

INVISCID INSTABILITY OF THE HYPERBOLIC-TANGENT VELOCITY PROFILE – SPECTRAL “TAU” SOLUTION

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Abstract: The paper presents a spectral solution of the Rayleigh equation for the case of parallel, free shear layer with the hyperbolic-tangent mean velocity profile. The expansion of the eigenfunction into the Chebyshev polynomial series allowed transformation of the differential eigenvalue problem into the general algebraic one. The standard algebraic eigenvalue problem was obtained by the use of Gary and Helgasson transformation. The results were compared with the shooting method. Although the calculations were carried out in order to validate the method, some additional study of the velocity ratio and momentum thickness influence on the temporal eigenmode growth rate was also performed.

Keywords: free shear layer, linear stability theory, spectral approximation, “tau” method, Q-R algorithm

1. Background and motivation

The instability development of the parallel mixing layer is initially dominated by a linear mechanism because disturbances of the velocity components can be treated as infinitesimal. Thus the linear hydrodynamic stability theory is expected to give, at least qualitatively, reasonable predictions for the initial region of the perturbation development. The mean laminar flow in x -direction with known velocity profile $U(y)$ is assumed to be influenced by a disturbance, which is composed of the plane waves propagated in the main flow direction. Since it is assumed that the perturbation is two-dimensional it is possible to introduce a stream function of the perturbation velocity field in the following form:

$$\Psi(x, y, t) = \text{Re}[\Phi(y) e^{i\alpha(x-\lambda t)}] \quad (1)$$

The amplitude function is assumed to be dependent on y -direction only because the mean flow depends on y alone. The components of the perturbation velocity can be expressed via stream function as:

$$u'_x = \frac{\partial \Psi}{\partial y} = \Phi'(y) e^{i\alpha(x-\lambda t)} \quad (2a)$$

$$u'_y = -\frac{\partial \Psi}{\partial x} = -i\alpha \Phi(y) e^{i\alpha(x-\lambda t)} \quad (2b)$$

Introduction of these velocity components into the linearized form of the Navier-Stokes equations leads to the Orr-Sommerfeld stability equation that in the inviscid limit, $Re \equiv \infty$ reduces to the Rayleigh equation [1]:

$$[U(y) - \lambda] \left(\frac{d^2 \Phi}{dy^2} - \alpha^2 \Phi \right) - \frac{d^2 U}{dy^2} \Phi = 0 \quad (3)$$

A solution of Equation (3), in the plane wave form of the perturbation (1), with real wave number α and positive imaginary part of the complex number λ is said to be an unstable linear eigenmode, in the sense that the amplitude of the disturbance is growing with time. Hence this type of instability is called the temporal one. Michalke [2] applied the temporal stability theory to the parallel shear layer with hyperbolic-tangent velocity profile:

$$U(y) = 0.5 [1 + \tanh(y)] \quad (4)$$

and the following boundary conditions:

$$\Phi(+\infty) = \Phi(-\infty) = 0 \quad (5)$$

It was noted very soon that the temporal stability theory assumed the instability development in time while in the case of shear layer the perturbation was growing in space. This observation stimulated a spatial stability theory, according to which, the quantity α was assumed to be complex and the product $\alpha\lambda$ was real. The spatial linear stability theory was used also by Michalke [3] to study the parallel shear layer with the same hyperbolic-tangent velocity profile. More recently, Monkewitz and Huerre [4] studied spatially growing waves in the case of shear layer characterised by more general velocity profile:

$$U(y) = \bar{U} \left[1 + R \tanh\left(\frac{y}{2\theta}\right) \right] \quad (6)$$

investigating a dependence of instability development on the velocity ratio R , defined as:

$$R = \frac{U_1 - U_2}{U_1 + U_2} \quad (7)$$

where: U_1 and U_2 – the velocities of upper and lower stream of the shear layer, respectively. The equivalent problem for axi-symmetric jets was studied by Michalke and Hermann [5]. A review of linear stability calculations using temporal and spatial approach was given by Ho and Huerre [6].

When the concept of absolute instability of the shear flows was proposed by Landau [7], it turned out that neither temporal nor spatial linear stability theories were able to predict such a perturbation development. Despite that in the case of absolute instability the perturbation was growing in time reaching a non-linear level in the location of its source, it was proved by many authors [8, 9] that the linear stability theory could still be useful if both parameters α and $\alpha\lambda$ were allowed to be complex. This approach is called the linear spatio-temporal stability theory that was used by Monkewitz and Sohn [10] to study the stability of low density axi-symmetric jet and by Jendoubi and Strykowski [11] to analyse constant and variable density jets with external flow.

