NUMERICAL SIMULATIONS OF THE EFFECT OF TIME-DEPENDENT RANDOM MASS DENSITY FIELD ON FREQUENCIES OF SOUND WAVES

KRIS MURAWSKI

Department of Environmental Physics,
Technical University of Lublin,
Nadbystrzycka 40, 20-618 Lublin, Poland
kamur@akropolis.pol.lublin.pl

(Received 16 February 2001; revised manuscript received 5 March 2001)

Abstract: Numerical simulations of the effect of random mass density field on frequencies of sound waves are considered for driven and impulsive sound waves which are described by one-dimensional hydrodynamic equations, with the ponderomotive force which depends stochastically both on space and time. The numerical results reveal frequency increase for short waves and both wave damping and amplification for the overall range of wavenumbers. Moreover, a space- and time-dependent random field leads to a generation of a wide frequency spectrum which contains both retarded and speeded up waves.

Keywords: convection – Sun, oscillations – turbulence

1. Introduction

Wave propagation in random media has been investigated by a number of authors. For example, Valley [1] discussed the scattering of the Alfvén wave by space-dependent fluctuations. His ideas were undertaken by Li and Zweibel [2] who considered decay of the Alfvén wave which propagated through a medium that contains time-dependent random density fluctuations. Lou and Rosner [3] showed that the Alfvén wave is damped owing to the time-dependent fluctuations. The theory of sound wave propagation in media with random sound speed, mass density and flow was reviewed by Ostashev [4]. It was shown by Lipkens and Blanc-Benon [5] that the nonlinear distortion of a pulse is weaker when turbulence is present. Linear fast magnetosonic waves, that were impulsively generated in a space-dependent random mass density medium, were discussed by Murawski, Nakariakov and Pelinovsky [6]. It was shown that the localized pulses experience a spatial delay and attenuation due to the random field. Linear sound waves propagating through the medium of a randomly flowing plasma, were considered by Murawski and Pelinovsky [7] and Murawski [8] to show that a space-dependent random flow is able to speed up and enhance sound waves. Nocera, Murawski and Mędrek [9] discussed the effect of a space-dependent random mass density field on the acoustic waves. The main conclusion was that such random density field alters the frequencies of the acoustic waves since these waves are affected by the scattering from random field (*e.g.* Howe [10]). Kawahara [11] showed that time-dependent random field, that is associated with bottom inhomogeneities, leads to amplitude attenuation and increase (decrease) of low (high) frequency self-modulated surface gravity waves. Numerical simulations of space-dependent random mass density fields were performed by Mędrek *et al.* [12]. The numerical results revealed that sound waves are retarded and attenuated by the random field.

Numerical simulations of sound waves propagation in time-dependent random media were performed by Juvé, Blanc-Benon and Wert [13] whose approach was based on the use of the Helmholtz equation. The obtained results demonstrate the high variability of a random wave field whose frequency spectrum shows that most of the energy is concentrated in the low frequency domain. On the other hand, Muzychuk [14] pointed out that space- and time-dependent fluctuations lead to a reduction of the mean field damping and eventually to enhancement of this field. Benilov and Pelinovsky [15] provided an example of a time-dependent random media whose high (low) frequency fluctuations lead to wave amplification (damping).

We also consider wave propagation in a medium with random density fluctuation, but our approach differs from the previous calculations in one important respect. By allowing the density fluctuations to vary with time we discuss the influence of arbitrary strength fluctuations on spectral properties of the sound waves. The basic motivation for the calculations presented in this paper is to show that a rigorous approach, such as numerical simulations of the Euler equations, can lead to more reliable results than those obtained with the use of the analytical method which is valid for a weak random field.

In this paper, we present a numerical approach to the problem of wave propagation in a medium whose mass density depends stochastically on time. Such a random field is excited by an external force which is represented by a source term in the mass continuity equation. Our work is stimulated by the fact that earlier treatments of weakly random media, based on Born (*e.g.* Howe [10]) and Rytov (Rytov, Kravtsov and Tatarsky [16]) approximations, are too approximate in accounting for multiple scattering, large fluctuations, and long propagation distances (*e.g.* Murawski [8], Murawski and Pelinovsky [7], Nocera, Murawski and Mędrek [9]).

The paper is organized as follows. The following Section is devoted to presentation of model equations. Numerical simulations for impulsively generated and driven waves are performed in Section 3. This paper is concluded by a short summary of the main results.

