NUMERICAL ASPECTS OF DYNAMIC, GEOMETRICALLY NON–LINEAR CALCULATIONS OF ELASTO–VISCOPLASTIC PLATES

PAWEŁ KŁOSOWSKI

Faculty of Civil Engineering,
Technical University of Gdansk,
Narutowicza 11/12, 80-952 Gdansk, Poland

Abstract: In the paper numerical aspects of elasto-viscoplastic dynamic calculations of circular plates are presented. Numerical results are compared with experiments carried out in the laboratory. Each plate is calculated using the self–built computer program based on the finite element method. The Gaussian quadtrature is used for the surface integration and some difficulties concerning the choice of the order of this integration are discussed. To integrate the constitutive equations (the first order differential equations), and the equations of motion (the second order differential equations), it is necessary to use proper value of the time step. Some remarks on selection of this value are given. Density of the finite element mesh used for discretisation of the plate is studied. It was found that the number of elements has bigger influence on the quality of the results than the number of layers used to describe the changes of stresses through the thickness of the plate. Finally, some comments on the choice of the model of damping are made. An approximate method of introduction of this factor in the calculations is presented.

Keywords: dynamics, viscoplasticity, finite element method

1. Introduction

Elasto-viscoplastic dynamic calculations of plates have big industrial importance and are subject of interest of many scientists and laboratories all over the world. Dynamically loaded plates are usually components of more complex structures like turbo generators, aeroplanes, spaceships and other constructions subjected to high temperatures. There are several ways to solve this problem, but none of them is exact. It is always necessary to make some approximations, but a question what is their influence on quality of the results arises. Between the numerical methods of calculation, the finite element method [1], [2] is the most important nowadays. The most substantial approximation, that has the basic influence on the solution, is division of a structure into finite elements and choice of the shape function. Both factors must be carefully studied. From the mathematical point of view, the dynamic problem can be treated as a set of the second order differential equations with respect of time. To solve them it is necessary to include into algorithm one of many methods of their integration. A special class of solving algorithms for this type of problems has been invented [3]–[6]. The time step of integration must be properly selected to fulfil the stability condition and reduce the range of errors. The elasto–viscoplastic constitutive equations are the first order differential equations. Usually, 3–4 equations of this type are linked and have to be solved simultaneously. The review of the types of the elasto–viscoplastic constitutive equations used in the structural analysis can be found in [7]. When the problem can be restricted to small strains only, the most often applied are: the Perzyna method ([9], [8]), the Chaboche model ([10], [11]), and the Bodner–Partom approach ([12]). Result of this integration depends also on the value of the time step used for integration.

Plates and shells are the structures having one dimension (thickness) much smaller than two others. The three-dimensional description of their behaviour must be modified into the two-dimensional one. This problem is very complex and has been a subject of interest of a large group of scientists for many years. First so--called "shell and plates theories" have already been invented in previous century [13], but the question is still open and investigated ([14]–[17]). In the present work the large magnitude of displacements is expected therefore the geometrically nonlinearity of the problem should be taken into account. In the numerical calculations, the first order shear deformation moderate rotation shell theory will be applied [18].

In this paper, the numerical aspects of the mentioned above assumptions and approximations will be discussed on an example of the dynamic calculations of the impulsively loaded circular plates. Estimation of the results of calculations will be based on experiments carried out on this type of plates[1]. They were subjected to a hit of nitrogen gas in the shock wave tube. In tests, steel 1 mm thick plates have been used in the room temperature. Full description of experiments can be found in [19].

2. Finite element mesh

To calculate any structure using the finite element method it is necessary to divide it into finite elements. In the literature, many types of elements, which can be applied in the algorithm, can be found. The choice of the certain type of an element is often determined by the computer program, which is going to be used. A researcher has two possibilities. To use a commercial program with a library of ready finite elements, usually having no possibilities to study all assumptions, which have been made, and having a restricted access to the way in which they are used. The other one is to build an own finite element code. This way is time consuming and the usage of the final product is restricted to a very narrow group of users. Computer programs are very large and need to be tested for many years to be sure that they are free of errors. The main advantage of self–built programs is the full access to their source code and a possibility of making necessary modifications.

