COMPUTER SIMULATIONS OF ONE-DIMENSIONAL COUPLED MAP LATTICES

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Abstract: We perform computer simulations of some one-dimensional models of coupled map lattices (CML) with symmetry and diffusive nearest neighbour coupling, to study Ising-type transitions. Such transitions appear to be related to the presence of a dip (minimum) in the plot of the Lyapunov dimension versus coupling parameter.

1. Introduction

For finite-dimensional dynamical systems, due to the exact definitions of some key statistical properties (ergodicity, mixing, etc.), and of geometrical objects such as homoclinic and heteroclinic points etc., transitions between order and chaos can be studied in a well developed theory. For extended (i.e., infinite dimensional) dynamical systems, instead, such as those describing fluid flows, an adequate mathematical theory was lacking for a long time. The analysis of experimental data for extended media was done in terms of notions such as turbulence, space-time chaos, and coherent structures, which remained vague, or were defined for particular situations. A whole theory was thus developed (see [1-3]), which was, however, somehow hindered by the lack of a precise definition of the main objects. The theory was essentially a collection of explicit phenomenological results concerning special classes of models of extended media. As an example we can quote the celebrated Kolmogorov theory of homogeneous isotropic turbolence (see [1]), or the theory of hydrodynamical stability (see [4]).

The situation has begun to change quite recently with the introduction of a class of extended dynamical systems that have been called coupled map lattices (CML's) [5]. They describe the evolution of a finite or infinite number of interacting "points", i.e., of finite dimensional dynamical systems that live at the sites of some lattice in a "physical" space. CML's are a sort of coupled oscillator systems, in which each oscillator has an "internal" state and a fixed coordinate in space.

With respect to the usual approach to the study of extended media, based on partial differential equations (PDE's), the CML's have the advantage of being generally easier to deal with, especially if one looks for invariant measures and attractors, and moreover they are particularly suited for computer simulations. In fact they can be used as discrete models for PDE's.

The mathematical analysis of the CML's led to a precise definition of the phenomenon of space-time chaos by Bunimovich and Sinai [6], and the same authors proved later its existence in some classes of CML's [7]. Moreover, in the same paper [6] Bunimovich and Sinai gave a definition of coherent structure, and suggested that such structures can arise in CML systems when the strength of the space interaction (order parameter) increases.

The definition of coherent structure given by Bunimovich and Sinai relies on the representation of the extended dynamical system (CML) as a lattice spin system of the type studied in Statistical Mechanics. The existence of space-time chaos, for weakly interacting strongly chaotic local maps, was obtained by connecting chaos to the absence of phase transitions in the lattice spin system. Here uniqueness of the phase for the spin system corresponds to space-time chaos and high temperature to weak coupling.

Phase transitions, as usual for dynamical systems, can be described as bifurcations that appear when the parameters of the space interaction vary. The emergence of new phases can be naturally interpreted as the appearence of coherent structures, i.e., by increasing the "coupling constant" we produce a transition from space-time chaos to some more organized type of motion. These ideas were confirmed by various computer simulations [10-13] and some analytical studies [15-16].

A natural question is whether or not the CML's can show the same type of phase transitions as the classical models of Statistical Mechanics, and first of all as the Ising model. A positive answer to this question has recently been obtained by I. Miller and D. Huse ([17], see also our later paper [18]), who have found Ising-type phase transitions in a two-dimensional square lattice of diffusively coupled piecewise linear expanding maps, symmetric with respect to reflection.

It is important to observe that while the one-dimensional lattice spin systems correspond to finite dimensional (nonextended) dynamical systems (the infinite one-dimensional lattice points are the values of discrete time, see e.g., [8-9]), the dimension of the spin systems corresponding to CML's equals at least two. Thus, in such systems phase transitions could occur for one-dimensional lattices. It turned out, however, that the one-dimensional CML of the same local maps as in [14], with analogous coupling, does not show such type of transitions, and this result was interpreted as an indication that phase transitions analogous to those in the lattice models of Statistical Physics can occur in CML's, only starting with dimension 2. It would seem that the difference between extended and nonextended dynamical systems pointed out in [6] is in some sense inessential.

The main motivation of our work, based essentially on computer simulations, was to show that this is not so, and that Ising-type transitions do occur in one-dimensional CML's.

We found in [18] evidence in favour of this, and we made a reasonable guess on the conditions needed for an Ising-type transition. Namely, it seems that an essential condition for them to occur is that there should be some "balance" between the local production of chaos (nonlinearity) and the diffusion coupling constant. This balance condition can be expressed by the behavior of the Lyapunov dimension, and appears to correspond to the presence of a dip in the plot of the Lyapunov dimension versus the (diffusive) coupling constant.