2. Model equations

The outline of our approach is as follows. We first present equations for the time evolution of the sound waves, propagating in the presence of density irregularities. As these equations are quite complicated, analytical solutions can only be obtained under some restrictive circumstances such as linear waves. As a consequence of that, we solve these equations numerically for the random waves, excited impulsively or driven by a periodic driver.

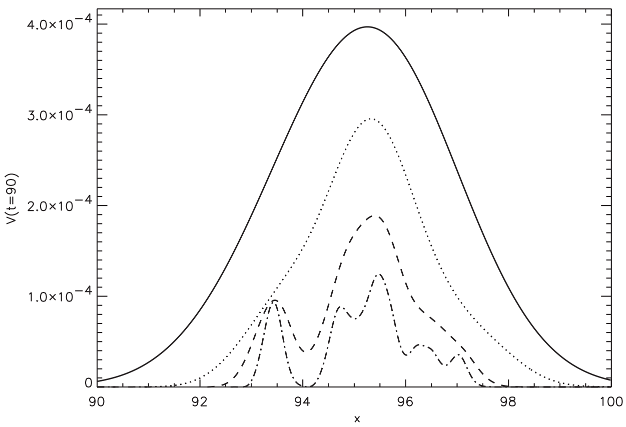

Figure 1. Ensemble averaged spatial profiles of the velocity $V(x)$ at $t=90\,l_t$ in the case of $\sigma=0.1$, $V_d=0$, and $V_0=10^{-3}c_0$. The curves from the top correspond to different initial pulse width w: $w=2\,c_0\,l_t$ (solid), $w=1\,c_0\,l_t$ (dotted), $w=0.5\,c_0\,l_t$ (broken) and $w=0.25\,c_0\,l_t$ (dash-dotted)

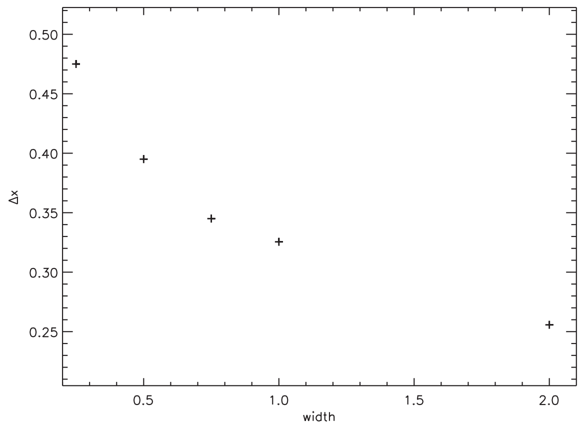

Figure 2. The difference between position of the pulse maximum at $t=90\,l_t$ for the random field with $\sigma=0.1$ and the deterministic medium as a function of the pulse width for the wave profiles of Figure 1

Consider one-dimensional sound wave propagation in a gravity-free medium. Then, the sound waves are described by hydrodynamic equations, *viz.*

$$\varrho_t+(\varrho V)_x=S_\varrho, \tag{1}$$

$$\varrho(V_t+V\,V_x)=-p_x, \tag{2}$$

$$p_t+(p\,V)_x=(1-\gamma)p\,V_x. \tag{3}$$

Here ϱ is the mass density, V is the x-component of the velocity vector, and p is the pressure. The spatial coordinate and time are denoted by x and t, respectively. The indices x and t denote the partial derivatives, *e.g.* $\varrho_t\equiv\frac{\partial\varrho}{\partial t}$. In Equation (1), S_ϱ represents the external mass flux which is described below.

In the limit of small amplitude waves we can linearize Equations (1) – (3) to obtain:

