DIGITAL INFORMATION PROCESSING: THE LIE GROUPS DEFINING THE FILTER BANKS OF THE COMPACT DISC

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Abstract: The versality of the compact disc (CD) has quickly become apparent to manufacturers and users alike. Exceeding the expectations of even its most ardent supporters, the CD holographic disc storage system has become one of the most successful consumer electronics products ever introduced. The phenomenal success of the audio CD on the eager worldwide marketplace has encouraged rapid development of CD technology and spawned entirely new high tech applications for the dimpled disc. The Mini Disc (MD), for instance, occupies about one-fourth the area of the standard CD-Digital Audio (CD-DA) format yet provides an identical playing time through efficient data reduction. The essence of digital audio lies in its numerical basis. It is the aim of the present paper to elaborate the mathematical principles underlying the audio CD as far as they are concerned to the format's electronic and holographic principles.

Keywords: Information Technology, CD-Digital Audio format, holographic data storage, Heisenberg Lie group, filter bank design

Dedicated to the memory of Claude Elwood Shannon (April 30, 1916 – February 24, 2001), the pioneer of digital information processing. He changed our real life.

1. Introduction

An important aspect of all information transmission is the storage and detection of encoded information. Information technology (IT) deals with the implementation of these modalities. Specifically storing audio information places great demands on a digital medium. A 60-minute musical program recorded in stereo channel modality with pulse-code modulation (PCM) at a standard time sampling frequency of 44.1 kHz and with 16-bit amplitude quantization level every 23μs, for instance, generates over

5 billion bits of information in all. Because error correction, synchronization and modulation are obligatory requirements for successful audio information storage and detection, the total capacity required with random access capability is over 15 billion bits. Thus digital audio's requirements are considerable. The storage of visual information for parallel image processing [1–3] is, of course, even more demanding than the storage of audio information. *Holographic* methods are particularly efficient for storing image data.

The original Compact Disc–Digital Audio (CD-DA) format was developed to meet these demands at low costs. A *frame* structure provides such a format. The frame as the irreducible unit of data representation in the CD format is the smallest recognizable section of data on a disc surface. It furnishes a means to distinguish the data types: audio data and its parity, synchronization word, and subcode. The information contained in a CD frame prior to modulation contains a 27-bit synchronization word, 8-bit subcode, 192 data bits, and 64 parity bits. CD frames are assembled when the master disc is encoded. Assembly of the frame involves several processing steps as well as modulation and the addition of merging bits.

The encoding process begins with placing serial strings of audio data on the CD data surface. Six 32-bit PCM audio sampling periods, alternating from 16-bit left and right channels, are grouped in a frame, the left channel preceding the right. Each 32-bit sampling period is divided to yield four 8-bit audio symbols. The original 16-bit number is called a word, and it is split into two 8-bit symbols. After grouping audio data into symbols, error correction takes place. This step employs a combination of interleaving and parity to make the data more robust against errors encountered during storage.

Following encoding, an 8-bit subcode symbol is added to each frame. Although the user cannot access the subcode directly, it provides much of the information for proper disc playback as well as front panel controls and display. To interpret and utilize this information, the CD player collects subcode symbols from ninety-eight consecutive frames to form a subcode block, with eight 98-bit words. Only two bits are used in audio CDs. Included is information specifying the total number of selections on the disc, their beginning and ending points and timings, the index points within a selection, the program lead-in and lead-out points, and updated information on the pickup's position as the disc is played. The other 6 bits are available for encoding other information on audio CDs. After the audio, parity, and subcode data are assembled, the bit stream is modulated using eight-to-fourteen modulation (EFM). Blocks of 8 data bits are translated into blocks of 14 bits, known as channel bits, using a read-only-memory (ROM) dictionary which assigns an arbitrary and unambiguous word of 14 bits to each 8-bit word. In EFM, each 8-bit word is translated to a 14-bit word selected for its specific bit pattern. With 14-bit words, more unique patterns can be selected. Thus EFM provides a kind of error correction.

The blocks of 14 bits are linked by 3 merging bits. Two merging bits, always 0s, are required to prevent the possibility of successive 1s between serial words. The additional merging bit, either a 1 or a 0, depending on the preceding and succeeding patterns, is added to each code pattern to aid in clock synchronization and to suppress the signal's low-frequency component. The resultant channel stream produces pits and lands on the CD data surface which are at least 3 channel bits and no more than 11 channel bits long. The tracks of pits on the CD data surface are the physical manifestation of data encoding, including multiplexing, interleaving, parity, error protection, and EFM encoding.

The high tech CD-DA system was introduced in Europe and Japan in the fall of 1982 and in the United States of America in the spring of 1983. It has become one of the most successful high precision microelectronic devices ever introduced. Specifically, more than one billion CD-DAs are sold every year, and LPs have all but vanished. Because it contains the same audio information, bit for bit, that was recorded in the studio, the audio CD has been growing rapidly in popularity since it was launched worldwide in the early 1980s. The commercially available CD players represent prime examples of the benefits of digital microelectronic chips and integrated optoelectronic systems. Due to the *spinorial* feature of the standard CD-DA format, they are perhaps the most sophisticated and microelectronically subtle high tech pieces of audio equipment to ever reach the consumer.