More work is needed for a sharper and more complete result on the occurrence of one-dimensional Ising-type transitions in CML's, which should include in particular evidence of an inverse power behavior of quantities such as the square of the "mean square magnetization" near the critical point, with an estimate of the critical exponent.

Our work is in progress, and we can only present here an account of the methods that we use, and an outline of some of the results obtained so far.

2. Description of the model

We consider CML's on the 1-dimensional lattice $\mathbf{Z}^1$, with diffusive coupling. The local systems are given by a one-dimensional expanding map f of the interval $I = [-1,1]$ into itself, with the further property that f is continuous and piecewise linear.

The phase space of the finite N-point approximation of our CML is

$$\Omega_N = \{x = \{x_k \in I : k \in \mathbf{Z}_N\}\}$$

where $\mathbf{Z}_N = \mathbf{Z}/(N\mathbf{Z})$ denotes the integers considered modulo N. On Ω_N we consider the maps Φ_ε and F with values in Ω_N and defined as

$$(\Phi_\varepsilon x)_k = (1-\varepsilon)x_k + \varepsilon \sum_{j \in \mathbf{Z}_N} a_{|j-k|} x_j, \quad (F(x))_k = f(x_k), \quad k \in \mathbf{Z}_N. \tag{1.1}$$

Here $\varepsilon \in (0,1]$ is the coupling constant, and $\sum_j a_j = 1$.

The dynamics is given by the composition

$$H_\varepsilon = \Phi_\varepsilon \circ F,$$

which maps Ω_N into itself. H_ε represents the successive action of the local map F and the coupling Φ_ε, and is explicitly written componentwise as follows

$$(H_\varepsilon x)_k = (1-\varepsilon) f(x_k) + \varepsilon \sum_{j \in \mathbf{Z}_N} a_{|j-k|} f(x_j)$$

Summarizing, we can say that the class of CML's under consideration corresponds to continuous piecewise linear one-dimensional expanding maps diffusively interacting on a finite one-dimensional lattice, with periodic boundary conditions.

Most numerical studies of CML's were done for the nearest neighbour coupling, and we also consider here only the nearest neighbour diffusive coupling, for which (1.1) is given by

$$(\Phi_\varepsilon x)_k = (1-\varepsilon)x_k + \frac{\varepsilon}{2}(x_{k-1} + x_{k+1}).$$

Moreover, we consider only one map, which is shown in Figure 1 and is hereafter denoted as f.

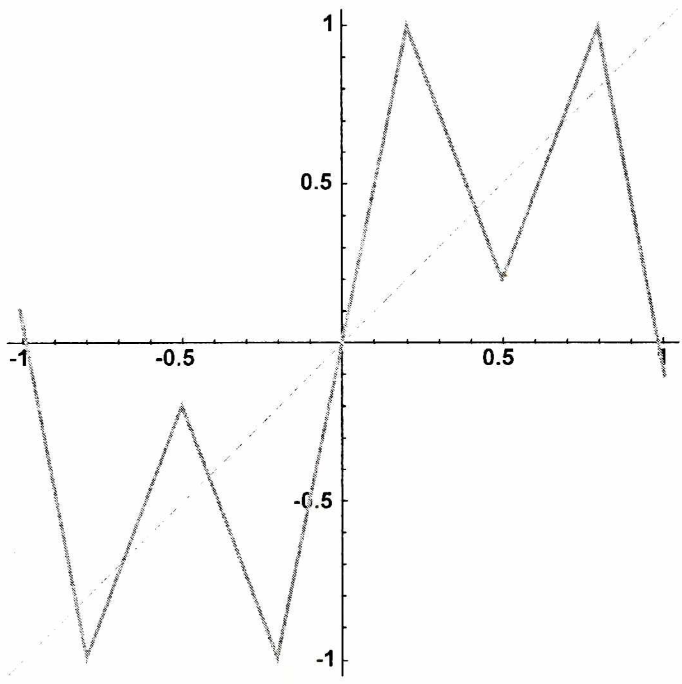

Figure 1. *The map f and the identical map*

3. Lyapunov exponents and Lyapunov dimension

We denote by λ_i, $i = 1,\ldots, N$ the i–th Lyapunov exponent of the CML system described in the preceding paragraph, labelled, as usual in decreasing order. The Lyapunov dimension (the integer part of it), is given by the first k such that the quantity

$$\sum_{i=1}^{k} \lambda_i$$

is negative. If there is no such $k < N$, then the Lyapunov dimension is set equal to the full dimension N.

