j}$, $l = 1, 2, \ldots, L-1$, $j = 1, 2, \ldots, J-1$. Then the solution $\tilde{\underline{c}}$ of (35) is computed as the unique minimizer of the following problem:

$$\min_{\underline{c}\in C}\left\{\left\|G\underline{c} - \underline{V}\right\|_{\mathbf{C}}^2 + \mu\left\|\underline{c}\right\|^2\right\}, \qquad (36)$$

where the constraint C is defined in a natural way, that is $C = \{(c_1, c_2, \ldots, c_{(L-1)J})' \in \mathbf{R}^{(L-1)J} : c_i > -1,\ i = 1, 2, \ldots, (L-1)J\}$ and $\mu > 0$ is a parameter.

We note that the numerical solution of an inverse problem requires special attention when synthetic data are used, because the so called *inverse crime* must be avoided, see [5] page 121. Roughly speaking the *inverse crime* is the reduction of an infinite dimensional inverse problem to a finite dimensional inverse problem. For example th *inverse crime* occurs when the direct problem (i.e. the computation of the data of the inverse problem) and the inverse problem are solved using the same discretization of the same mathematical formulation. Here the *inverse crime* cannot occur, in fact the direct problem is solved using a finite difference approximation of problem (18), (19) and the inverse problem is solved using the integral equation (25) which is an integral formulation of the linearized version of problem (18), (19).

In the numerical experience we consider seven examples. These examples correspond to different choices of the data and the unknown of Problem 1.

Example 1 Let us consider $N_\omega = 1$, $\omega_1 = 100$, the heat flux $f(\underline{x}) = 1$, $(\underline{x}) \in \partial D$, the thermal diffusivity of the unperturbed medium given by the following function:

$$k(\underline{x}) = \begin{cases} 1, & 0 \le \|\underline{x}\| \le \frac{1}{2}, \\ 2, & \frac{1}{2} < \|\underline{x}\| \le 1, \end{cases} \qquad \underline{x} \in D^c, \tag{37}$$

and the perturbation given by the following function:

$$c(\underline{x}) = \frac{1}{2}\gamma(\rho, \tfrac{3}{5}, \tfrac{4}{5})\gamma(\theta, \tfrac{\pi}{6}, \tfrac{\pi}{3}), \ \ \underline{x} = \rho(\cos\theta, \sin\theta)', \ \ \rho \in [0,1], \ \theta \in [0, 2\pi), \tag{38}$$

where, given t_1, $t_2 \in \mathbf{R}$ with $t_1 < t_2$, we define:

$$\gamma(t, t_1, t_2) = \begin{cases} 4\,\frac{(t_1 - t_2)(t_2 - t)}{(t_2 - t_1)^2}, & t \in [t_1, t_2], \\ 0, & t \notin [t_1, t_2], \end{cases} \qquad t \in \mathbf{R}. \tag{39}$$

Example 2 Let us consider $N_\omega = 1$, $\omega_1 = 100$, the heat flux $f(\underline{x}) = 1$, $\underline{x} \in \partial D$, the thermal diffusivity of the unperturbed medium given by the function (37) and the perturbation given by the following function:

$$\begin{gathered} c(\underline{x}) = \frac{1}{2}\gamma(\rho, \tfrac{3}{5}, \tfrac{4}{5})\gamma(\theta, \tfrac{\pi}{6}, \tfrac{\pi}{3}) + \frac{1}{2}\gamma(\rho, \tfrac{3}{5}, \tfrac{4}{5})\gamma(\theta, \tfrac{2}{3}\pi, \tfrac{17}{18}\pi), \\ \underline{x} = \rho(\cos\theta, \sin\theta)', \quad \rho \in [0,1], \quad \theta \in [0, 2\pi). \end{gathered} \tag{40}$$