In the present work, an own Fortran code has been used. The nine-node

[1] All experiments have been performed in laboratories of Institute of General Mechanics of RWTH Aachen (Germany)

isoparametric finite element, which had been tested in elastic dynamic calculations of plates and shells, has been applied ([20], [21]). Because of the rotational symmetry of geometry and loading only quarter of the plate has been divided into the finite element mesh (Figure 1). This kind of mash is determined by the type of used finite elements and the Cartesian co–ordinate system, which is used for description of the boundary conditions.

The boundary conditions and surface distribution of loading are settled by the support system used in the experiments and has been checked in static calculations of the example. The support system models the plate, wchich is not fully clamped and the load on the outside ring of the plate is equivalent to pressure acting on active part of the plate. The number of elements and nodes has an essential influence on

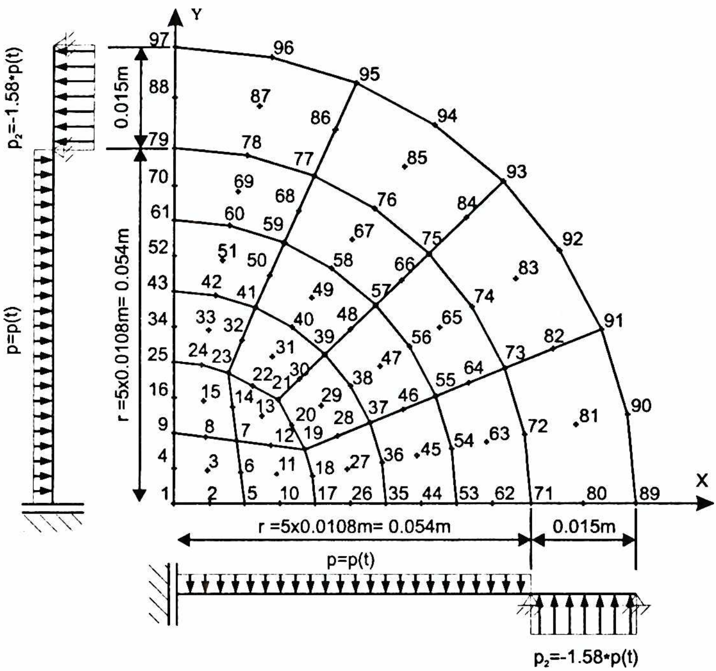

***Figure 1**. The finite element mesh and load distribution on the circular plate.*

results of the calculations. On the one hand larger number of elements increases accuracy, on the other it also increases calculations time, which in dynamic approaches is very long. Usual way of examining the accuracy of results, with respect to number of elements, is performing tests with different numbers of them. In the present work, tests began with 12 elements and 61 nodes. Later in each test additional ring of 4 elements has been added in the inner part of the plate. The number of elements is assumed to be sufficient, when results of two following calculations are similar enough.

3. Integration over the volume of the finite element

In the finite element method, to describe displacements inside the finite element as a function of the nodal displacements, a so-called shape function must be chosen. This function often depends on co-ordinates of the point where the displacements are calculated. Moreover, in each node of the element many components of displacements (often with different physical meaning like translations and rotations) appear. All relations between the nodal quantities and their equivalents in the space of an element can be gathered in a matrix of the shape functions **N**. Derivatives of the components of this matrix are used later in expressions for strains **E** and stresses **s**. In the elasto-viscoplastic approach a convenient way of proceeding with the inner element relation is division of the element into layers [22]. Number of layers determines the distribution of strains and stresses through the plate thickness and has an influence on accuracy of the results. In this paper, it is assumed that the strains and stresses are constant through the layer thickness.