The linear stability theories, based on the Rayleigh equation, represent a differential eigenvalue problem. In all of the papers mentioned above this problem was solved by the “shooting” method where integration of the stability equation was carried out by the Runge-Kutta scheme and the Newton-Raphson procedure was applied to guess the eigenvalues. The

shooting method reveals two important disadvantages. Firstly, the convergence rate of the rootfinder is dependent on the initial eigenvalue guess. Moreover, in the hydrodynamic stability problem we are always interested in the least stable mode, thus the procedure has to be performed many times. An alternative method to the shooting one is the matrix method [12], which transforms the differential eigenvalue problem into the algebraic one by the discretization of the stability equation. The primary advantage of the matrix method is the existence of a very efficient and reliable method for solving the algebraic eigenvalue problem, namely the Q-R algorithm [13], which gives all the eigenvalues of the matrix. However, since the computation time for the Q-R algorithm increases as the cube of the matrix order, it is important to use an accurate discretization scheme. It is recommended by Boyd [14], Gottlieb and Orszag [15] and Canuto *et al.* [16] to use a spectral approximation based on the series of Chebyshev polynomials. Orszag [17] solved the problem of the plane Poiseuille flow stability by the approximation of the eigenfunctions using this method. The aim of the present paper is to apply the spectral method to free shear layer stability analysis using the temporal instability concept. Bearing in mind that this theory does not predict the convective perturbation development in space, this problem is treated rather as the test-case which is expected to prove efficiency and reliability of the method used, not as an investigation which can bring any physical insight into the problem considered. Because the linear spatio-temporal stability theory turned out to be valid even in the case of absolute instability and can predict correctly the parameters for which the transition from convective to absolute regime appears, it seems to be justified to develop and test new, more reliable methods to solve the hydrodynamic stability equations.

2. Outline of the method

According to the spectral approximation based on the series of Chebyshev polynomials the eigenfunction of the stability Equation (3) is expressed as:

$$\Phi(\bar{y}) = \sum_{m=0}^N a_m T_m(\bar{y}) \quad (8)$$

where: $T_m(\bar{y})$ – the Chebyshev polynomial of degree m defined as:

$$T_m(\bar{y}) = \cos(m \arccos \bar{y}) \quad (9)$$

The argument \bar{y} is the non-dimensional distance from the range $\langle -1, +1 \rangle$ related to lateral direction of the shear layer as:

$$\bar{y} = \frac{y}{Y_{\text{inf}}} \quad (10)$$

where: Y_{inf} stands for the location where the boundary conditions are formulated:

$$\Phi(y = -Y_{\text{inf}}) = \Phi(\bar{y} = -1) = \Phi(y = Y_{\text{inf}}) = \Phi(\bar{y} = 1) = 0 \quad (11)$$

The Rayleigh stability formula (Equation (3)) expressed in the non-dimensional distance \bar{y} takes the form:

$$U(\bar{y}) \frac{d^2 \Phi}{d\bar{y}^2} \frac{1}{Y_{\text{inf}}^2} - \alpha^2 U(\bar{y}) \Phi(\bar{y}) - \frac{d^2 U}{d\bar{y}^2} \Phi(\bar{y}) \frac{1}{Y_{\text{inf}}^2} - \lambda \left(\frac{d^2 \Phi}{d\bar{y}^2} \frac{1}{Y_{\text{inf}}^2} - \alpha^2 \Phi \right) = 0 \quad (12)$$

The discretization of the Equation (12) requires a spectral approximation of the given mean velocity profile. The most convenient way is to use also the series of the Chebyshev

polynomials. The coefficients of the Chebyshev spectral approximation can easily be calculated using the mean velocity values at the Chebyshev-Gauss-Lobatto points [16], defined as:

$$\bar{y}_k = \cos\left(\frac{k\pi}{N}\right) \quad (13)$$

It can be shown that if the coefficients are calculated according to the following formula

$$a_m^{(U)} = \frac{2}{N} \sum_{k=0}^N \bar{c}_k U(\bar{y}_k) T_m(\bar{y}_k) \quad (14)$$