Example 3 Let us consider $N_\omega = 1$, $\omega_1 = 100$, the heat flux $f(\underline{x}) = 1$, $\underline{x} \in \partial D$, the thermal diffusivity of the unperturbed medium given by the following function:

$$k(\underline{x}) = \begin{cases} 1, & 0 \le \|\underline{x}\| \le \frac{1}{2}, \\ 2, & \frac{1}{2} < \|\underline{x}\| \le \frac{3}{4}, \\ 1, & \frac{3}{4} < \|\underline{x}\| \le 1, \end{cases} \qquad \underline{x} \in D^c, \tag{41}$$

and the perturbation given by the following function:

$$c(\underline{x}) = \frac{1}{2}\gamma(\rho, \tfrac{3}{5}, \tfrac{7}{10})\gamma(\theta, \tfrac{4}{9}\pi, \tfrac{5}{9}\pi), \ \ \underline{x} = \rho(\cos\theta, \sin\theta)', \ \ \rho \in [0,1], \ \theta \in [0, 2\pi). \tag{42}$$

Example 4 Let us consider $N_\omega = 1$, $\omega_1 = 100$, the heat flux $f(\underline{x}) = 1$, $\underline{x} \in \partial D$, the thermal diffusivity of the unperturbed medium given by the function (41) and the perturbation given by the following function:

$$c(\underline{x}) = \frac{1}{2}\gamma(\rho,\tfrac{3}{5},\tfrac{7}{10})\gamma(\theta,\tfrac{4}{9}\pi,\tfrac{5}{9}\pi) + \frac{1}{5}\gamma(\rho,\tfrac{4}{5},\tfrac{9}{10})\gamma(\theta,\tfrac{13}{9}\pi,\tfrac{14}{9}\pi),$$
$$\underline{x} = \rho(\cos\theta,\sin\theta)^t, \quad \rho\in[0,1], \quad \theta\in[0,2\pi). \tag{43}$$

Example 5 Let us consider $N_\omega = 3$, $\omega_1 = 25$, $\omega_2 = 50$, $\omega_3 = 100$ the heat flux $f(\underline{x}) = 1$, $\underline{x} \in \partial D$, the thermal diffusivity of the unperturbed medium given by the function (41) and the perturbation given by the function (42).

Example 6 Let us consider $N_\omega = 3$, $\omega_1 = 25$, $\omega_2 = 50$, $\omega_3 = 100$, the heat flux $f(\underline{x}) = 1$, $\underline{x} \in \partial D$, the thermal diffusivity of the unperturbed medium given by the function (41) and the perturbation given by the function (43).

In Examples 1–6 the direct problem is solved using the following parameters: $\lambda = 10^{-12}$, $N_\rho = 1000$, $N_\theta = 120$. The inverse problem is solved using the following parameters: $N_g = 45$, $L = 48$, $J = 50$, $\mu = 10^{-4}$. In Figures 1–6 are reported the numerical results obtained for Examples 1–6, respectively.

We note that the support and the sign of the perturbation are satisfactory reconstructed for Examples 1, 2, but the value of the reconstructed perturbation has a smaller value than the original one (see Figures 1 and 2). Moreover, in Examples 3, 4 we have a not satisfactory reconstruction of the support and the value of the