To describe equilibrium equations (which in dynamic calculations are represented by the equations of motion) the vector of the balanced forces **Q** and the vector of the nodal forces related to the loading **R** have to be calculated. They are obtained by integration of stresses **s** or distributed loading **p** over the element volume *v*. Typical relations for a single element *e* have the form:

$$\mathbf{Q}^e = \int_v \mathbf{B}^T \mathbf{s}\, dv; \qquad \mathbf{R}^e = \int_v \mathbf{B}^T \mathbf{p}\, dv. \tag{1}$$

Integration over the volume v can be divided into integration over the thickness *h* and over the midsurface of the plate/shell finite element $\mathcal{A}$. As the vector **s** has been assumed to be constant through the layer thickness and material of the plate is assumed to be isotropic, integrals can be easily calculated over the thickness of the whole element. Integration over the midsurface demands application of a numerical procedure. The same procedure can be also applied in an estimation of the pressure loading on the outside surfaces of the elements. Here the Gaussian scheme of integration is applied. Moreover, in the isoparametric element, for transformation of them, the same matrix of the shape functions **N** as in calculations of displacements is used. The co-ordinates transformation is included in the process of the midsurface integration. The elementary surface area $d\mathcal{A}$ can be expressed in the form:

$$d\mathcal{A} = \det(\mathbf{J})\, d\zeta\, d\vartheta, \tag{2}$$

where: det(**J**) is the determinant of the Jacobi matrix **J**, which denotes the relation between the partial derivatives of the shape function N in the original curved co–ordinate system s, ϕ and the natural co-ordinates ζ and ϑ :

$$\begin{Bmatrix} \dfrac{\partial N}{\partial \zeta} \\ \dfrac{\partial N}{\partial \vartheta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial s}{\partial \zeta} & \dfrac{\partial s}{\partial \vartheta} \\ \dfrac{\partial \phi}{\partial \zeta} & \dfrac{\partial \phi}{\partial \vartheta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial N}{\partial s} \\ \dfrac{\partial N}{\partial \phi} \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} \dfrac{\partial N}{\partial s} \\ \dfrac{\partial N}{\partial \phi} \end{Bmatrix} \tag{3}$$

The integral of any function $\overline{\mathcal{F}}(s,\phi)$, which can be expressed by a function $\mathcal{F}(\zeta,\vartheta)$, is numerically calculated using the expression (Gaussian quadrature):

$$\int_{\mathcal{M}} \overline{\mathcal{F}}(s,\phi)\, d\mathcal{M} = \int_{-1}^{1}\int_{-1}^{1} \mathcal{F}(\zeta,\vartheta)\det(\mathbf{J})d\zeta\, d\vartheta = \sum_{i=1}^{n}\sum_{j=1}^{n} W_i W_j \mathcal{F}(h_i,h_j), \tag{4}$$

where h_i, h_j are abscissas and W_i, W_j weight coefficients, n is the order of the numerical integration. The values of abscissas and weight coefficients can be found in [1]. The order of integration n is related to the used type of the finite element. For the nine-node isoparametric element, three types of integration can be used. None of them is faultless. When $n = 2$, a so-called reduced integration is introduced. Its main drawback is that the mass matrix, which is necessary for the dynamic applications is singular and can not be inverted. The full integration ($n = 3$) leads to overestimation of stiffness of shear terms for thin structures like plates and shells and therefore gives wrong values of displacements. The compromise between these two methods is a uniformly reduced integration, in which for bending terms the full integration is used and for shear ones the integration is reduced. Unfortunately this method often gives (especially for long narrow structures) a so-called "hour-glass effect" — neighbouring elements have opposite signs of the rotational degrees of freedom. For circular plates the "hour--glass effect" usually does not appear and this type of integration is used in the present work.

4. Integration of the constitutive equations

The constitutive equations are relations between strains and stresses. For isotropic elastic material only two coefficients: E – the Young modulus, and ν – the Poisson coefficient, are necessary to write this relation down. If additionally, according to the shell theories, the plane state of stress in each layer parallel to the midsurface is assumed, this relation can be written in the incremental form useful in geometrically non-linear applications:

$$\dot{\mathbf{s}} = \begin{Bmatrix} \dot{\sigma}_x \\ \dot{\sigma}_y \\ \dot{\tau}_{xy} \end{Bmatrix} = \mathbf{D}\cdot\dot{\mathbf{E}}^e = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{Bmatrix} \dot{\varepsilon}_x \\ \dot{\varepsilon}_y \\ \dot{\gamma}_{xy} \end{Bmatrix}, \tag{5}$$

where: $\dot{\mathbf{s}}$ stands for increment of stresses and $\dot{\mathbf{E}}^e$ for increment of elastic strains.