$$V_t=-\frac{1}{\varrho_e}p_x, \tag{4}$$

$$p_t=-\gamma\,p_0\,V_x, \tag{5}$$

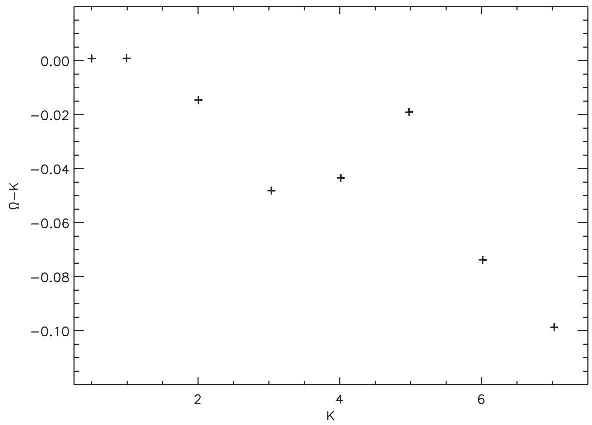

Figure 3. Ensemble averaged frequency difference for the linear random waves in the case of the time-dependent random mass density field with its strength $\sigma = 0.1$ and the periodic driver with $V_d = 10^{-3}c_0$. Here $K = kc_0 l_t$ and $\Omega = \omega l_t$ are the normalized wavevector and frequency. The correlation time is denoted by l_t

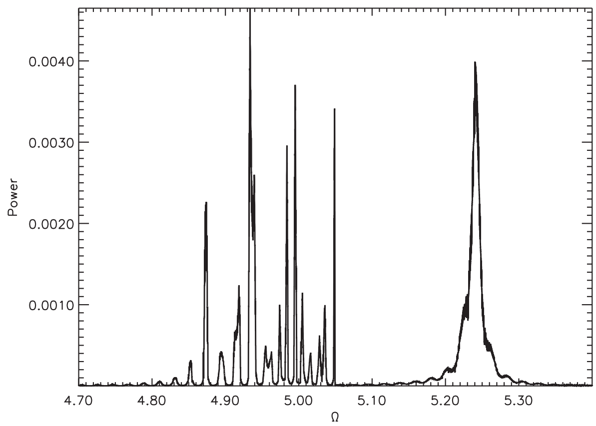

Figure 4. Ensemble averaged frequency spectrum of the random sound waves for space- and time-dependent random mass density field with its strength $\sigma = 0.1$. The periodic driver acts at $x = 0$ with its frequency $\omega_d = 5/l_t$ and amplitude $V_d = 10^{-3}c_0$

where $\varrho_e(x,t)$ and $p_0 = const.$ correspond to the equilibrium mass density and pressure, respectively. $V(x,t)$ denotes the perturbed velocity and $p(x,t)$ is the pressure.

From Equations (4) – (5), we can derive the wave equation:

$$V_{tt} - c_e^2 V_{xx} + \frac{\varrho_{et}}{\varrho_e} V_t = 0. \tag{6}$$

A qualitative inspection of this equation reveals that the last term leads to local damping or amplification in dependence on the sign of the time derivative ϱ_{et}. From this equation it follows that the wave frequency is reduced by the time-dependent mass density, and the wave is attenuated (amplified) for $\varrho_{et} > 0$ ($\varrho_{et} < 0$).

We assume now that the equilibrium mass density can be written as follows:

$$\varrho_e(x,t) = \varrho_0 + \varrho_r(x,t), \tag{7}$$

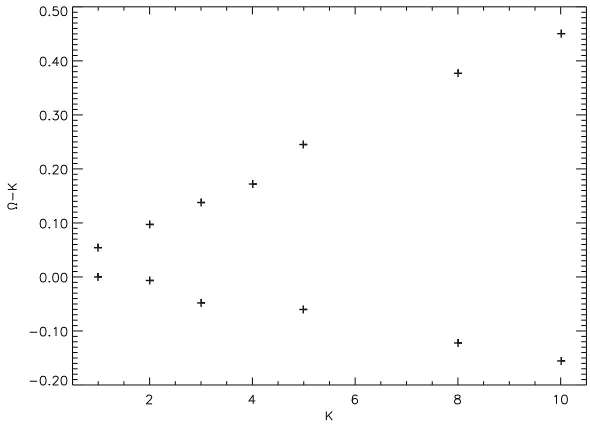

Figure 5. Ensemble averaged frequency difference for the random waves in the case of space- and time-dependent random mass density field with its strength $\sigma = 0.1$ and the periodic driver with $\omega_d = 5/l_t$ and $V_d = 10^{-3} c_0$. Only two most prominent frequencies of Figure 4 are presented