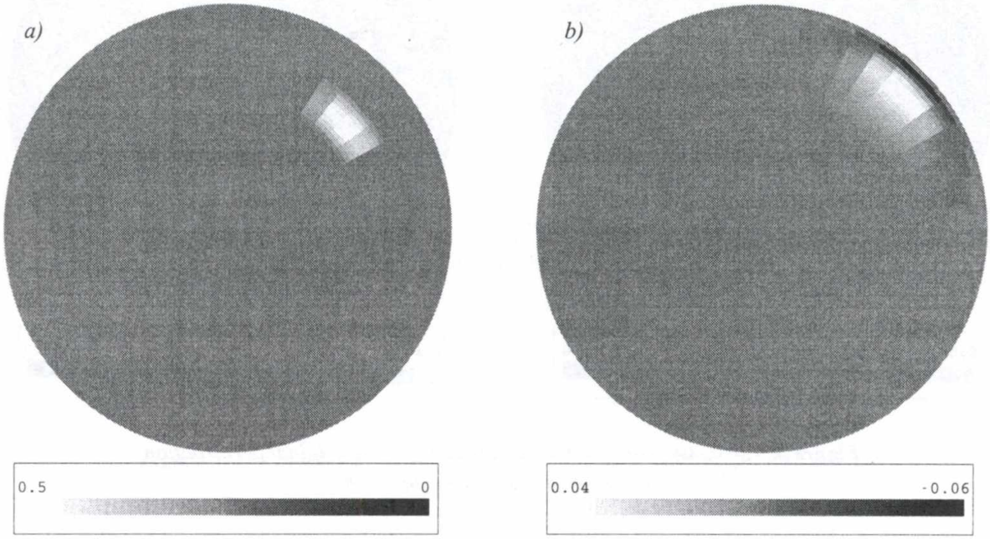

__Figure 1.__ The numerical result for Example 1: (a) the original perturbation, (b) the reconstructed perturbation

perturbation (see Figures 3 and 4). In particular using a large value for ω, i.e. $\omega = 100$, we have a not satisfactory reconstruction of the part of the perturbation having support far from the boundary of the disk D. The reconstruction of this part

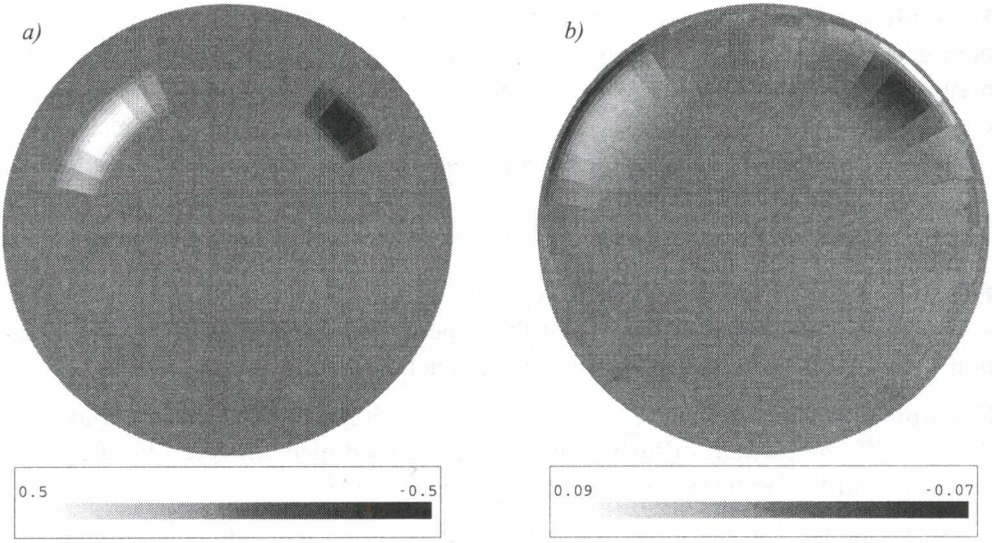

Figure 2. The numerical result for Example 2: (a) the original perturbation, (b) the reconstructed perturbation