In the viscoplastic domain, the stresses–strain relations are more complicated. Their three forms, for the most popular types of description, are given in Table 1.

Each column in Table 1 is a set of the first order differential equations with respect of time. The first obstacle is that to use them it is necessary to determine many material parameters. When this step is done, the problem of a solution method of these equations still remains. The explicit methods like: the Euler, middle point or

Table 1. Constitutive equations

Model	Perzyna	Chaboche	Bodner-Partom
inelastic strain rate	$\dot{\mathbf{E}}^I = \frac{3}{2}\dot{p}\frac{\mathbf{s}'}{J(\mathbf{s}')}$	$\dot{\mathbf{E}}^I = \frac{3}{2}\dot{p}\frac{\mathbf{s}'-\mathbf{X}'}{J(\mathbf{s}'-\mathbf{X}')}$	$\dot{\mathbf{E}}^I = \frac{3}{2}\dot{p}\frac{\mathbf{s}'}{J(\mathbf{s}')}$
deviatoric part of the stress tensor	$\mathbf{s}' = \mathrm{dev}(\mathbf{s}) = \mathbf{s} - 1/3 \cdot \mathrm{tr}(\mathbf{s})\mathbf{I}$		
cumulated inelastic strain rate	$\dot{p} = \gamma\left\langle\frac{J(\mathbf{s}')-k}{k}\right\rangle^n$	$\dot{p} = \gamma\left\langle\frac{J(\mathbf{s}'-\mathbf{X}')-R-k}{K}\right\rangle^n$	$\dot{p} = \frac{2D_0}{\sqrt{3}}\exp\left[-\left(\frac{Z}{J(\mathbf{s}')}\right)^{2n}\frac{n+1}{2n}\right]$
	$j(\mathbf{a}) = (3/2\,\mathbf{a}:\mathbf{a})^{\frac{1}{2}} = (3/2\,a^{ij}:a_{ij})^{\frac{1}{2}}$ $\langle x\rangle = \frac{1}{2}(x+\lvert x\rvert)$		$Z = R + \mathbf{X}:\frac{s}{J(s)}$ $\dot{W}^I = \mathbf{s}:\dot{\mathbf{E}}^I$
isotropic hardening	—	$\dot{R} = b(R_1 - R)\dot{p}$ $R(t=0) = 0$	$\dot{R} = m_1(R_1 - R)\dot{W}^I$ $R(t=0) = R_0$
kinematic hardening	—	$\dot{\mathbf{X}} = \frac{2}{3}a\dot{\mathbf{E}}^I - c\mathbf{X}\dot{p}$	$\dot{\mathbf{X}} = m_2\left(\frac{3}{2}D_1\frac{s}{J(s)} - \mathbf{X}\right)\dot{W}^I$
material parameters	γ, k, n	k, K, n, a, c, b, R_1, γ	n, D_0, D_1, R_0, R_1, m_1, m_2

Runge-Kutta method, or implicit like: the trapezoidal rule or the Adams--Multon–Beshforth algorithms can be applied. In the considered problem the differential equations, which are going to be solved, have a complex form and are linked together. During integration, geometrical non-linearity of the problem must also be taken into account. For this reason, to reduce the storage of data and increase the speed of calculations, only the simplest methods are useable. Here the trapezoidal rule of integration, as relatively well stable and because of iterations reducing the range of errors, has been chosen.

If for any function $f(x, y)$ one wants to find a function $y(x)$, which is a solution of the differential ordinary equation:

$$\frac{dy}{dx} = f(x,y) \text{ with initial condition } y(a) = c, \tag{6}$$

using the trapezoidal method of solving, he have to apply the following numerical expression:

$$y_{n+1} - y_n = \tfrac{1}{2}h\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right], \qquad n = 0,1,\ldots. \tag{7}$$

As the term y_{n+1} appears on both sides of the equation the iterations have to be performed on each step n. For the k-th iteration the relation has a form:

$$y_{n+1}^{k+1} = \tfrac{1}{2} h f\left(x_{n+1}, y_{n+1}^{k}\right) + y_n + \tfrac{1}{2} h f\left(x_n, y_n\right), \tag{8}$$

where h is a step of integration.

