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STOCHASTIC FEM ANALYSIS OF STRIP FOUNDATION

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Abstract: The paper offers a stochastic description of a random soil medium subjected to external loading. The strip foundation on a soil layer in the 3D and the 2D strain states is analysed. It is assumed that the soil medium is statistically homogeneous and its mechanical behaviour governed by the linear elasticity theory. It is also assumed that elastic parameters can be modelled as the multidimensional random fields. Stochastic 2D and 3D finite element methods (SFEM), based on the Monte Carlo technique were used. The influence of elements supports attached along vertical planes on standard deviations of displacements and stresses is discussed. Local averages of random field of elastic parameters are introduced. The convergence of applied in SFEM simulation algorithm was tested. The analysis performed enables determination of the standard deviations of components of the stress tensor and the displacement vector for the 3D state, based on the solution for the 2D plane strain state.

Keywords: soil medium, stochastic models, finite–element description

1. Introduction

One of the basic tasks of soil mechanics is to determine the state of strains and stresses in a soil medium, subjected to external loading. In nature, soils intrinsically involve randomness and uncertainty. Thus, one of the fundamental decisions is whether the model should be deterministic or stochastic. Deterministic models are useful, but stochastic models are more realistic. The problem is that the stochastic models are difficult to handle mathematically. Considerable effort has been recently made to improve models of the soil properties by describing them as stochastic processes, or more generally as random fields.

Usually the state of strains and stresses in soil mechanics is calculated for the assumption of a linear, rarely non-linear elasticity theory. In case of stochastic medium, the elasticity theory becomes random and is governed by stochastic differential equations. Though such equations have been studied and developed by many researches in different disciplines of physics and engineering, it is only recently that attempts have been made to develop the theory of stochastic equations. In fact, this field has not yet been sufficiently explored, and almost all results that have been obtained so far refer to specific situations.

For an isotropic and homogeneous elastic soil medium, Young's modulus and Poisson's ratio can be taken as a pair of elastic parameters. The spatial variability of these parameters can be efficiently modelled as a multivariate and multidimensional random field (Wilde [1981], Vanmarcke [1983]).

A solution of the stochastic partial differential equations governing random elasticity, leads to displacements and stresses, which are also multivariate, three-dimensional random fields. This solution is, in most cases, obtained numerically, using mainly the stochastic finite element method. Many variants of this method have recently been developed (Bucher and Shinozuka [1988], Deodatis [1990], Liu et al. [1987], Shinozuka [1987], Spanos and Ghanen [1991], Yamazaki and Shinozuka [1988]).

In many geotechnical engineering problems, e.g. retaining walls, strip foundations, or slopes and embankments, the plane strain analysis is widely used. Such analysis is reasonable for elongated bodies of uniform cross sections subjected to uniform loading along their longitudinal axes. This means that in a stochastic soil medium full correlation in this direction exists. Usually this is not so, and some soils, depending on their origin, can exhibit a significant horizontal variability of the material's elastic parameters. For taking such variabilities into account, the three-dimensional stochastic analysis is necessary.

In the paper, the strip foundation on the soil layer in the 3D and the 2D strain states are analysed. The stochastic 2D and 3D finite element methods, based on the Monte Carlo technique are used. The resulting components of stresses and strains for the 3D state are compared with those for the 2D state. The performed analysis enabled determination of the standard deviations of components of the stress tensor and the displacement vector for the 3D state, based on the solution of the 2D plane strain state.

2. Stochastic soil description

It is assumed in this paper that the soil medium is a linearly elastic and isotropic body, so its response is defined by two elastic parameters: Young's modulus E and Poisson's ratio ν. Randomness of these parameters influences the distribution of displacements and stresses in soil medium. It is well known that the relative variation of Poisson's ratio is much smaller than the variation of modulus of elasticity. Thus, it is further assumed that, for simplicity, ν is taken as a deterministic constant, while E is a homogeneous random field, which for a three-dimensional space can be presented in the following form:

$$E = \overline{E} + \tilde{E}(x_1, x_2, x_3) = \overline{E}[1 + \alpha\beta(x_1, x_2, x_3)], \tag{1}$$

where: α — coeffiecient of variation,
$\beta(x_1, x_2, x_3)$ — normalized, homogeneous random field,
$\langle\beta(x_1, x_2, x_3)\rangle = 0$, $\mathrm{Var}[\beta(x_1, x_2, x_3)] = 1$,
$\langle\ldots\rangle$ — averaging operator.