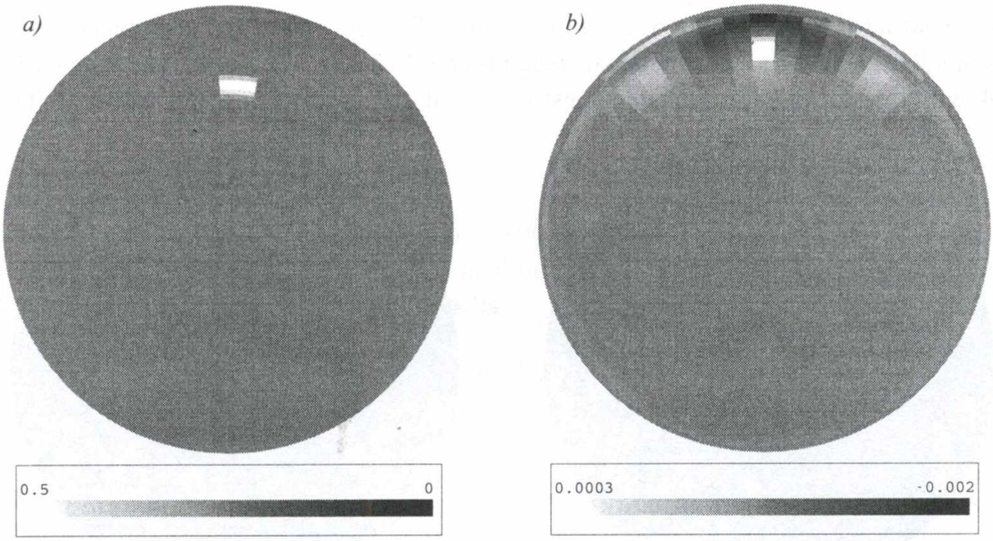

Figure 3. The numerical result for Example 3: (a) the original perturbation, (b) the reconstructed perturbation

of the perturbation improves using either smaller values for the parameter ω or various values for the parameter ω. In Examples 5, 6 three different values for ω are considered, i.e. ω = 25, 50, 100 (see Figures 5 and 6). The reconstructions obtained in Examples 5, 6 improve the reconstructions obtained in Examples 3, 4, respectively. However, as in Examples 1, 2 the value of the reconstructed perturbation is smaller than the original one.

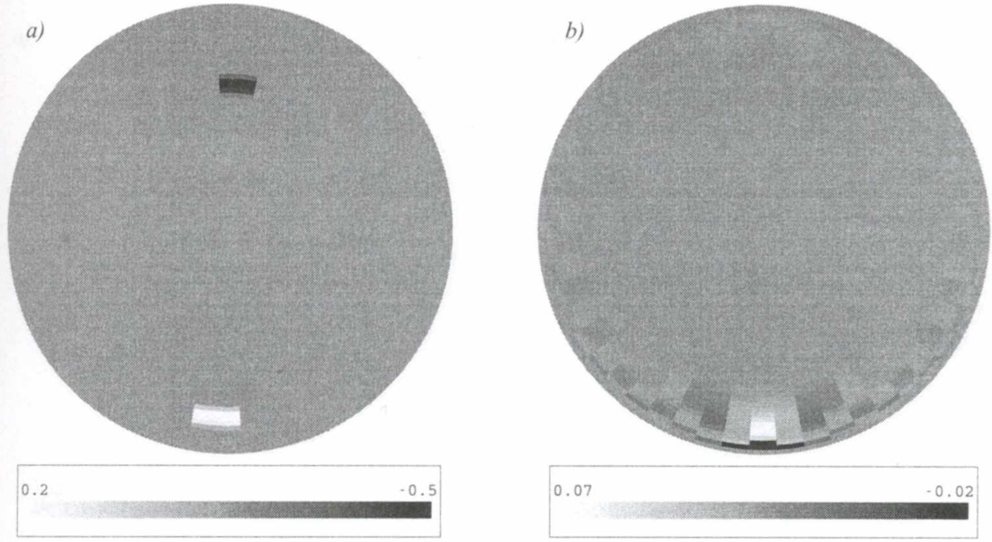

Figure 4. *The numerical result for Example 4: (a) the original perturbation, (b) the reconstructed perturbation*