For the first iteration the starting procedure is necessary:

$$y_{n+1}^{0} = y_n + h f\left(x_n, y_n\right). \tag{9}$$

In the literature, the following condition of stability of the method, for a single non-linear differential equation, can be found [23]:

$$\frac{1}{2} h \left\| \frac{\partial f}{\partial y_n} \right\| < 1. \tag{10}$$

It has to be satisfied for each equation given in Table 1, at any point in the plate volume, for arbitrary time step n and iteration k. In practice that means that small enough time step $\Delta t = h$ has to be used in calculations.

5. Integration of the equations of motion

The chosen method of integration of the constitutive equations needs small value of the time step. As for integration of the equations of motion the same time step Δt will be applied, the central difference method seems to be the most effective. It bases on a set of two equations describing relation between the accelerations of the nodal points $\ddot{\mathbf{q}}_t$, the nodal velocities $\dot{\mathbf{q}}_t$ and the nodal displacements $\mathbf{q}$ at time $t - \Delta t$, t, and $t + \Delta t$

$$\begin{aligned} \ddot{q}_t &= \frac{1}{\Delta t^2}\left(q_{t-\Delta t} - 2q_t + q_{t+\Delta t}\right), \\ \dot{q}_t &= \frac{1}{\Delta t}\left(-q_{t-\Delta t} + q_{t+\Delta t}\right). \end{aligned} \tag{11}$$

To apply this method, it is convenient to write the equation of motion at time t in the form:

$$\mathbf{M} \cdot \ddot{\mathbf{q}}_t + \mathbf{C} \cdot \dot{\mathbf{q}}_t + \mathbf{Q}_t = \mathbf{R}_t, \tag{12}$$

where: **M**, **C** stand for the mass and damping matrices respectively; $\mathbf{Q}_t$ is the vector of nodal balanced forces including all non-linear effects; $\mathbf{R}_t$ is the vector of nodal loading forces.

The central difference method is conditionally stable even in its linear variant. The convergence of the calculations is guaranteed, when the time step Δ_t fulfils the condition (the sufficient condition):

$$\Delta t \le \Delta t_{cr}; \qquad \Delta t_{cr} = 2 / \omega_{max}, \tag{13}$$

where ω_{max} is the highest frequency of free vibrations of the structure.

The value of the highest frequency ω_{max} in pesented variant of geometrically non-linear calculations does not change significantly. It mainly depends on the size of the used finite elements: for smaller elements, smaller time step must be used. Condition can be checked in the initial configuration. It is enough to build the procedure of the highest frequency calculation. In this work, a self–built procedure based on the Jacobi algorithm has been used.

The central difference method becomes especially effective when the mass matrix **M** and the damping matrix **C** are diagonal. The well-known nodal quadrature method of lumping of both matrices on the level of a single element can be found in [1], [2].

$$\left[M_{ij}^{e}\right]=\iint_{\mathcal{M}}\left(\left[N_{i}\right]^{T}\mathbf{i}\left[N_{j}\right]\right)d\mathcal{M}=\sum_{q=1}^{9}W_{q}\left(\left[N_{i}\right]^{T}\mathbf{i}\left[N_{j}\right]\right)_{q}\det(J)_{q}, \tag{13}$$

$$\left[C_{ij}^{e}\right]=\iint_{\mathcal{M}}\left(\left[N_{i}\right]^{T}\boldsymbol{\mu}\left[N_{j}\right]\right)d\mathcal{M}=\sum_{q=1}^{9}W_{q}\left(\left[N_{i}\right]^{T}\boldsymbol{\mu}\left[N_{j}\right]\right)_{q}\det(J)_{q}, \tag{14}$$

where **i** and $\boldsymbol{\mu}$ are matrices of inertia and damping properties of a finite element.