For the convenience of further analysis, only the separable correlation structure of random field β is considered. The following correlation functions were taken into account:

$$R(x,y,z)=e^{-a_x|x|-a_y|y|-a_z|z|}, \tag{2a}$$

$$R(x,y,z)=\left(1+c_x\cdot|x|\right)e^{-c_x|x|}\left(1+c_y\cdot|y|\right)e^{-c_y|y|}\left(1+c_z\cdot|z|\right)e^{-c_z|z|}, \tag{2b}$$

$$R(x,y,z)=\cos\left(d_x\cdot|x|\right)\cos\left(d_y\cdot|y|\right)\cos\left(d_z\cdot z\right)\cdot e^{-d_x\cdot|x|-d_y\cdot|y|-d_z\cdot|z|}, \tag{2c}$$

where: $x = x_i - x_j$, $y = y_i - y_j$, $z = z_i - z_j$,
$a_x, a_y, a_z, c_x, c_y, c_z, d_x, d_y, d_z$ — correlation decay coefficients.

3. Stochastic finite element method

In the finite element method, the medium is discretized into a finite number of elements connected by nodes. A shape function, determining displacements inside elements, is assumed and then the stiffness matrix is determined. Taking into account all elements, a global stiffness matrix is built up. Introducing boundary conditions, nodal displacements are sought. Eventually, they allow to compute strains and stresses inside elements.

The linear finite element equations are:

$$\mathbf{K}\cdot\mathbf{U}=\mathbf{F}, \tag{3}$$

where **K** is the global stiffness matrix, **U** is the vector of nodal displacements and **F** is the global nodal load vector.

Due to the uncertainty in the basic variables, the quantities in the equation (3) are uncertain too.

The stochastic finite element method based directly on the Monte Carlo technique is the simplest and the oldest variant of SFEM. First, a set of soil properties in all elements, for a given random field describing soil medium is generated. Next, the deterministic computations for each realisation are performed. As a result, a finite number of realisations of displacements and in consequence strains and stresses are obtained. Then, statistical parameters such as average values, variances or correlation functions are determined.

The way of simulation of random field is essential in SFEM. An effective way of such simulation was proposed by Wilde [1981] and Shinozuka [1987]. In the present paper, the simulation algorithm proposed by Skowronek [1985] and developed by Bielewicz, Górski and Walukiewicz [1994] is used. The Majority of calculations, including 3D SFEM, were performed at TASK Computer Centre (Gdansk, Poland).

4. Strip foundation on random subsoil

Let us consider an infinite strip foundation of width B, laid on the elastic, random horizontal stratum, resting on a smooth, rigid base. A load acting on the strip is uniform and flexible of intensity p. The random horizontal stratum is assumed to be weightless. So, this is a case of the classic 3D boundary problem, that for the deterministic soil medium can be analysed in the plane strain state. In practice, however, the described problem of the strip foundation is generally analysed in the plane strain state also for a case of stochastic soil medium. It is caused by technical difficulties resulting from the lack of the proper 3D numerical programs, as well as time and memory limitations of available computers.

The finite element mesh applied in a plane strain analysis is shown in Figure 1. It consists of 256 square 1.25 × 1.25 m elements. The supports of elements at the base allow only for the horizontal displacements, whereas supports at both vertical sides of the mesh allow only for the vertical ones. The numerical calculations were performed for the following data: external loading from strip foundation $p = 10$ [kPa], thickness of soil stratum $h = 20$ [m], width of strip foundation $B = 2.5$ [m], the

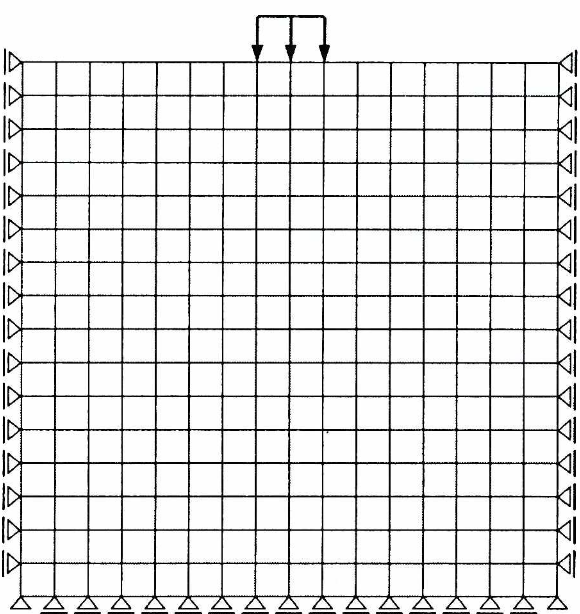

Figure 1. *Finite element mesh applied in* ***2-D*** *analysis*

average value of Young's modulus $\overline{E} = 100$ [MPa], Poisson's ratio $\nu = 0.3$, coefficients of variation $\alpha = 0.1$ and three values of decay coefficient $\lambda = 1$, 2 and 5 [m^{-1}] ($\lambda = c$ according to 2b).