To compute the Lyapunov exponents, we use the method developed by Benettin et al. [19].

The linearization of the map H_ε, at a point $\mathrm{x} = \{x_i\}_{i \in Z_N}$ is expressed by the matrix:

$$A(\mathrm{x}) \equiv \begin{pmatrix} (1-\varepsilon)\xi_0 & \frac{\varepsilon}{2}\xi_1 & 0 & \dots & 0 & \frac{\varepsilon}{2}\xi_{N-1} \\ \frac{\varepsilon}{2}\xi_0 & (1-\varepsilon)\xi_1 & \frac{\varepsilon}{2}\xi_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{\varepsilon}{2}\xi_0 & 0 & 0 & \dots & \frac{\varepsilon}{2}\xi_{N-2} & (1-\varepsilon)\xi_{N-1} \end{pmatrix}.$$

Here $\xi_k = \xi_k(\mathrm{x}) = f'(x_k)$, is the derivative of the map computed at the point x_k.

If we consider the particular trajectory of the map H_ε on $\mathbf{Z}_N$ starting from $\mathbf{x}(0)$, i.e. the set $\{\mathrm{x}(k) : k = 1, 2,\dots\}$, where $\{\mathrm{x}(k) = H_\varepsilon^\kappa \mathrm{x}(0)\}$, the linearization map along it is given by

$$\mathbf{z}(k+1) = A(\,\mathbf{x}(k))\,\mathbf{z}(k). \tag{2.1}$$

As a conseguence of Oseledec theorem

$$\lambda_1 + \dots \lambda_n = \lim_{T\to\infty} \frac{1}{T} \ln\left(\left\| \mathrm{z}^1(T),\dots,\mathrm{z}^n(T) \right\|\right),$$

where $\| \mathbf{z}^1(T),\dots, \mathbf{z}^n(T) \|$ is the volume of the open parallelepiped generated by the n vectors $\mathbf{z}^1(T),\dots,\mathbf{z}^n(T)$ which are the evolution of any initial set of n independent vectors $\mathbf{z}^1(0),\dots, \mathbf{z}^n(0)$.

In computer simulations it is impossible, in practice, to compute the sequence $\mathbf{z}(k)$ with the help of formula (2.1) beyond a few steps. In fact the vectors $\mathbf{z}^j(k)$ diverge (if $\lambda_1 > 0$) typically as exp $(\lambda_1 k)$, so that the computer overflow limit is reached after a few iterations. Moreover, we have also precision problems because the angle between the vectors $\mathbf{z}^i(T)$ and $\mathbf{z}^k(T)$ goes to zero as exp $(-|\lambda_1 - \lambda_2| T)$.

To overcome these problems we replace, after each l iterations, the vectors $\mathbf{z}^i(l)$ with the vectors $\mathbf{w}^i(l)$ obtained by the $\mathbf{z}^i(l)$ with some orthonormalisation method.

We have computed the Lyapunov exponents for values of N up to $N = 400$ and several values of ε. We have verified that the method converges very fast, and the result does not depend on the initial conditions, which are chosen taking for any x_k a random number uniformly distributed in the interval $[-1, 1]$, independently for different sites $k \in \mathbf{Z}_N$.

The plot of the Lyapunov dimension versus ε tends to stabilize for large N and is stable with an excellent confidence level around $N = 600$. Figure 2 shows the behavior of the Lyapunov dimension as a function of ε for the map f of Figure 1, and $N = 600$. As we have said in the introduction, we believe that the dip in the plot, which occurs for our map around $\varepsilon = 0.55$, is a key condition for Ising-type transitions. This value turns