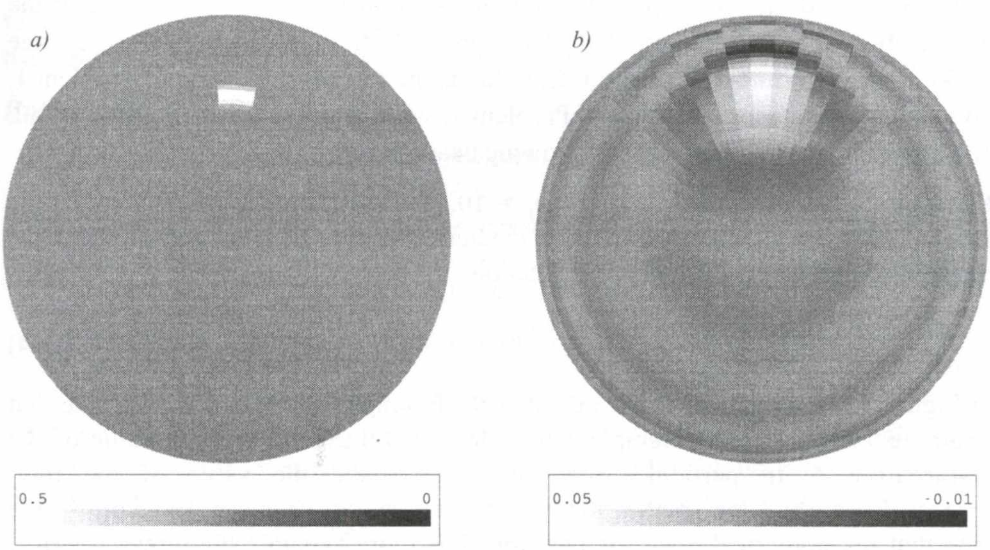

Figure 5. *The numerical result for Example 5: (a) the original perturbation, (b) the reconstructed perturbation*

From some numerical experience which is not reported here we know that the reconstruction of the perturbation does not improve greatly considering slight different values for the approximation parameters: N_θ, N_g, L, J, μ. So we can assert that the linear system (34) gives an accurate approximation of the solution of integral equation (25) when we choose the value of the approximation parameters as described above. The considerable difference between the values of the original

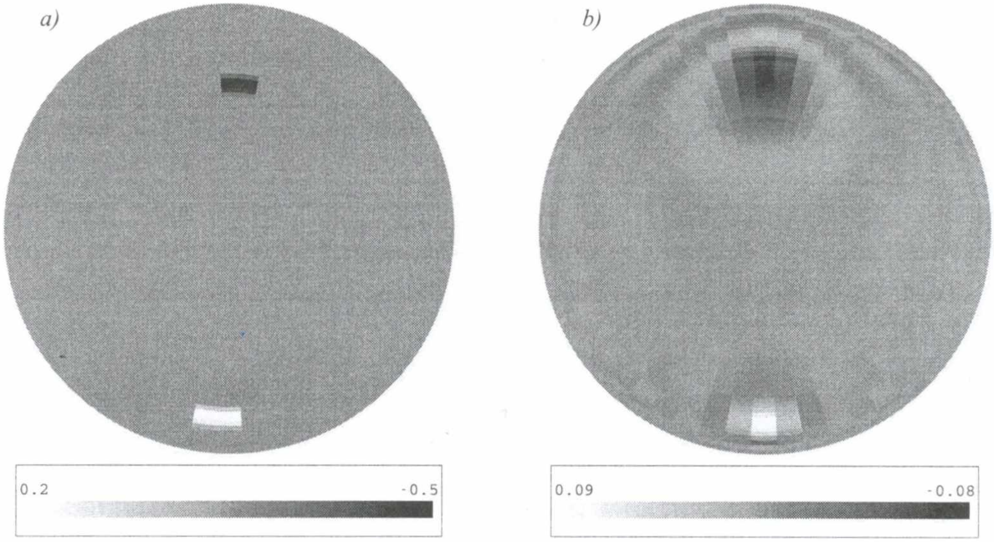

Figure 6. The numerical result for Example 6: (a) the original perturbation, (b) the reconstructed perturbation

perturbation and the reconstructed perturbation seems to be the consequence of the linearization of the inverse problem. More precisely from our numerical experience we know that the integral equation (25), that is the linearized version of Problem 1, gives an accurate approximation of Problem 1, when c and ω are sufficiently small. To show this fact we consider the following example.