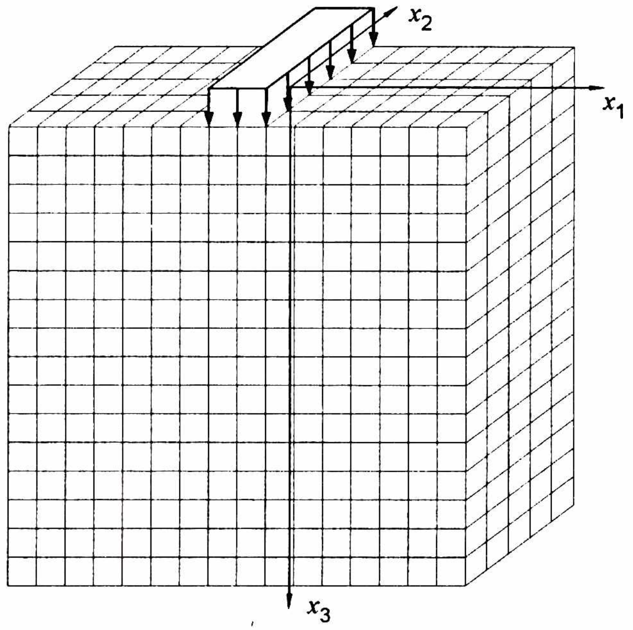

Figure 2. *Finite element mesh applied in 3-D analysis*

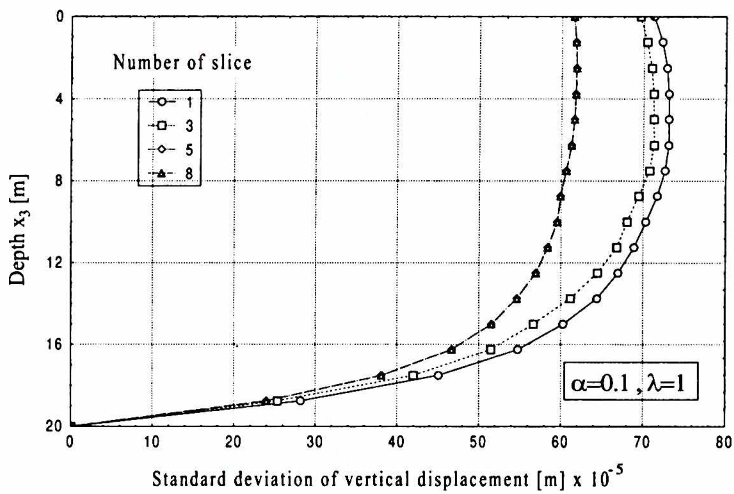

Figure 3. *Change of standard deviation of vertical displacement with depth, for different slices*

The strip loading and the FEM mesh in 3D is presented in Figure 2. The assumed finite element mesh consists of 1280 (16 × 16 × 5) cubic elements, and each side is 1.25 m long. The displacements at the vertical sides and at the bottom of the mesh are analogous to those presented for the 2D state in Figure 1.

In the 3D analysis, only five slices of finite elements in the longitudinal (x_2)

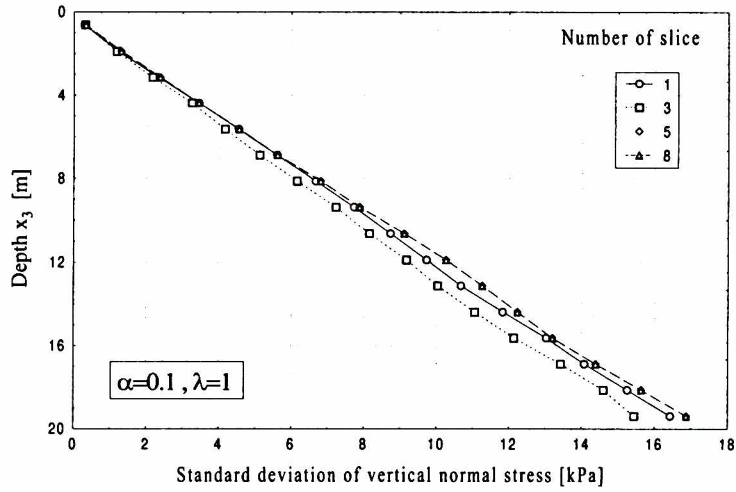

Figure 4. *Change of standard deviation of vertical normal stress with depth, for different slices*

direction were considered. It is justified as for the assumed parameters characterising randomness of the soil, the influence of element supports in vertical planes x_1x_3 on standard deviation of displacements and stresses is meaningful only in two border slices of elements. It is visible in Figure 3 (displacements) and Figure 4 (stresses), where calculations were performed for the mesh presented in Figure 1 and for the infinite, uniformly distributed vertical external loading. The influence of element supports vanishes pretty fast with distance to vertical boundary planes. This influence depends on the coefficient of variation of elasticity modulus α and dicay coefficient λ. Some additional calculations shown that it is directly proportional to α and inversely proportional to λ.