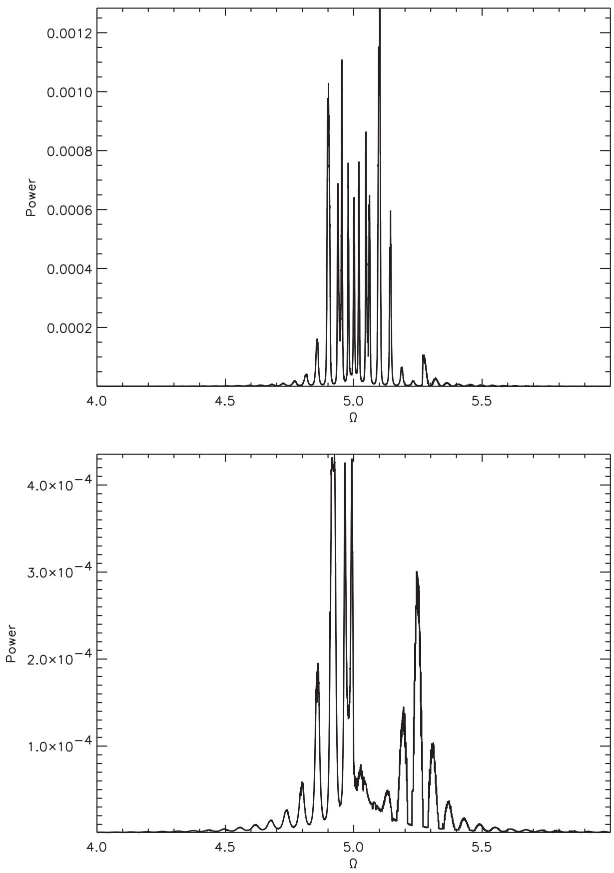

Figure 6. Ensemble averaged frequency spectrum of the random waves at $x = 0.2\,x_{max}$ (upper panel) and $x = 0.8\,x_{max}$ (lower panel) in the case of the periodic driver with $\omega_d = 5/l_t$ and $V_d = 10^{-3} c_0$ as well as the space- and time-dependent random mass density field with its strength $\sigma = 0.1$

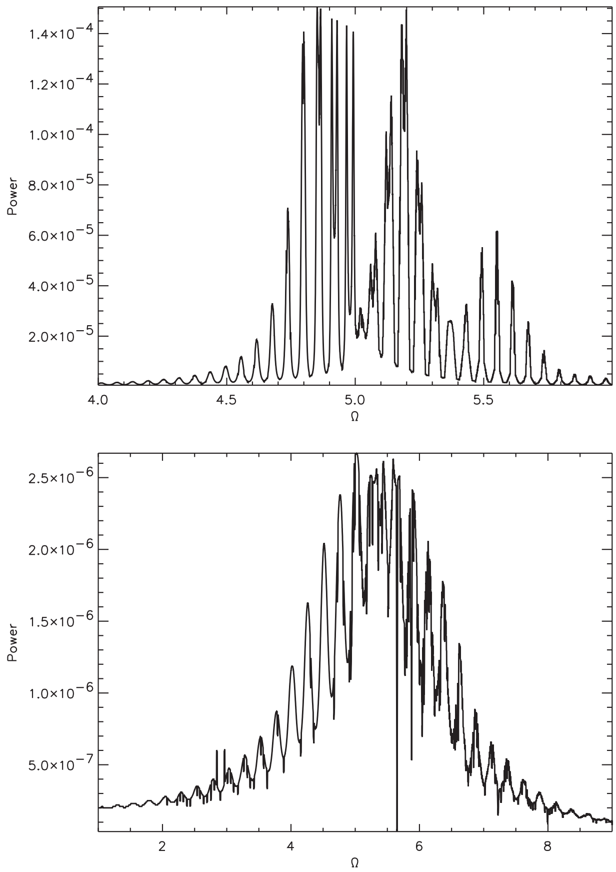

Figure 7. Ensemble averaged frequency spectrum of the random waves in the case of space- and time-dependent random mass density field with its strength $\sigma = 0.2$ (upper panel) and $\sigma = 0.4$ (lower panel) and the periodic driver with $\omega_d = 5/l_t$ and $V_d = 10^{-3} c_0$

where $\varrho_0 = const.$ and ϱ_r is a random function such that its ensemble average (*e.g.* Sobczyk [17]) is zero, *i.e.* $\langle \varrho_r \rangle = 0$. The density contrast $\varepsilon(x,t)$ is defined as the ratio of the random mass density ϱ_r to the uniform mass density ϱ_0, *i.e.*