Example 7 Let us consider $N_\omega=1$, $\omega_1 = 10$, the heat flux $f(\underline{x}) = 1$, $\underline{x} \in \partial D$, the thermal diffusivity of the unperturbed medium given by the function (37) and the perturbation given by the following function:

$$c(\underline{x}) = \frac{1}{20}\gamma(\rho,\tfrac{3}{5},\tfrac{4}{5})\gamma(\theta,\tfrac{\pi}{6},\tfrac{\pi}{3}),\quad \underline{x} = \rho(\cos\theta,\sin\theta)^t,\quad \rho\in[0,1],\ \theta\in[0,2\pi). \tag{44}$$

In Figure 7 the numerical result obtained for Example 7 is reported. We note that Example 7 differs from Example 1 only for the value of ω and the value of the perturbation c. In particular the value of ω and the value of the ratio c/ω in Example 7 are smaller than the corresponding values in Example 1. We note that the numerical result of Example 7 is more accurate than the numerical result of Example 1.

In the numerical solution of Problem 1 we have approximated the function w_c, where $u + w_c$ is the solution of problem (18), (19), with the solution υ_c of problem (20), (21). This approximation resèmbles the classical Born approximation in the wave equation theory, for details see [21], page 361. A large number of numerical methods for the solution of inverse scattering problems are based on the Born approximation, see for example [22] and [23], page 547. These numerical methods usually are improved considering slight variations of the Born approximation, such as for example *the iterated Born approximation* and *the distorted Born*

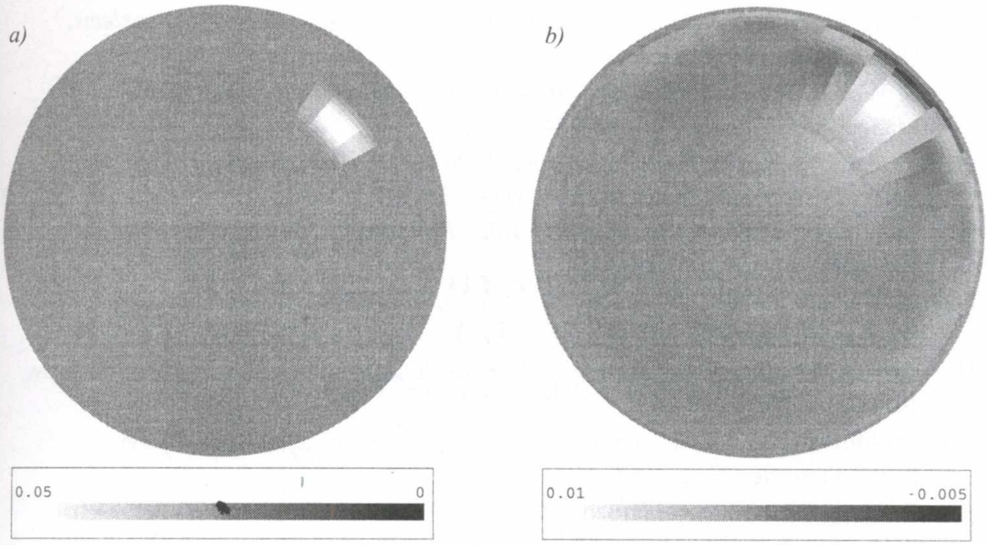

Figure 7. *The numerical result for Example 7: (a) the original perturbation, (b) the reconstructed perturbation*

approximation. The numerical results presented in this paper are promising for futher studies, where iterative techniques can be considered, in particular based on the ideas of the iterated Born approximation and the distorted Born approximation.