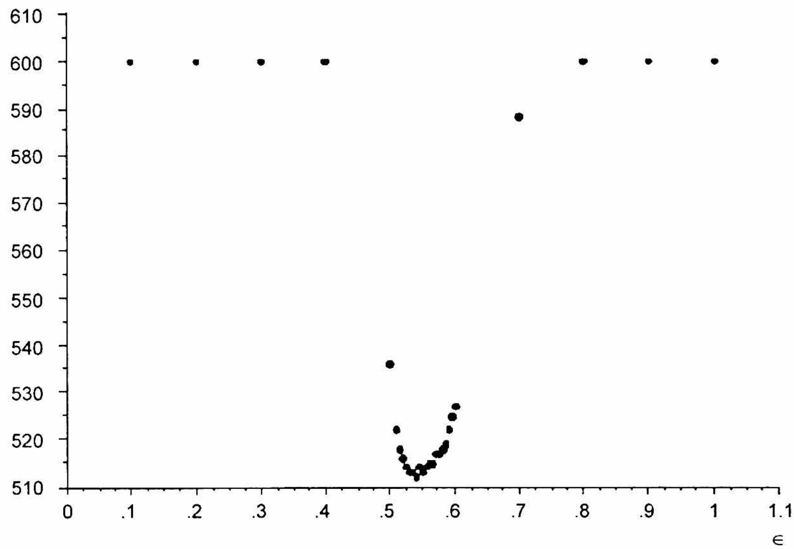

Figure 2. *The Lyapunov dimension, N=600 Time=500*

out to be close to the critical value of ε for which the transition appears.

4. Evidence for transition to an ordered pattern: spatial averages

We denote by $\overline{\mathbf{x}} = \{\overline{x}_k : k \in \mathbf{Z}_N\}$ the initial point, which is chosen at random as described in section 2, and by

$$\mathbf{x}(t) = \mathbf{x}(t, \overline{\mathbf{x}}) = H_\varepsilon^t\, \overline{\mathbf{x}}$$

the evolution of our system at the discrete time t with initial condition $\overline{\mathbf{x}} \in \Omega_N$.

The most obvious parameters in following the transition to an ordered behavior are the empirical spatial average $\mathbf{Ex}(t, \mathbf{x})$ and the empirical dispersion $\mathbf{Dx}(t, \mathbf{x})$, which are written as

$$\mathbf{Ex}(t, \overline{\mathbf{x}}) \equiv \frac{1}{N} \sum_{k=1}^{N} x_k(t, \overline{\mathbf{x}})$$

$$\mathbf{Dx}(t, \overline{\mathbf{x}}) \equiv \frac{1}{N} \sum_{k=1}^{N} \left(x_k(t, \overline{\mathbf{x}}) - \mathbf{Ex}(t, \mathbf{x})\right)^2$$

At the initial time, since the initial data are random and symmetric in sign, $\mathbf{Ex}(t)$ is close to 0 and $\mathbf{Dx}(t)$ is close to $\frac{1}{4}$. As time goes on, if ε is small, the situation remains approximately the same, as expected, because the system must show space-time chaos [6], and the stationary measure should be close to the product measure obtained by factorising the invariant measure for f, which is symmetric with respect to sign change.

When we increase ε beyond the supposed critical value the situation is quite different. The behavior of **Ex**(t) and **Dx**(t) as times goes on is shown in Figures 3 and 4. After some transient time **Ex**(t) takes a definite positive or negative value close to ± 0.55, and stays there with small oscillations. With respect to the random distribution of the initial data the probability of ending up to positive or negative values is the same. The dispersion **Dx**(t) also shows a dramatic drop after at the same transition time,

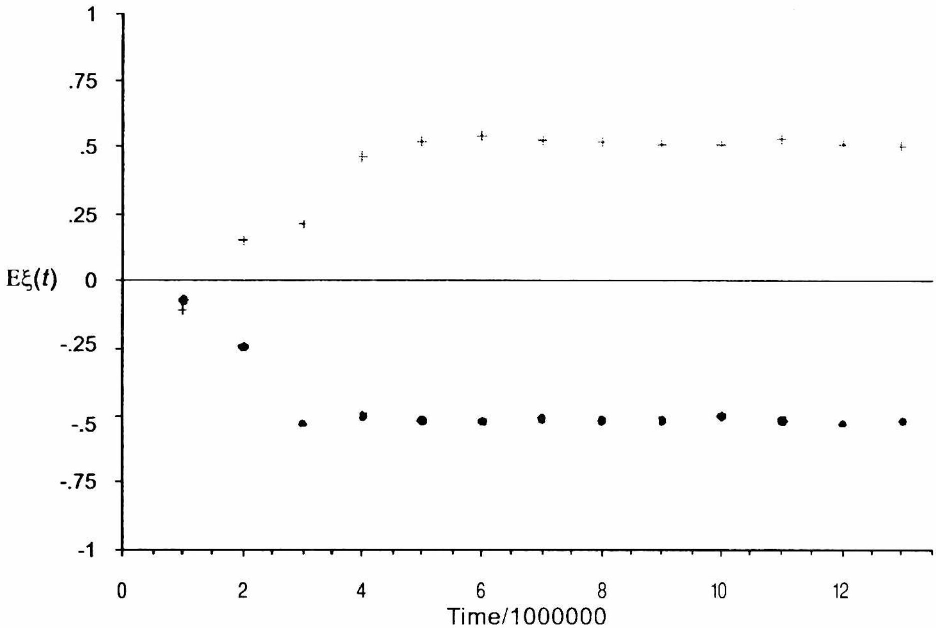

Figure 3. *Behaviour of* **E**$\xi(t)$ *versus time with* $\varepsilon = .6$ *and* $N=1000$ *for two different initial conditions*