$$\varepsilon(x,t) = \frac{\varrho_r(x,t)}{\varrho_0}. \tag{8}$$

3. Numerical results

In this Section, we present the results of the numerical simulations for nonlinear Equations (1–3). These simulations are performed with the use of the CLAWPACK code (LeVeque [18]), which is a packet of Fortran routines for solving hyperbolic equations. The code utilizes the Godunov-type method (*e.g.* Murawski and Tanaka [19], and references therein). Initially, at $t = 0$, the equilibrium state is set as follows:

$$\varrho_0 = const., \qquad V_0 = 0, \qquad p_0 = const. \tag{9}$$

Here ϱ_0, V_0, and p_0 are the background mass density, velocity, and pressure, respectively. Waves are excited by the periodic driver which acts at $x = 0$, *i.e.*

$$V(x=0,t) = V_d sin(\omega_d t) \tag{10}$$

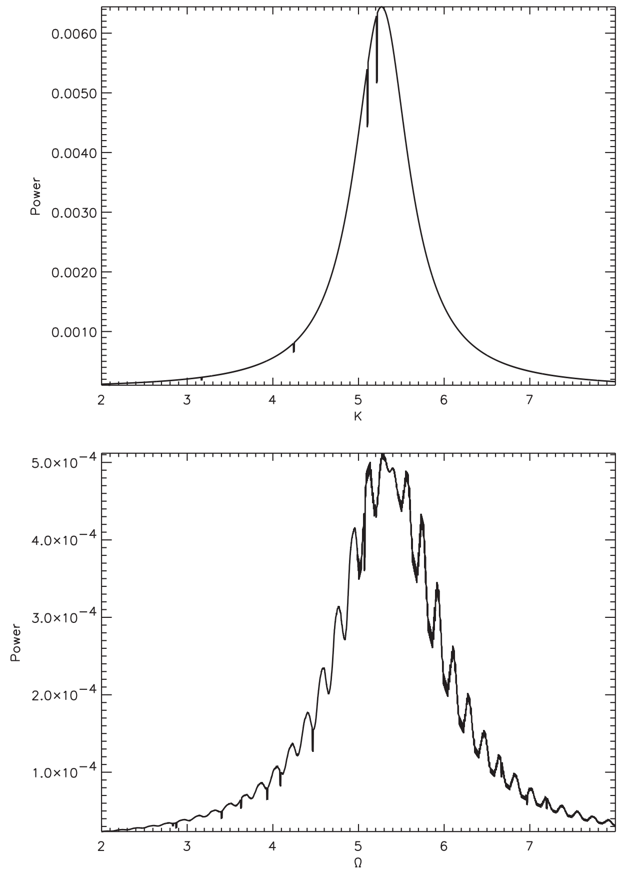

Figure 8. Ensemble averaged wavenumber (upper panel) and frequency (lower panel) spectra of the random waves in the case of time-dependent random mass density field with its strength $\sigma = 0.1\varrho_0$ the periodic driver with $\omega_d = 5/l_t$ and $V_d = 10^{-2}c_0$

or impulsively through the initial condition:

$$V(x,t=0) = V_0 e^{-(x-5c_0 l_t)^2/w^2}, \tag{11}$$

where V_0 is the amplitude of the initial pulse, w is its width, V_d denotes the amplitude of the driver, and ω_d is its frequency.

The Fourier transform F of initial profile (11) is given by:

$$FV(x,t=0) = V_0 \frac{w}{2\pi} e^{-k^2 \frac{w^2}{4}}, \tag{12}$$

In the case of driven waves, the free-boundary conditions are applied at $x = 70\,c_0 l_t$ such that:

$$\frac{\partial \varrho}{\partial x} = \frac{\partial V}{\partial x} = \frac{\partial p}{\partial x} = 0, \quad \text{at} \quad x = 70\,c_0 l_t. \tag{13}$$

In the case of impulsively generated waves the simulation region is extended from $x = 0$ and $x = 100\,c_0 l_t$ and the free-boundary conditions are applied there.