References

[1] K. Bryan, L.F. Caudill, *An Inverse Problem in Thermal Imaging*, SIAM Journal on Applied Mathematics, **56**, 1996, 715–735

[2] C. Giordana, M. Mochi, F. Zirilli, *The Numerical Solution of an Inverse Problem for a Class of One–dimensional diffusion equations with piecewise constant coefficients*, SIAM Journal on Applied Mathematics, **52**, 1991, 428–441

[3] M. Mochi, G. Pacelli, M.C. Recchioni, F. Zirilli, *An inverse problem for a class of two dimensional parabolic equations with piecewise constant coefficients*, to appear in Journal of Optimization Theory and Applications

[4] A. Pierce, *Unique Identification of Eigenvalues and Coefficients in a Parabolic Problem*, SIAM Journal on Control and Optimization, **17**, 1979, 494–499

[5] D. Colton, R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, Springer–Verlag, Berlin, 1992

[6] P. Maponi, M.C. Recchioni, F. Zirilli, *The use of optimization in the reconstruction of obstacles from acoustic or electromagnetic scattering data, Large–Scale Optimization with Applications*, Part I, L.T. Biegler, T.F. Coleman, A.R. Conn, F.N. Santosa Editors, IMA volumes in mathematics and its applications, Springer–Verlag, New York, **92**, 1997, 81–100

[7] P. Maponi, L. Misici, F. Zirilli, *A Numerical Method to Solve the Inverse Medium Problem: an Application to the Ipswich Data*, IEEE Antennas & it Propagation Magazine, **39**, 1997, 14–19

[8] A.E. Yagle, B.C. Levy, *Layer–stripping solutions of multi–dimensional inverse scattering problems*, Journal of Mathematical Physics, **27**, 1986, 1701–1710

[9] S. He, S. Strom, V. Weston, *Time Domain Wave–Splittings and Inverse Problems*, Oxford Science Publication, 1998

[10] M.V. de Hoop, *Generalization of the Bremmer coupling series*, Journal of Mathematical Physics, **37**, 1996, 3246–3282

[11] M. Asch, W. Kohler, G. Papanicolaou, M. Postel, B. White, *Frequency content of randomly scattered signals*, SIAM Review, **33**, 1991, 519–625

[12] O.A. Ladyzhenskaya, *The Boundary Value Problems of Mathematical Physics*, Springer–Verlag, New York, 1985

[13] E. Zauderer, *Partial Differential Equations of Applied Mathematics*, John Wiley & Sons, New York, 1989

[14] Carslaw, J.C. Jaeger, *Conduction of Heat in Solids*, Oxford University Press, Glasgow, 1959

[15] M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions*, Dover Publications, New York, 1965

[16] R. Kress, *Linear Integral Equations*, Springer–Verlag, Berlin, 1989

[17] W.H. Press, S.A. Teukolsky, W.T. Vetterling, B.P. Flannery, *Numerical Recipes in Fortran 77*, vol. I, Cambridge University Press, New York, 1992

[18] I.M. Gel'fand, Shilov, *Generalized Functions*, Academic Press, New York, 1964

[19] E.C. Titchmarsh, *The Theory of Functions*, Oxford University Press, London, 1939

[20] *NAG Fortran Library*, NAG Ltd, Oxford, 1997

[21] A. Nayfeh, *Perturbation Methods*, John Wiley & Sons, New York, 1973

[22] P.M. Van Der Berg, R.E. Kleinman, *Gradient Methods in Inverse Acoustic and Electromagnetic Scattering*, Large–Scale Optimization with Applications, Part I, L.T. Biegler, T.F. Coleman, A.R. Conn, F.N. Santosa Editors, *IMA volumes in mathematics and its applications*, Springer–Verlag, New York, **92**, 1997, 173–194

[23] W.C. Chew, *Waves and Fields in Inhomogeneous Media*, IEEE Press, New York, 1995