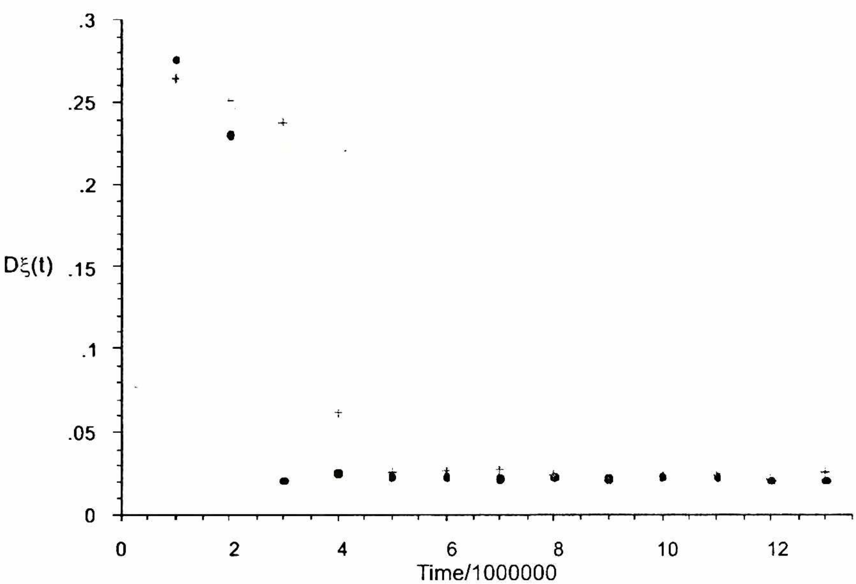

Figure 4. *Behaviour of empirical dispersion with* $\varepsilon = .6$, $N=1000$ *and the same initial conditions of the previous figure*

showing that the system "is frozen" around the values $x_k \approx \pm 0.55$.

It is worth to observe that the jump of the dispersion is very sharp and gives a reliable estimate of the transition time.

The behavior of $\mathbf{E}x(t)$ and of $\mathbf{D}x(t)$ is a clear indication of a transition from a state corresponding to an invariant measure which is symmetric with respect to sign change to a more ordered situation in which the symmetry is broken.

In order to make closer analogies with Ising transitions one should investigate more closely what happens near the critical value. We report on this in the next paragraph.

5. Mean square magnetization near the critical point

It is well known that the mean square magnetization in Ising models behaves, as the inverse temperature β gets close to the critical value β_c as $(\beta_c - \beta)^{-\gamma}$, where $\gamma > 0$ is a constant that is called the "critical exponenent". Huse and Miller [14] have shown that Ising-type transitions for CML's behave similarly and found an estimate for the critical exponent.

In our case the role of β is played by the parameter ε and the mean square magnetizations is

$$M_\varepsilon(t) \equiv \left\langle \frac{1}{N}\left(\sum_{k=0}^{N} \mathrm{sign}(x_k(t))\right)^2 \right\rangle,$$

where $\langle\cdot\rangle$ represents averaging over the distribution of the initial data $\bar{\mathbf{x}}$, which is distributed as described above, and sign(x) denotes the sign of x i.e. 1 if x is positive and -1 otherwise. $M_\varepsilon(t)$ is related to the correlation function of the system, defined as

$$r_j(t) \equiv \langle \mathrm{sign}(x_k(t))\mathrm{sign}(x_{k+j}(t)) \rangle .$$

(This quantity does not depend on k by translation invariance.)