The influence of element supports in vertical planes was also confirmed in the 3D state computations, in the initial verification tests, where twelve slices of elements in the longitudinal direction were assumed. In order to eliminate this influence utmost slices can be omitted and in the following, only results for the middle slice in the 3D state are presented.

5. Local averages of random field

The random field of elastic parameters is in fact a continuos field. The medium discretisation by square or cubic finite elements imposes the field discretisation through an assumption of a finite, multivariate random variable. The random variables in each element are correlated with random variables in other finite elements. They all form the multivariate random field that is characterised by probability density function, which is used in the simulation procedure. The easiest way to create such variable is through an assumption that there is a full correlation inside each element and the correlations between elements are equal to correlations between variables occurring in geometric centres of elements. Such way of discretisation does not take into account the dimensions of elements and spatial correlations occurring inside those elements. Usually it can significantly influence calculation results. In order to include those two factors in the analysis, the local average of random field should be performed. Such average not spoil the Gaussian distribution, but it causes a change of parameters describing probability density function.

The local averages procedure, introduced by Vanmarcke [1983], also described and developed by Knabe, Przewłócki, Różyński [1998] was adopted in the present paper. One-dimensional elasticity modulus $E(z)$, averaged over an interval Δz can be written in the following form:

$$E_{\Delta z} = \frac{1}{\Delta x}\int_{\Delta z} E(z)dz \tag{4}$$

The averaged elasticity modulus depends on the position of the interval over which integration is performed and it is the random variable. Its expected value is constant and equal to the one determined for an input realisation. Its variance depends on element's dimensions Δz. For increasing dimensions more fluctuations in the averaging process are reduced so the variance of average random value decreases. In order to gain independence of element dimensions, a variance function $G(\Delta z)$ is introduced:

$$G(\Delta z) = \frac{Var[E_{\Delta z}]}{Var[E]} = \frac{\sigma^2_{E_{\Delta z}}}{\sigma^2_E}\ , \tag{5}$$

where: $Var[E_{\Delta z}] = \sigma^2_{E_{\Delta z}}$ — variance of averaged random variable $E_{\Delta z}$,

$Var[E] = \sigma^2_E$ — point variance determined in the input set of realisation.

The variance function can be computed for any correlation function:

$$G(\Delta z) = \frac{1}{(\Delta z)^2}\int_0^{\Delta z}\int_0^{\Delta z} R(z_1 - z_2)dz_1dz_2 = \frac{2}{\Delta z}\int_0^{\Delta z}\left(1 - \frac{z_2}{\Delta z}\right)R(z_2)dz_2, \tag{6}$$

and for example for (2b) it equals:

$$G(\Delta z) = \frac{2}{c\cdot\Delta z}\left[2 + e^{-c\cdot\Delta z} - \frac{3}{c\cdot\Delta z}\left(1 - e^{-c\cdot\Delta z}\right)\right]. \tag{7}$$

The variance function for great values of Δz, is inversely proportional to the length of interval and can be written:

$$G(\Delta z) = \frac{\theta}{\Delta z}\ , \tag{8}$$

where θ is a scale of fluctuations and it characterise spatial variability of soil properties.
This parameter is related to the correlation function:

$$\theta = 2\int_0^{\infty} R(\tau)\,d\tau\,. \tag{9}$$

The variance function will be also used in SFEM for determining covariance between averaged random values. Such covariance for different intervals of the same length and for the correlation function (2b) was found to be as follows:

$$Cov\left[E_i, E_j\right] = \left\{\left(c\cdot\left|z_j - z_i\right| + 3\right)\left[\cosh\left(c\cdot\Delta z\right) - 1\right] - c\cdot\Delta z\cdot\sinh\left(c\cdot\Delta z\right)\right\}\frac{2e^{c\cdot\left|z_j - z_i\right|}}{\left(c\cdot\Delta z\right)^2}\,, \tag{10}$$

where z_i and z_j are co-ordinates of intervals centres.

The influence of averaging on standard deviation of normal vertical stresses, in the case of 2D analysis, is shown in Figure 5 and Figure 6. Two values of decay coefficients $\lambda = 1$ and $\lambda = 5$ and two different mesh sizes a = 0.5 m and a = 1.25 m are considered. It is seen in these figures that for small elements and small decay coefficients (high correlation) the averaging insignificantly influences the results. However, this influence is much more significant for biggest elements and smaller correlations.