where

$$\bar{c}_k = \begin{cases} 0.5 & \text{when } k=0 \text{ or } k=N \\ 1 & \text{otherwise} \end{cases} \quad (15)$$

then the velocity profile approximation takes the form:

$$U(\bar{y}) = \sum_{m=0}^N \bar{c}_m a_m^{(U)} T_m(\bar{y}) \quad (16)$$

In Equation (12) eigenfunction $\Phi(\bar{y})$ and velocity profile $U(\bar{y})$ appear with their second derivatives. According to the properties of the Chebyshev polynomials the second derivatives can be expressed also in the form of the Chebyshev polynomial series as follows:

$$\frac{d^2 \Phi(\bar{y})}{d\bar{y}^2} = \sum_{m=0}^N a_m^{(2)} T_m(\bar{y}) \quad (17a)$$

$$\frac{d^2 U(\bar{y})}{d\bar{y}^2} = \sum_{m=0}^N a_m^{(U2)} T_m(\bar{y}) \quad (17b)$$

where

$$a_m^{(2)} = \frac{1}{c_m} \sum_{\substack{p=m+2 \\ p+m \\ \text{even}}}^N p(p^2 - m^2) a_p \quad (18a)$$

$$a_m^{(U2)} = \frac{1}{c_m} \sum_{\substack{p=m+2 \\ p+m \\ \text{even}}}^N p(p^2 - m^2) a_p^{(U)} \bar{c}_p \quad (18b)$$

and coefficient c_m defined as

$$c_m = \begin{cases} 2 & \text{for } m=0 \\ 1 & \text{for } m>0 \end{cases} \quad (19)$$

In the first three terms of Equation (12) the products of the functions approximated with the polynomial series appear, which can be generally written as:

$$v(\bar{y}) w(\bar{y}) = \sum_{m=0}^N \sum_{n=0}^N b_n a_n T_m(\bar{y}) T_n(\bar{y}) \quad (20)$$

It can be proved that the product of the Chebyshev polynomials can be eliminated from the r.h.s. of the last formula that leads to the expression:

$$v(\bar{y}) w(\bar{y}) = \frac{1}{2} \sum_{m=0}^N \frac{1}{c_m} \sum_{n=-N}^N c_{|n|} c_{|m+n|} a_{|n|} b_{|m+n|} T_m(\bar{y}) \quad (21)$$

A spectral approximation of the Rayleigh stability equation is obtained if the eigenfunction and velocity profile approximations are applied. The method of weighted residuals with the

Chebyshev polynomials used as the test functions leads to the following system of linear equations:

$$\begin{aligned} & \frac{1}{2Y_{\text{inf}}^2} \sum_{n=-N}^N \bar{c}_{|n|} c_{|n|} a_{|n|}^{(U)} \sum_{\substack{p=m+|n|+2 \\ p+|m+n| \\ \text{even}}}^N p [p^2 - (m+n)^2] a_p - \frac{\alpha^2}{2} \sum_{n=-N}^N c_{|m+n|} c_{|n|} \bar{c}_{|n|} a_{|n|}^{(U)} a_{|m+n|} - \\ & - \frac{1}{2Y_{\text{inf}}^2} \sum_{n=-N}^N c_{|m+n|} c_{|n|} a_{|n|}^{(U2)} a_{|m+n|} - \lambda \left(\frac{1}{Y_{\text{inf}}^2} \sum_{\substack{p=m+2 \\ p+m \\ \text{even}}}^N p (p^2 - m^2) a_p - \alpha^2 c_m a_m \right) = 0 \end{aligned} \quad (22)$$

where: $m = 0, \dots, N-2$

According to the “tau” method two additional equations are formed by the use of the boundary conditions:

$$\begin{aligned} \sum_{m=0}^N a_m T_m(-1) &= \sum_{m=0}^N a_m (-1)^m = 0 \\ \sum_{m=0}^N a_m T_m(1) &= \sum_{m=0}^N a_m = 0 \end{aligned} \quad (23)$$

The system of linear equations described by Equations (22) and (23) can be rewritten in the form of the general algebraic eigenvalue problem:

$$(\mathbf{A} - \lambda \mathbf{B}) \mathbf{a} = 0 \quad (24)$$

where the eigenvector \mathbf{a} consists of the coefficients of the eigenfunction spectral approximation (see Equation (8)).

The general eigenvalue problem (24) was transformed into the standard one using the Gary and Helgason algorithm [12] and the standard problem was solved using the Q-R method in the form for real Hessenberg matrices, as published among others in [18, 19].