The method is a variant of the Gaussian quadrature with abscissas in the nodal points. In this case, at each node i all shape functions except N_i have zero value, and that leads to diagonalisation of the matrices **M** and **C**.

6. Examples of test calculations

Several testing calculations of circular steel plates (h = 1 mm, E = 215.7 GPa, ν = 0,33) have been performed to check the numerical aspects mentioned before. It is difficult to separate each problem, therefore while one aspect of the calculations is examined, the other parameters are assumed to be fixed.

To start the inelastic calculations it is necessary to choose the time increment of integration. As it has been mentioned before, the same time step Δt is used for both time integrations. For the finite element mesh presented in Figure 1 the highest frequency of free vibrations is $\omega_{max} = 1.1345179\cdot 10^{7}\ s^{-1}$. The corresponding critical time step is $\Delta t_{cr} = 1.7628\cdot 10^{-7}$ s. The lowest free frequencies are: $\omega_1 = 4.77228\cdot 10^{3}\ s^{-1}$ and $\omega_2 = 1.670810\cdot 10^{4}\ s^{-1}$, and are much smaller than ω_{max}. Therefore, in the calculations even a slightly smaller than Δt_{cr} value of the time step gives very exact results. The time step estimation with respect of integration of the constitutive equations is more difficult. It is necessary to examine each differential function at each point, where it is calculated. It is more convenient to make several calculations with different values of the time step. The divergence of the results is clearly visible – after several time steps displacements are going to infinity and the calculations are broken. When small enough values of the time step are selected results are very similar. That means that local and global errors in this case are small.

In the present work, it has been found that the time step $\Delta t = 1.5\cdot 10^{-7}$ s guarantees convergence of both types of the integration.

As it can be seen from Table 1, the Chaboche model of the constitutive equations is the extension of the earlier invented Perzyna model. Both models have the same definition of the yield limit and the inelastic strains. Only the form of the hardening functions is different. The Bodner–Partom constitutive relations assume that the viscous effects are present from the beginning of the deformation process. They have the same type of description of hardening functions as in the Chaboche model, but inner relations used in them are different. For these reasons, calculations of tested examples were performed with the Chaboche and the Bodner–Partom types of the constitutive equations. In the Bodner–Partom approach the following values of the material parameters have been used:

$$E = 215661 \text{ MPa}, \quad D_0 = 10000\, s^{-1}, \quad n = 9.61, \quad D_1 = 21.35 \text{ MPa},$$

$$m_1 = 0.068 \text{ MPa}^{-1}, \quad m_2 = 1.82 \text{ MPa}^{-1}, \quad R_0 = 259.38 \text{ MPa}, \quad R_1 = 422.90 \text{ MPa}.$$

For the Chaboche equations, the material parameters have values:

$$E = 215661 \text{ MPa}, \quad n = 9.51, \quad k = 210.15 \text{ MPa}, \quad K = 14.085 \text{ MPa},$$
$$c_1 = 37730, \quad a_1 = 685882 \text{ MPa}, \quad R_1 = 138.26 \text{ MPa}, \quad b = 17.64.$$

In the first set of tests, the number of layers has been investigated. The midpoint deflection functions are compared to experimental results in Figure 2. In these calculations, the Chaboche model of constitutive equations and the mesh shown in Figure 1 have been used. The presented time functions of displacements show that number of layers has a secondary influence on the quality of the results. Even for four layers, the results are very similar to the experimental ones. Six and ten layers give almost the same response.