The results presented in this paper following were obtained for the correlation function (2b). For other functions, the results more or less differ. Usually, the correlation function is assumed apriori and, based on measurements, its parameters

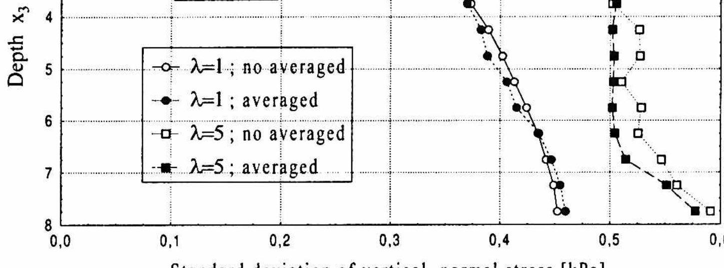

Figure 5. *Change of standard deviation of vertical normal stress with depth for mesh spacing a=0.5 m*

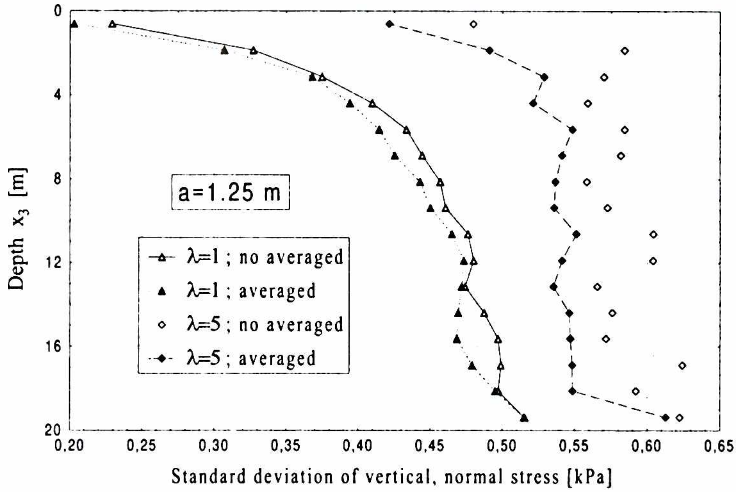

***Figure 6.** Change of standard deviation of vertical normal stress with depth for mesh spacing a=1.25 m*

are estimated. However, for measured realisation of soil properties, different functions can be assumed. The results obtained by SFEM should depend on a real correlation and not on the assumed correlation function. In the presented averaging formulation the parameter θ does not depend on the shape of the correlation function. The change of standard deviation of vertical normal stress with depth for different correlation functions is shown in Fig.7. Calculations were performed for two values of scale of fluctuations $\theta = 1$, $\theta = 5$ and three different correlation functions R1(t), R2(t), R3(t), given respectively by (2b), (2a) and (2c). The parameters of considered correlation functions $R_1(\tau)$, $R_2(\tau)$, $R_3(\tau)$ expressed by the

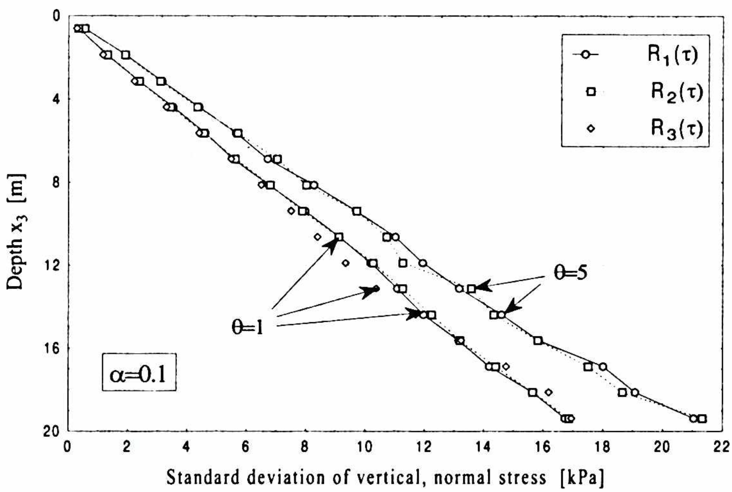

***Figure 7.** Change of standard deviation of vertical normal stress with depth for different correlation functions*

scale of fluctuation, in one—dimensional case, are as follows: $a = 2 \cdot \theta$, $c = 4 \cdot \theta$ and $d = \theta$. It is seen in Figure 7 that the influence of the shape of correlation function on the standard deviations of normal stresses is insignificant.