In fact

$$M_\varepsilon(t) = \frac{1}{N}\sum_{k=0}^{N}\sum_{l=0}^{N} \langle \mathrm{sign}(x_k(t))\mathrm{sign}(x_l(t)) \rangle = \sum_{i=0}^{N} r_i(t).$$

For large t $M_\varepsilon(t)$ stabilizes in t after some transient time, and so do the quantities $r_j(t)$, the limiting values are denoted by M_ε^* and r_j^*. In analogy with the Ising model we believe, and we could actually check, that if ε is small, i.e., under some critical value ε_c, which is close to the value of ε for which the Lyapunov dimension attains its minimum, the correlation function r_j^* is summable, and actually decays exponentially as const $e^{-j/l}$, where $\ell = \ell(\varepsilon)$ is the "correlation length". Therefore, if $N > \ell(\varepsilon)$ the (asymptotic) mean square magnetization M_ε^* does not depend on N. For $\varepsilon > \varepsilon_c$, instead, $r_j^* > \mathrm{r} > 0$

for all j and is M_ε^* is of the order N. Moreover, we can expect that for $\varepsilon < \varepsilon_c$, but close to it we have

$$M_\varepsilon^* \approx \frac{\text{const}}{(\varepsilon_c - \varepsilon)^\gamma}, \tag{5.1}$$

where γ is the critical exponent.

The main aim of our work in progress is a reliable check of the behavior (5.1) and a reasonable estimate of the exponent γ. One of the main difficulties in computing the behavior of the mean square magnetization near ε_c is that, as we have said, we have to take N reasonably larger than the correlation length, which grows to infinity as ε approaches ε_c from below. We take as a confidence value $N \approx 2l(\varepsilon)$.

Figure 5 (a-c) show tipical behaviours of $M_\varepsilon(t)$ versus t for different values of ε. One can see that the value grows for some transient time, and then oscillates around some limiting value. The transient time seems to grow as $\varepsilon \to \varepsilon_c$, although in a rather moderate way. Nevertheless, this gives additional difficulties because we have to increase the running time.

The oscillating behavior is due to the finite size of the sample that we consider, which is typically between 200 and 600. The oscillation indicate that some large "islands" of negative values of $x_k(t)$ appear in a "sea" of positive values, or vice versa, and these

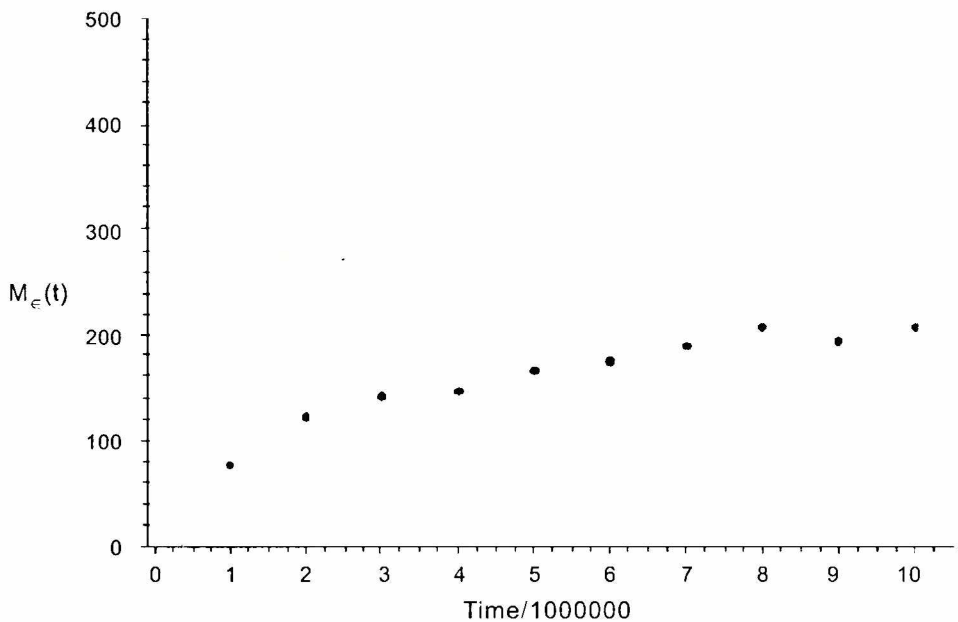

Figure 5a. *Behaviour of $M_\varepsilon(t)$ versus t with averaging over 400 different initial data. N=1000 and ε = .571*