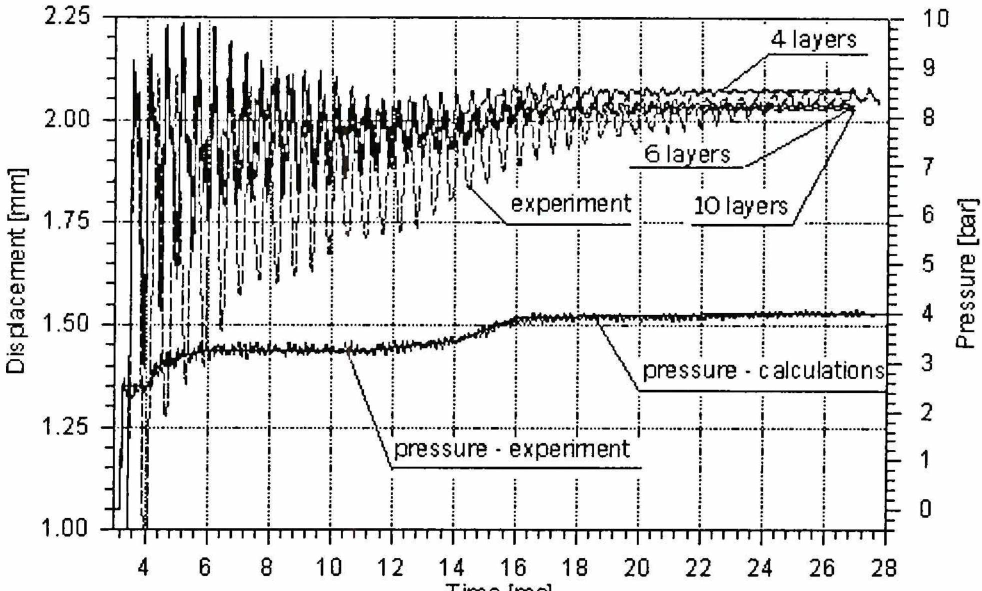

***Figure 2.** The time functions of the middle point deflection for different number of layers.*

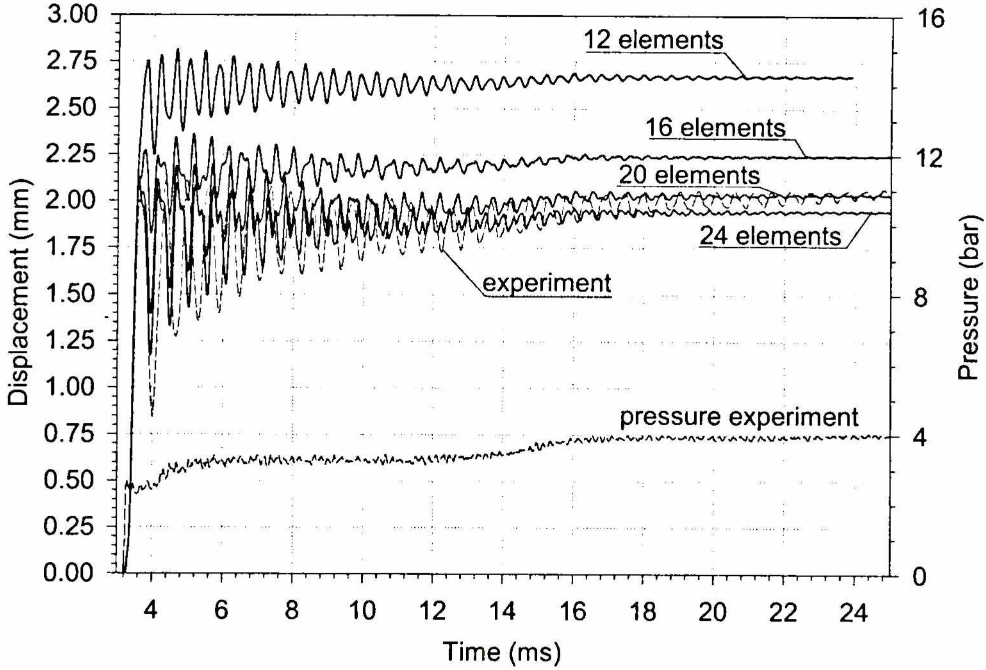

Figure 3. *The time functions of the middle point deflection for different number of elements — the Chaboche model.*