6. Convergence of simulation algorithm

The stochastic finite element method based on the Monte Carlo technique is more effective if a smaller number of random field generations is necessary to gain stabilisation of the solution. It should be emphasised that the global stiffness matrix must

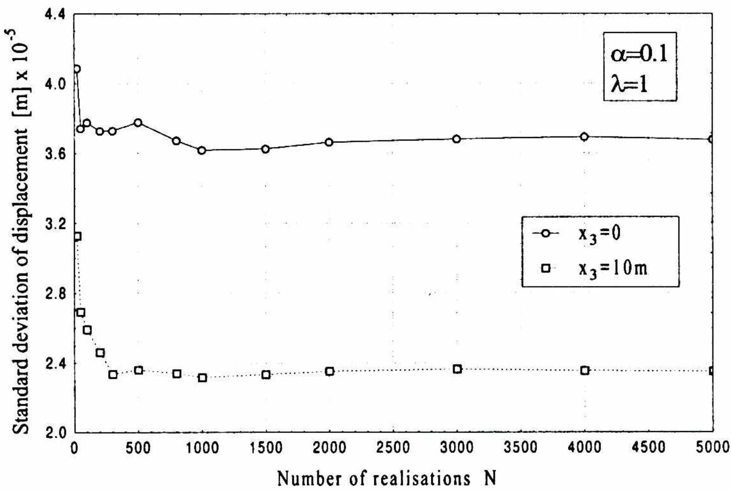

Figure 8. *Standard deviation of vertical displacement vs. number of realisations*

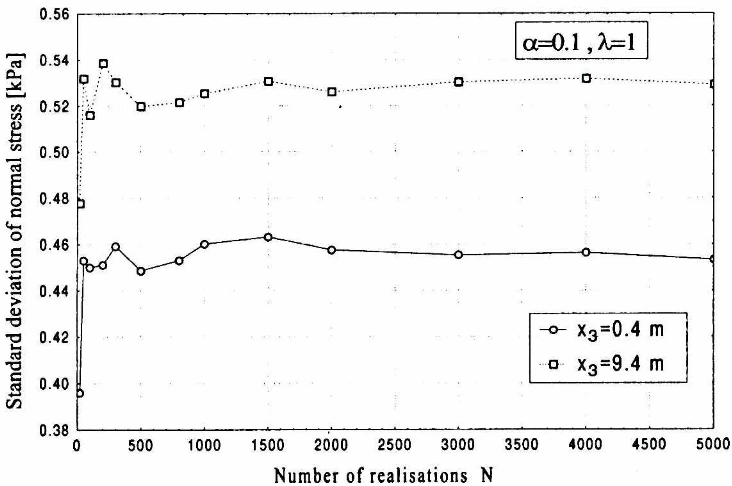

Figure 9. *Standard deviation of vertical normal stress vs. number of realisations*

be inverted for each realisation. For a great number of finite elements such operation, especially in the 3D analyses, becomes time—consuming, even for fast computers. The algorithm assumed in the computing programme simulation, and based on the local propagation scheme, was tested by Bielewicz, Górski and Walukiewicz [1994] and appeared to be good enough. The standard deviations of vertical displacement and normal stress vs. the number of random field generations, for two different depths, are shown in Figure 8 and Figure 9, respectively. The presented results were obtained for the coefficient of variation $\alpha = 0.1$ and dicay coefficient $\lambda = 1$.

It is seen in those figures that a good stabilisation of results is gained already after 2000 realisations, whereas it is good enough after 1000.

7. Numerical results of 2D and 3D analysis

The stresses and displacements in the subsoil subjected to the external loading due to the strip foundation (Figure 1 and Figure 2) were computed using 2D and 3D stochastic finite element method, for three values of decay coefficient $c = \lambda = 1, 2, 5$. The differences between average values of stresses and displacement in 2D and 3D states are rather negligible, whereas differences between respective standard deviations can be significant. The standard deviations of vertical displacements and vertical normal stresses are strongly influenced by the statistical parameters characterising soil randomness. All results in the following are presented in dimensionless co-ordinates.

The change of the standard deviation of the vertical displacement with depth in the 2D and 3D states, for the symmetry axis ($x_1 = 0$, $x_2 = 0$) and for the decay coefficient $\lambda = 5$ is shown in Figure 10. In this case, the results for both states analysed differ considerably. Only at the base of the layer do the standard deviations