3. Numerical results

In the method presented above two parameters governing the calculation accuracy can be pointed out. The first one is the number $N+1$ of terms in spectral approximations of the eigenfunction Equation (8) and the velocity profile Equation (16). The second parameter is the distance Y_{inf} where the boundary conditions are formulated. Because the exact boundary conditions for a free shear layer are formulated in the infinity, then the greater the Y_{inf} -distance is, the more accurate is the solution. On the other hand the greater is Y_{inf} -distance, the more terms in the spectral approximation are required for a correct approximation of the eigenfunctions and the velocity profile.

The influence of these two parameters is presented in Figure 1 that shows the imaginary part of the eigenvalue λ for the least stable mode for two wave numbers $\alpha = 0.2$ and $\alpha = 0.5$ as a function of both the Y_{inf} -distance and number of terms N . In the case of lower wave number (longer wavelength of the disturbance) the Y_{inf} -distance required for the correct evaluation of the eigenvalue is greater (Figure 1a) than in the case of the higher wave number shown in Figure 1b. In the first case the slope of the $\lambda_i(Y_{\text{inf}})$ is very low for the Y_{inf} -parameter higher than 20, while for higher wave number $\alpha = 0.5$ (see Figure 1b), the approximation of the eigenvalue obtained for $Y_{\text{inf}} > 10$ is accurate enough. This behaviour is

justified because the longer wave of the perturbation requires a longer distance in order to approximate accurately the boundary conditions posed in infinity. The Y_{inf} -parameter is also limited from above and this limit is higher for greater numbers of terms N used in spectral approximation. The examples shown above suggest that the Y_{inf} -distance for which good approximation is obtained is close to the wavelength because $l = 2\pi/\alpha = 31.4$ in the case $\alpha = 0.2$ and $l = 12.57$ for $\alpha = 0.5$. Therefore for the rest of a calculations presented later on in the paper the Y_{inf} -distance was chosen to be equal to the wavelength of the perturbation considered *i.e.*:

$$Y_{\text{inf}} = \frac{2\pi}{\alpha} \quad (25)$$

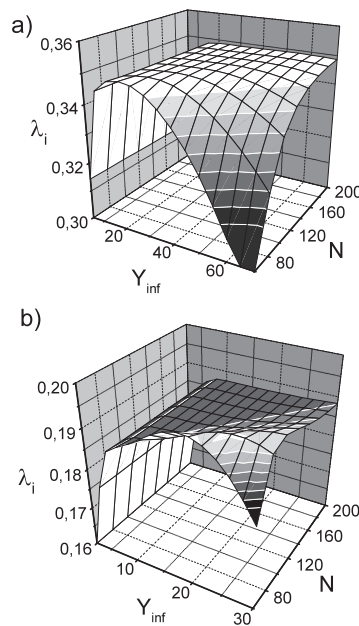


Figure 1. Approximated value of the imaginary part of the eigenvalue λ as a function of the Y_{inf} -distance and number of terms used in approximation for $\alpha=0.2$ (a), and $\alpha=0.5$ (b)

Table 1 shows the comparison of the λ_i -values of the least stable eigenmode calculated for the Y_{inf} -distance determined according to Equation (25), and a number of terms in spectral approximation $N = 100, 200, 300$, respectively, with the results obtained by Michalke [2].

It can be seen from Table 1 that $N=200$ gives the same results as shooting method with the accuracy to four digits apart from the eigenvalues obtained for extreme wave numbers $\alpha=0.1$ and $\alpha=0.9$, which represent weakly amplified perturbations as can be seen from the growth rates presented versus wave number in Figure 2.

The calculations carried out with the use of spectral method served only as a numerical test-case to validate the correctness of the method, which is confirmed by the results shown in Table 1 and Figure 2. However, some additional computations were carried out to study an influence of the velocity ratio R and momentum thickness θ of the mean velocity profile on the maximum growth rate $\alpha\lambda_i$. Figure 3 shows the velocity profiles for which the

Table 1. Comparison of the eigenvalues of the least stable mode obtained by shooting and spectral methods