We choose the mass source term S_ϱ in Equation (1) that depends stochastically on space and time as follows:

$$S_\varrho = \sigma \sin\left(\frac{\pi}{x_r - x_l}(x - x_l)\right) \sin(2\omega_g t), \qquad x_l \le x < x_r, \tag{14}$$

where σ is the amplitude of the density fluctuations and ω_g is the frequency, which is associated with the characteristic time, T_g, *i.e.* $T_g = \frac{2\pi}{\omega_g}$. The left and right positions of the cell are given by x_l and x_r, respectively. In this model, σ, x_l and x_r are chosen randomly, with Gaussian statistics, such that 200 km $\le x_r - x_l \le$ 2000 km. The characteristic time T_g is set equal to 600 s and held fixed during the simulations. As a consequence of the term $\sin(2\omega_g t)$ in Equation (14), the random density field is initially zero. Then, it grows in the time for $t < T_g/2$ and subsequently decreases to zero at $t = T_g$. A spatial pattern of the random field is chosen randomly every time $t = nT_g$, where $n = 0,1,2,\ldots$. For the case of the time-dependent only random field the term $\sin\left(\frac{\pi}{x_r - x_l}(x - x_l)\right)$ in Equation (14) is dropped.

3.1. Time-dependent random field

3.1.1. Impulsively generated waves

In this case the random field depends on time only, $\varrho_r(t)$, and waves are generated impulsively by setting initial condition (11). The random field is excited through the source term S_ϱ which attains values of $\varrho_r(t)$, chosen randomly (Murawski [8]) at each time-step of the numerical simulations. Figure 1 presents the wave profiles for the initial amplitude $V_0 = 10^{-3} c_0$. For such small amplitude nonlinear effects are negligibly small. The random field strength $\sigma = 0.1$. Note that the initial impulses excite packets of waves which contain higher K for lower pulse width w. See Equation (12). In the deterministic medium the pulse would reach the point $x = 95\, c_0 l_t$. The presence of the random field leads to pulse acceleration (splitting) in the case of high (low) values of w. The splitted pulses are either slowed down or sped up by the random field. These pulses are damped and this damping is inversely proportional to the pulse width; broader pulses are damped the least.

3.1.2. Driven waves

We consider now the case of a time-dependent random field of its strength $\sigma = 0.1$ and a low amplitude driver with $V_d = 10^{-3} c_0$. Figure 3 presents the results of the spectral analysis of $V(x = 60\, c_0 l_t, t)$ and $V(x, t = 150\, l_t)$. As a result of the presence of the random field the random frequency Ω is shifted in comparison to the non-random frequency $\Omega_0 = K$. Note that $\Omega - K$ is positive for low K. For $K \ge 2$ the frequency difference is negative, suggesting that the waves are retarded. These results, however, agree with the results for the impulsively generated waves (Figure 1) which shows that wide (of high w) impulses are accelerated while narrow impulses for which contributions from high k are higher, split into retarded and accelerated waves.

3.2. Space- and time-dependent random field

Figure 4 presents the frequency spectrum which was obtained by averaging over 15 realizations of space- and time-dependent random fields with their strength $\sigma = 0.1$. The periodic driver is settled at the spatial coordinate $x = 0$. Its frequency $\omega_d = 5/l_t$ and amplitude $V_d = 10^{-3} c_0$. It is interesting that while a space-only-dependent random field excites a narrow spectrum which is centered around $\omega = 5.34/l_t$ (not shown), a space- and time-dependent random field generates a multitude of spectral lines which are organized in two groups:

the lines for which $\omega < \omega_d$ and the lines with $\omega > \omega_d$. At the same time, the wavenumber spectrum consists of an essentially single line at $k = 5/(c_0 l_t)$ (not shown). The lines for which $\omega < \omega_d$ ($\omega > \omega_d$) correspond to slowed down (speeded up) waves. Indeed, Figure 5 displays the normalized frequency difference $\Omega - \Omega_0$ as a function of the normalized wavenumber K. The normalized non-random frequency $\Omega_0 = K$. Only the lines which correspond to two strongest maxima of the spectral power are presented. The waves which are sped up or slowed down as a consequence of the random field are clearly visible. The frequencies depart one from another and this departure grows with K. For low K, Ω is close to Ω_0, while it departs from Ω_0 for high K.

Figure 6 displays the frequency spectra which were made from signals, collected at $x = 0.270\, c_0 l_t$ (upper panel) and $x = 0.870\, c_0 l_t$ (lower panel). We conclude that the spectrum gets broader with a distance of the wave propagation, x; the two groups of frequencies split apart at $x = 0.870\, c_0 l_t$ (lower panel). The effect of random field grows with a propagation distance.