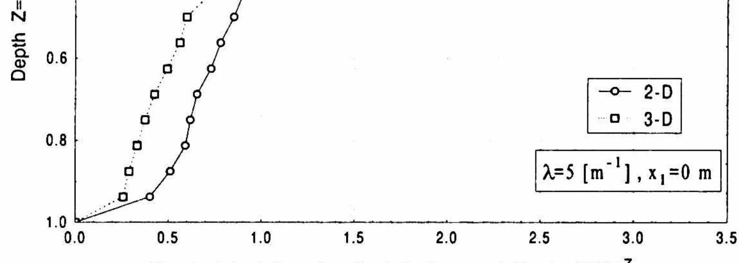

***Figure 10.** Standard deviation of vertical displacement vs. depth*

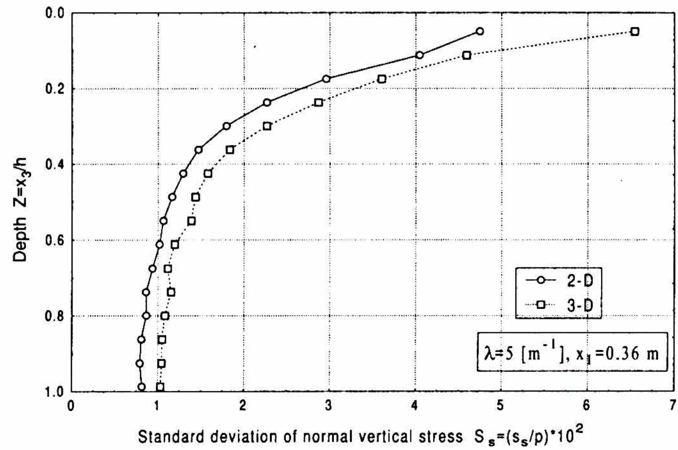

Figure 11. Standard deviation of normal vertical stress vs. depth

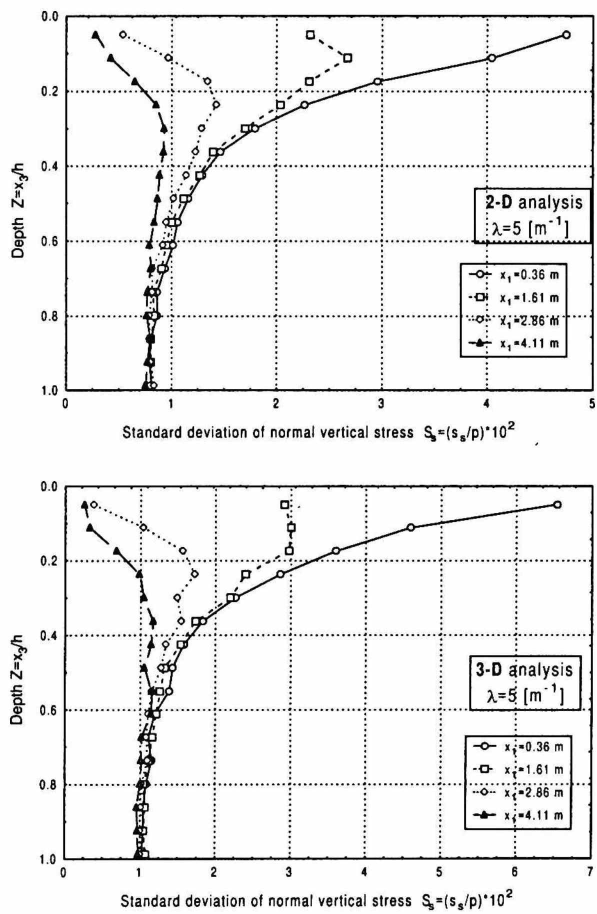

Figure 12. Standard deviation of the normal vertical stress vs. depth

in the 2D and 3D states tend to each other. The standard deviations of vertical displacements are there equal to zero due to the boundary conditions. In the plane strain analysis the standard deviations of vertical displacements are higher than in the 3D state.

The change of the standard deviation of the normal vertical stress with depth, in the 2D and 3D states, for decay coefficient $\lambda = 5$ and along vertical line of x_1 is shown in Figure 11. It can be seen there that the character of changes of the considered standard deviations in both states is similar. Respective curves, however, are shifted in respect to each other.

It can be seen in Figure 11 that the standard deviation of the normal vertical stress in the 3D state is higher than for the plane strain analysis and that the corresponding curves are almost parallel. In other words, the differences between respective standard deviations are such that their ratio is almost constant.

The change of the standard deviation of the normal vertical stress with depth, for decay coefficient $\lambda = 5$ and along four different vertical lines is shown in Figure 12, separately for the 2D and 3D states.

The character of relationships of standard deviations of displacements and stresses with depth, for other values of dicay coefficients are similar to the already presented ones. In general, differences between 2D and 3D curves decrease for higher correlations of elasticity modulus.