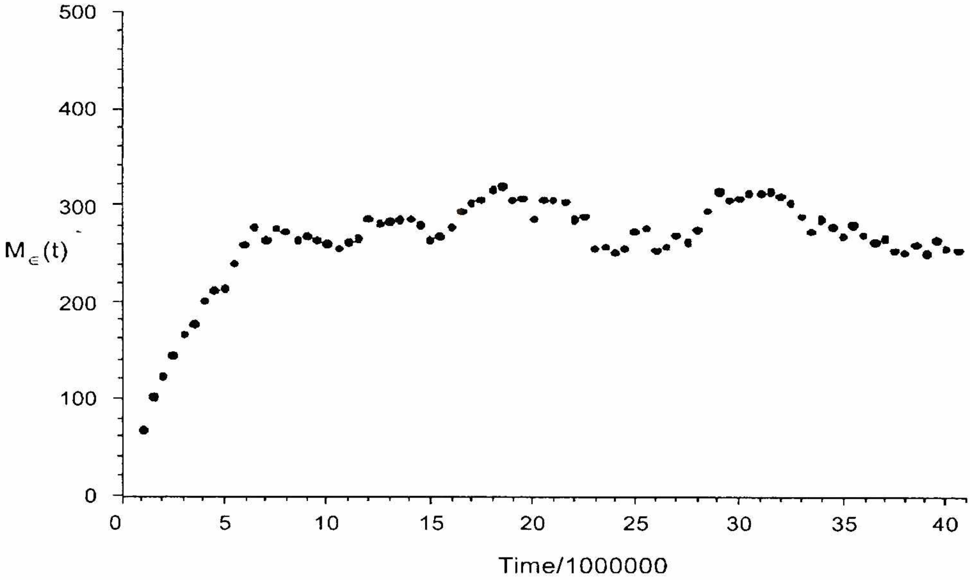

Figure 5b. *Behaviour of $M_\varepsilon(t)$ versus t with averaging over 200 different initial data, $N=1000$ and $\varepsilon = .573$*

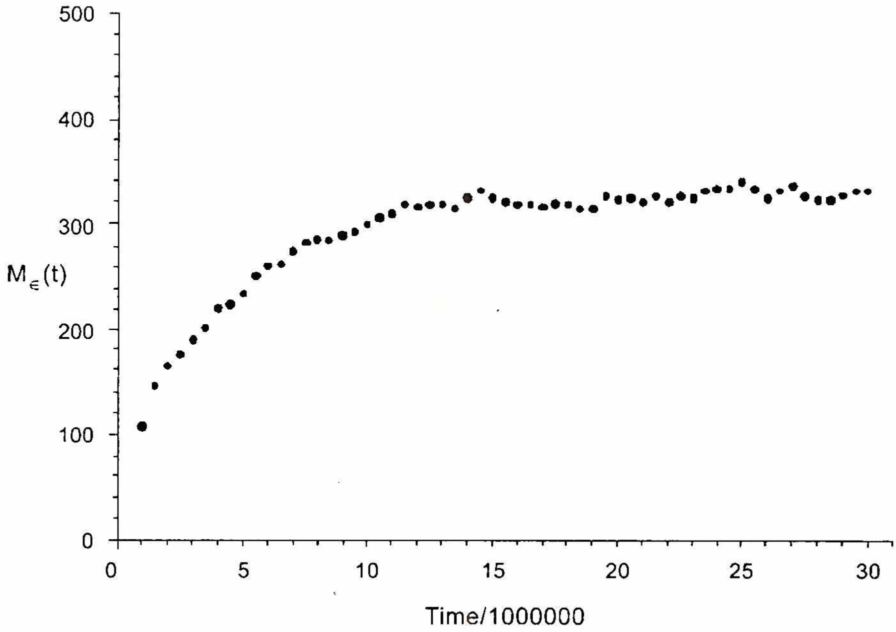

Figure 5c. *Behaviour of $M_\varepsilon(t)$ versus t with averaging over 550 different initial data, $N=1000$ and $\varepsilon = .57375$*