Wave number A	Eigenvalue λ_i shooting method Michalke [2]	Eigenvalue λ_i spectral method method $N = 100$	Eigenvalue λ_i spectral method method $N = 200$	Eigenvalue λ_i spectral method method $N = 300$
0.1	0.4184	0.4153	0.4181	0.4182
0.2	0.3487	0.3495	0.3488	0.3487
0.3	0.2885	0.2891	0.2885	0.2884
0.4	0.2352	0.2357	0.2352	0.2352
0.5	0.1875	0.1880	0.1875	0.1875
0.6	0.1442	0.1448	0.1442	0.1442
0.7	0.1044	0.1056	0.1043	0.1043
0.8	0.0674	0.0702	0.0674	0.0673
0.9	0.0327	0.0392	0.0339	0.0329

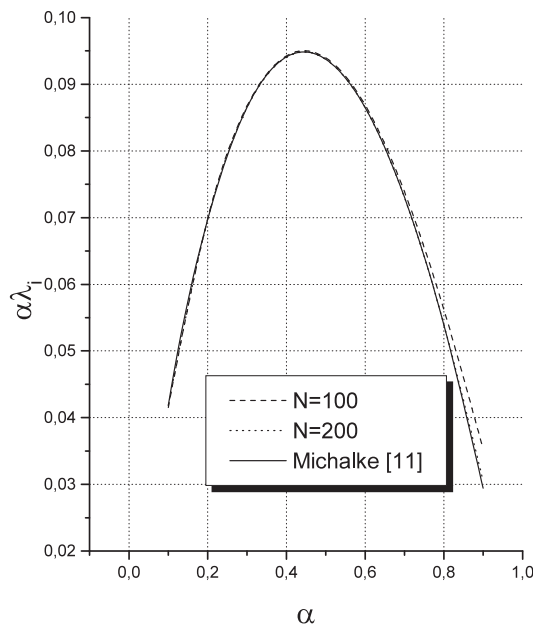


Figure 2. The growth rate vs wave number calculated with the use of spectral method compared with results of shooting method applied by Michalke [2], mean velocity profile, $R = 1$ and $\theta = 0.5$

computations were carried out for velocity ratios $R = 0.25, 0.5, 0.75$ and 1 . All the profiles have the same momentum thickness $\theta = 0.5$ like in the calculations of Michalke [2]. The growth rate versus wave number for these velocity profiles is presented in Figure 4. It can be observed that the velocity ratio influences significantly the value of the maximum growth rate while it changes very weakly the wave number of the most amplified disturbance. This conclusion confirms the observation of Monkewitz and Huerre [4] based on the spatial theory.

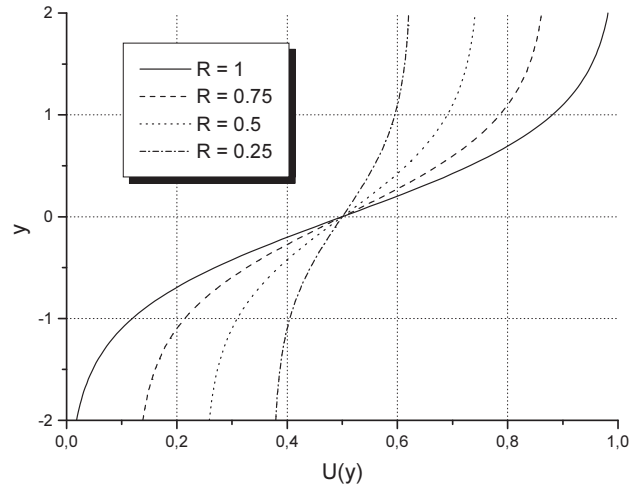


Figure 3. Mean velocity profiles for various values of the velocity ratio, $\theta = 0.5$

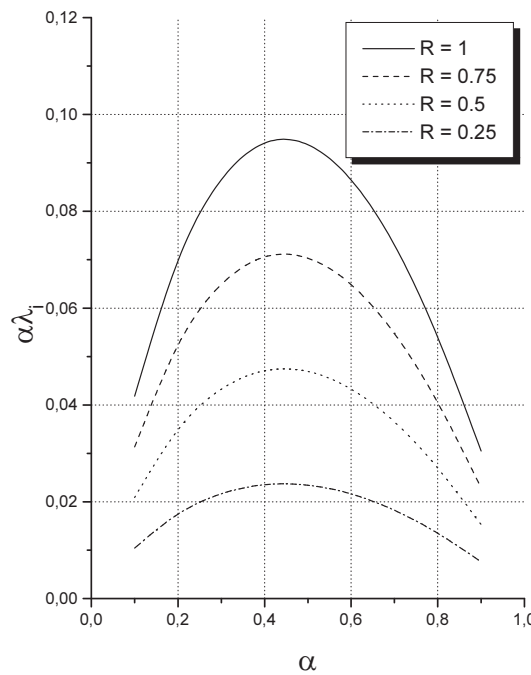


Figure 4. Influence of the velocity ratio on the growth rate vs wave number for various velocity ratios R