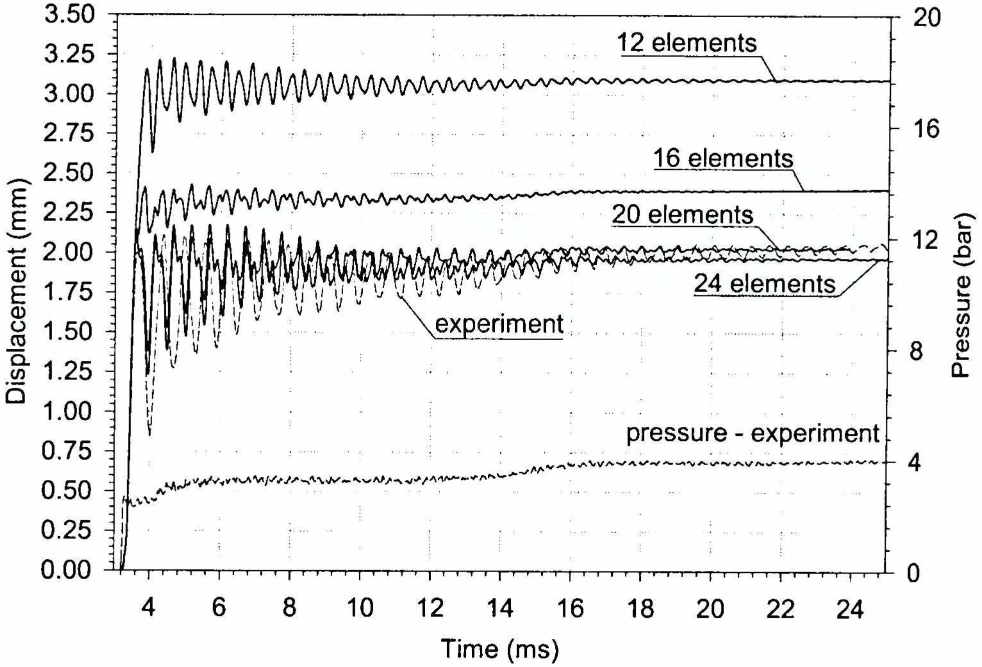

Figure 4. *The time functions of the middle point deflection for different number of elements — the Bodner–Partom model.*

The number of finite elements is studied in Figure 3 (the Chaboche model), and in Figure 4 (the Bodner–Partom model). The number of layers has been restricted to six. In both cases, solutions for 12 and 16 elements do not coincide with the experimental result. A better solution has been obtained for 20 and 24 elements. Both types of the finite element mesh produce solutions similar to each other and to the function recorded experimentally. The calculations are more sensitive on the number of elements than on the number of layers.

Usage of the consistent mass matrix instead of the lumped one has a small influence on the dynamic response of a structure. Here, both calculations produce the same displacement functions. It is only necessary to mention that the application of the consistent mass matrix changes the value of the highest free frequency and consequently a smaller time step of the time integration must be used.

Influence of the different damping matrix forms has not been studied because of lack of sufficient data. To find values of the damping coefficients, it is necessary to make additional difficult experiments. Here, the elastic experiments have been compared with the calculations with a single value of the damping coefficient used in the lumped damping matrix for all degrees of freedom describing displacements in the normal to the midsurface direction. It has been assumed that the value of this coefficient is the outside pressure proportional:

$$\mu_{33} = \mu^0 \cdot \left(1 + p / p_0\right), \tag{16}$$

where p is the actual value of the pressure, p_0 is the atmospheric pressure (1 bar), $\mu^0 = 600\mathrm{N \cdot s/m^3}$ is an assumed value of a damping coefficient.

7. Conclusions

The presented above calculations show, that not all numerical approximations used in the finite element method have the same influence on the quality of the results. Except of the proper identification of coefficients of the constitutive equations the most important are the selection of the type of the finite elements and their mesh. Dynamic calculations are time consuming. To calculate a single example it is necessary to use a high-performance workstation for more than 18 hours. In addition, storage of the results requires a large amount of space on a hard disk. Very often analysis must be restricted to the time functions of displacements in some point or configurations of the plate in some special time moments. An extremely large amount of data is necessary in the analysis of the stress distribution in the plate volume. In the dynamic calculations, it is necessary to keep a balance between increasing number of elements and layers on the one hand, and the calculations time and storage space on the other.

A problem of identification of the damping properties of the plate material and surrounding space, having a big influence on the result, is still open. The damping description used in the paper can be treated as the first approximation and still more efficient method or reworking of it is necessary.

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