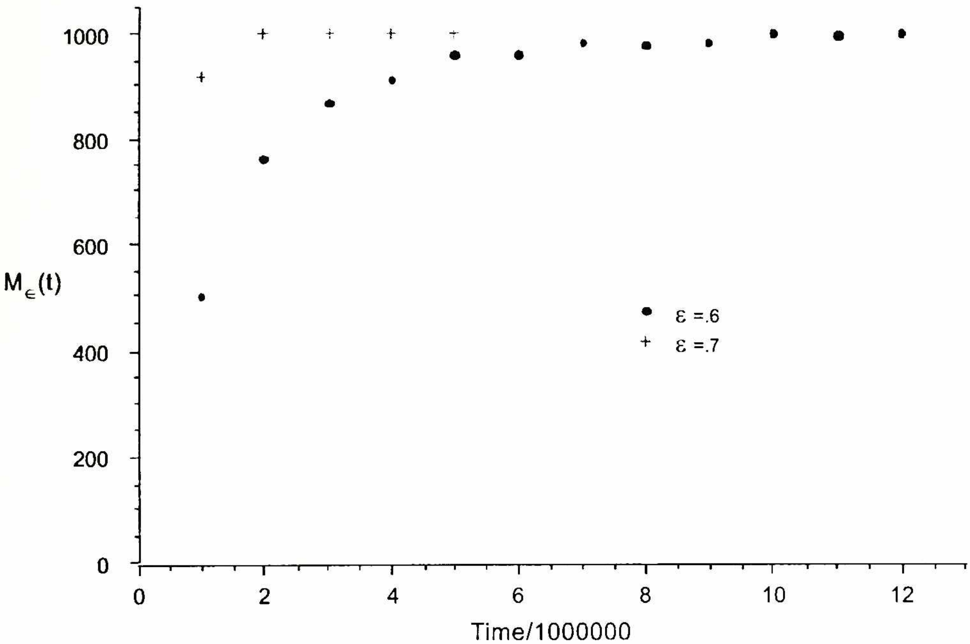

Figure 6. *Behaviour of $M_\epsilon(t)$ versus t with averaging over 50 different initial data and N=1000*

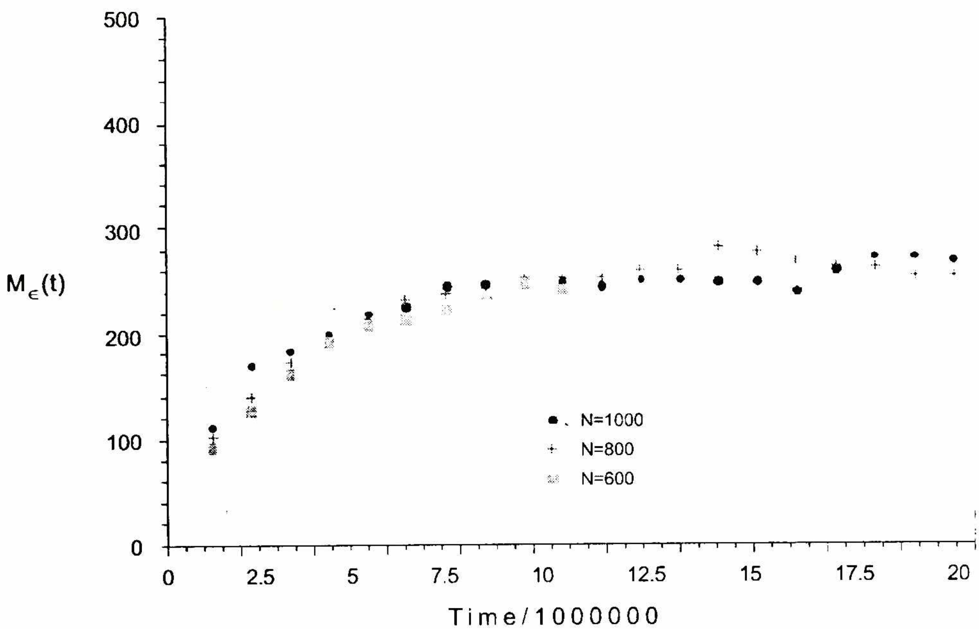

Figure 7. *For N=600 averagisng over 550 initial data. For N=800 averaging over 800 initial data. For N=1000 averaging over 400 initial data. ε =.5725*

islands have a rather long lifetime. Anyway, with the sample size that we choose oscillations are kept within some 10% of the average value.

Figure 6 reports $M_\varepsilon(t)$ for $\varepsilon = 0.6$ and 0.7. It is seen that M_ε^* is close to N, which implies $r_j^* \approx 1$ for all j. This means that almost all x_k have the same sign.

In Figure 7 we report the behaviour of $M_\varepsilon(t)$ for a fixed ε versus t for different value of the dimension N of the CML. We have always $N > 2M_\varepsilon(t)$, and one can see that the "stationary value" M_ε^* does not depend on N.

At present we can guess that $\varepsilon_c \approx .578$ and that the "stationary values" M_ε^* seem to diverge as $\varepsilon \to \varepsilon_c$, in a way which is compatible with the power law (5.1), but more data are needed in order to say that we have a real check of the power law (5.1).

Acknowlegment

The work has been partially supported by C.N.R. (G.N.F.M.) and M.U.R.S.T. research funds. The numerical calculations have been partially performed at TASK Computer Centre, Gdansk, Poland.

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