Figure 7 shows the effect of a stronger random field on frequency spectrum of the sound waves. In the case of $\sigma = 0.2\varrho_0$ (upper panel), the strongest line of $\Omega < \Omega_0$ is shifted to the left in comparison to the corresponding line of Figure 4. The strongest line of $\Omega > \Omega_0$ of Figure 4 is split into two lines: the line at $\Omega \simeq 5.2$ is slightly shifted to the left; the line at $\Omega \simeq 5.55$ is shifted to the right. Note that for $\sigma = 0.4\varrho_0$ (lower panel), the spectrum is wider ($1 < \Omega < 9$) than the corresponding spectrum for the weaker ($\sigma = 0.1\varrho_0$) random field (Figure 4), for which $4.5 < \Omega < 5.5$. A stronger random field leads to line splitting and broadening of the spectrum.

Figure 8 displays the effect of higher wave amplitude on the wavenumber (upper panel) and frequency (lower panel) spectra. These spectra are centered around $\simeq 5.3$. Note that the frequency spectrum is wider than in the corresponding case of low amplitude waves (Figure 4).

4. Summary

In this paper we have presented a numerical study of the propagation of the sound waves in a medium with space- and time-dependent random mass density fluctuations. The main findings can be summarized as follows. The time-dependent random mass density field leads to frequency increase (decrease) of long (short) waves. The space- and time-dependent field leads to a multitude of frequencies which depart from the driving frequency ω_d. A frequency $\omega < \omega_d$ ($\omega > \omega_d$) corresponds to slowed down (sped up) waves. As a consequence of the presence of these frequencies, a spectrum of the sound waves is broader than in the case of a space-dependent random field. The corresponding wavevector spectrum is narrow. The effect of the space- and time-dependent random field grows with a propagation distance and with a strength of the random field. The nonlinearity leads to a generation of higher harmonics (not shown) and to a broader frequency spectra.

As the analytical method (Howe [10]) is essentially valid for a weak random field only, the numerical simulations provide a potential tool for a detailed description of an interaction between waves and a random field. It has been already proved that this tool is very powerful to represent random flows (Nocera, Murawski and Mędrek [9]) and sound waves in space-dependent random density plasma (Murawski [8], Murawski, Nakariakov and Pelinovsky [6], Mędrek *et al.* [12]).

K.M's work was financially supported by the State Committee for Scientific Research in Poland, KBN grant no. 2 PO3D 017 17. They acknowledge the support of the Italian Research Council. The numerical simulations have been performed on a SUN workstation at the Department of Complex Systems, the Institute of Physics, UMCS Lublin and on a SGI power station at the Poznań Supercomputing Center.

References

[1] Valley G C 1974 *ApJ* **186** 181
[2] Li H-S and Zweibel E G 1987 *ApJ* **322** 248
[3] Lou Y Q and Rosner R 1986 *ApJ* **309** 874
[4] Ostashev V E 1994 *Waves in Random Media* **4** 403
[5] Lipkens B and Blanc-Benon Ph 1995 *C. R. Acad. Sci. Paris* **320** 477
[6] Murawski K, Nakariakov V N and Pelinovsky E N 2001 *A & A* **366** 306
[7] Murawski K and Pelinovsky E N 2000 *A & A* **359** 759
[8] Murawski K 2000 *Acta Astron.* **50** 269
[9] Nocera L, Murawski K and Mędrek M 2001 *A & A* (in print)
[10] Howe M S 1971 *J. Fluid Mech.* **45** 785
[11] Kawahara T 1976 *J. Phys. Soc. Japan* **41** 1402
[12] Mędrek M, Michalczyk J K, Murawski K and Nocera L 2001 *Waves in Random Media* (submitted)
[13] Juvé D, Blanc-Benon Ph and Wert K 1999 *Theoretical and Computational Acoustics '97*, Y.-C. Teng *et al.* (eds.), World Sc. Pub. Co., p. 653
[14] Muzychuk O V 1975 *Radiofizika* **19** 1193
[15] Benilov E S and Pelinovsky E N 1989, in *Res. Rep. in Phys.*, Nonlinear Waves in Active Media, Springer
[16] Rytov S M, Kravtsov Yu A and Tatarski V I 1987 *Principles of Statistical Radiophysics* Springer-Verlag
[17] Sobczyk K 1985, *Stochastic wave propagation* Elsevier, PWN
[18] LeVeque R J 1997 *J. Comp. Phys.* **131** 327
[19] Murawski K and Tanaka T 1997 *Astrophys. Space Sc.* **254** 187