8. Comparison analysis

Comparing the results obtained, including different values of λ, one can assume that the relationships of the standard deviations of vertical displacements and normal stresses (for the 2D and the 3D states) versus the decay coefficients are of an

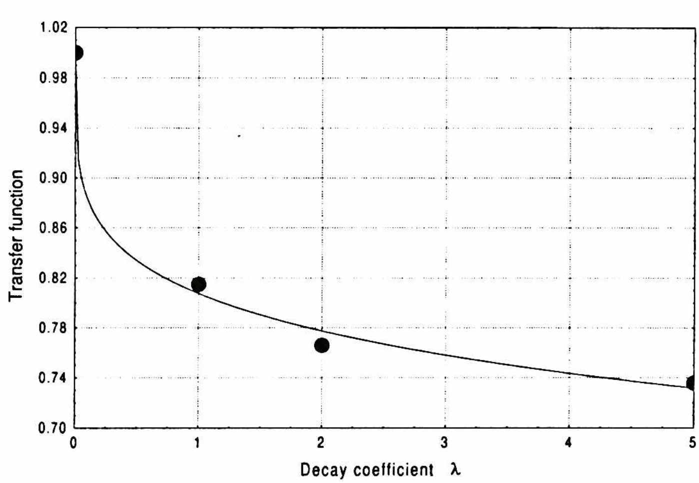

Figure 13. *Transfer function between standard deviations of vertical displacement in the 2-D and 3-D states*

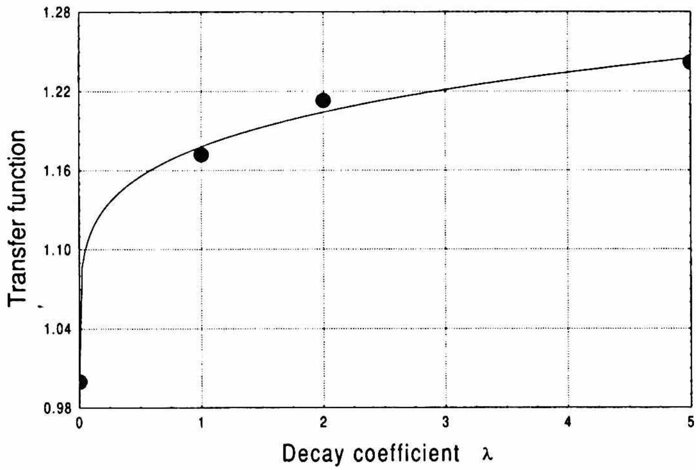

Figure 14. *Transfer function between standard deviations of normal vertical stress in the 2D and the 3D states*

exponential type. It may be written as follows:

$$\sigma^{III} = \sigma^{II} \cdot \exp\left(b \cdot \lambda^k\right), \tag{11}$$

where: σ^{III} and σ^{II} are 3D and 2D standard deviations, respectively, $\exp(b \cdot \lambda^k)$ can be called a transfer function.

The detailed procedure of calculating coefficients appearing in (11) is presented in the book by Przewłócki [1998]. In the case of displacements, coefficients were found to be $b = -0.21$, $k = 0.24$, whereas in the case of stresses $b = 0.16$, $k = 0.18$. The transfer functions given by expression (11) for displacements and stresses are shown in Figure 13 and Figure 14, respectively.

It is seen in Figure 13 that the standard deviation of the vertical displacement, for the decay coefficient $\lambda = 5$, is in the 3D state about 26% less than under the assumption of the plane strain. For the decay coefficient $\lambda = 0$ the standard deviations of displacements for both states should be identical. The most considerable change of the standard deviation takes place for relatively high correlations, so for the decay coefficient λ smaller than one.

In the case of stresses, the standard deviation of the normal vertical stress obtained in the 3D state is about 24% higher than the one computed in the plane strain analysis (for the decay coefficient $\lambda = 5$). For $\lambda = 0$ this standard deviation should be equal to zero, and the corresponding directional coefficient b equal to one. It is also seen that as in the case of displacement, the most considerable change of the standard deviation takes place for high correlations i.e. for a decay coefficient λ smaller than one.

9. Conclusions

The paper deals with an infinite strip foundation founded on the elastic, random subsoil. The 2D and 3D stochastic finite element method was used to analyse the state of displacements and stresses in soil medium was used. In order to gain independence of elements dimensions, this method requires averaging procedure. The parameter called the scale of fluctuations, which was additionally introduced to the method, allows to obtain results that depend on real correlation and not on the assumed correlation function.

The considerable influence on the standard deviations of both displacements and stresses have two parameters viz. the coefficient of variation of elasticity modulus α and the decay coefficient of the correlation function λ.

The standard deviation of the vertical displacement for the plane strain state is greater than for 3D state. Contrary to this, the standard deviation of the normal vertical stress for the 3D state is greater than for 2D. For the full correlation it is equal to zero and increases to constant value with the decrease of the correlation.

For the covariance function considered, the greatest changes of the standard deviations for both stresses and displacements take place for the relatively small decay coefficients, varying from $\lambda = 0$ to $\lambda = 2$.

The analysis of variances (standard deviations) as a measure of reduction of the space dimension is, from the engineering point of view, sufficient. The relationships suggested in the paper enable expression of the standard deviations of displacements and stresses in the 3D state by respective standard deviations obtained in the 2D analysis.

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