Recent experimental work on absolute instability in variable density round jets carried out by Kyle and Sreenivasan [20] revealed that one of the key parameters influencing the instability development was the boundary layer thickness. It was suggested by Monkewitz *et al.* [13] that the global modes observed in variable density jets result from a break off the convective coherent structures. It seems therefore that the boundary layer thickness should also affect the characteristics of the constant density shear layer instability. Figure 5 shows three velocity profiles with somewhat different momentum thickness $\theta = 0.4, 0.5$ and 0.6

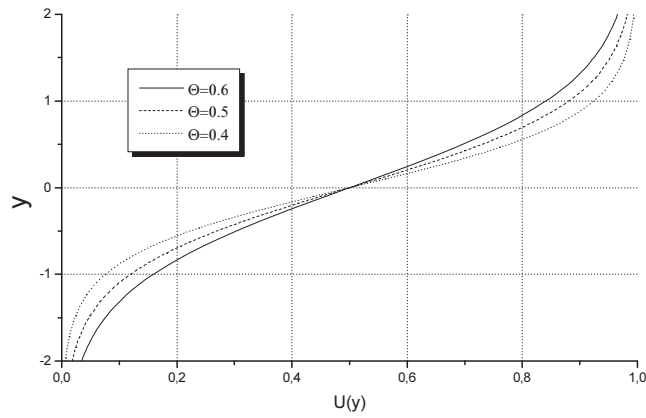


Figure 5. Mean velocity profile for various momentum thickness, $R = 1$

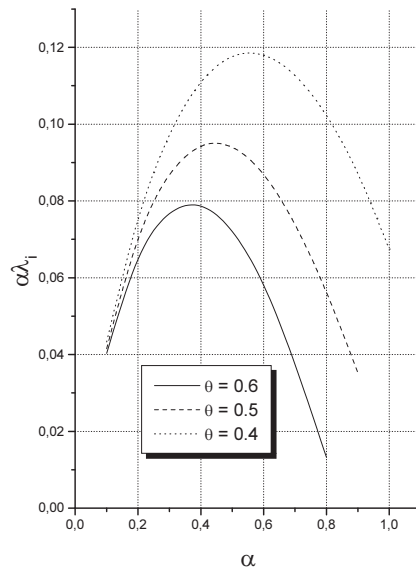


Figure 6. The growth rate vs wave number for various values of the mean velocity momentum thickness

respectively while Figure 6 presents corresponding growth rate curves. These results show that slight changes in the boundary layer thickness influence significantly the wavelength of the most amplified disturbance (see Figure 6). Moreover, it can be seen that the thinner is the boundary layer the higher is the wave number of the most amplified mode. One may conclude that the thinner boundary layers are more sensitive to the disturbances characterised by shorter wavelengths. This is in qualitative agreement with the measurements of Kyle and Sreenivasan [20] who observed that the Strouhal number of the global mode is increased for thinner boundary layer.

4. Concluding remarks

The linear stability theory turns out to be still in the focus of interest because an application of the spatio-temporal theory can predict correctly a transition from convective

to absolute regime as a function of many governing parameters as for example the density and velocity ratios. The huge amount of numerical work done in this field used the shooting method which has important shortcomings because, first of all a convergence rate is influenced by an initial eigenvalue guess and secondly one integration procedure leads to establishing only one eigenvalue, and in order to find the least stable mode the computations have to be repeated many times. An alternative is a matrix method which transforms a differential problem into the algebraic one. The paper presents an example of the application of the matrix method used with Chebyshev spectral approximation of the eigenfunction. The problem chosen as the numerical test-case was the temporal instability of the inviscid shear layer. The results obtained with the use of spectral method were compared with the results of shooting method given by Michalke [2], which confirms the correctness of the spectral calculations. The computations carried out are treated primarily as a test-case but some additional numerical examples are shown which illustrate the influence of the velocity ratio and boundary layer thickness on the wavelength of the most amplified temporal eigenmode.

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