Although the CD-DA system has prospered beyond the wildest dreams of its inventors, it does not signal the end of development in audio technology. Today, the compact disc family encompasses alternative format specifications such as the Compact Disc-Read Only Memory (CD-ROM) for professional databases and mass storage for computer-related applications, Digital Versatile Disc-Read Only Memory (DVD-ROM) for advanced high density storage of high fidelity audio-video frames, and various other types of CD formats among which the interactive compact disc (CD-I) format is a special specification of the CD-ROM format [4, 5]. The CD-I concept was introduced in 1986, the same year that over three million CD-DA players and over fifty-three million audio CDs were sold in the USA.

The CD-I system which permits the storage of a simultaneous combination of digital audio, video, graphics, text, and data, all functioning in an interactive format, with user control over presentation, on a 12-cm diameter optical disc, was launched in 1991. The CD-I system is thus a multimedia extension of digital audio found on CD-DA discs. It proves that the compact disc is highly suitable for non-audio applications. Because CD-I players reproduce conventional CD-DA discs, CD-I forms also an upscale CD-DA system.

The CD-I format defines both hardware and software standards, much like the audio CD format. Although CD-ROM can also store text, graphics, video and audio, CD-I defines a special integration of such functions. Because CD-I represents an interactive medium, its information may be accessed through a dialogue procedure.

The CD-ROMs which store 600 megabytes on one side of the 5 inch disc are the logical extension of the CD-DA format toward the broader application of information storage on a digital medium. The CD-ROM standard, unlike the CD-DA standard, does not link CD-ROM to any specific application of IT. The spinorial format is thus transparent and offers a cost-effective way of distributing large amounts of information, especially information not requiring frequent updating. Of course, various more advanced concepts such as the computationally highly demanding mini disc (MD) adaptive transform acoustic coding (ATRAC 3) format, supported by the technological *and* signal theoretic progress made since the launching of the original

CD-DA format, are under the way to open a new market for the information storage and signal retrieval industry.

The ATRAC encoder divides the PCM data into segments in intervals up to 20 ms long. Fast Fourier transform (FFT) software analyzes the wavelet data in each segment and generates corresponding frequency component data. Using psychoacoustic modeling, the system identifies the audio components that are audible and encodes them, assigning bits as needed according to the amplitude of audible frequency components. Other inaudible data is discarded. Following ATRAC encoding, data undergoes Cross Interleave Reed-Solomon Code (CIRC) and EFM encoding and is recorded to disc along with subcode and address information.

Audio data reduction techniques such as ATRAC are based on the working of the human ear and assume that the information capacity of the ear is less than the CD standard provides. To ascertain which data may be discarded, psychoacoustic models of the ear have been devised which demonstrate that in theory a data rate of less than 100 kbps per channel may be adequate, if properly encoded.

Elementary signal theoretic techniques such as the Shannon time sampling process and amplitude quantization modes of IT form the basis of digital audio. Without them, the CD-DA system would not be a viable reality. Both sampling and quantization are parameters which determine the limitations of an audio digitization transducer. Therefore all digital audio system architectures use these parameters to record and reproduce signals.

There are various different ways to encode serial strings of digital data [6]. Modulation is the process of encoding source information prior to transmission and detection via information channels and storage. Among these techniques, PCM is one of the most efficient high performance encoding methods. The PCM hardware design is routinely used for telemetry of images from space vehicles and forms the most popular digital audio system architecture, owing to its error-free properties.

PCM is a modulation process in which the instantaneous amplitude of an analog signal is converted to a binary number by a A/D converter and then transmitted as a serial string of bits. The encoded signal is fully compatible with digital circuitry which is usually designed to operate with a binary code. Because of its efficient use of bandwidth and its compatibility with off-the-shelf circuitry, PCM has proven to be an expedient means of representing audio data for recording and signal retrieval. The adaptive delta PCM (ADPCM) combines elements of PCM encoding and delta modulation (DM) encoding. Because of its ability to store digital data with fewer bits, ADPCM is extremely efficient. On the other hand, ADPCM requires additional processing steps beyond regular PCM for both encoding and decoding.

The PCM format like various other coding schemes of IT requires wider bandwidth than the corresponding analog signal. However, PCM data is easily multiplexed, that is, several data channels may be merged to form *one* channel of data. Therefore the majority of digital recordings are mastered on PCM digital audio recorders. At the output of a CD player which provides access to any part of the audio program within a second or less, the data returns to its PCM format at the D/A converter. A D/A converter does the opposite IT job of an analog-to-digital (A/D) converter. It takes a train of binary-coded words as its input and produces a continuous-time output proportional to the value of the digital input by means of the impulse response of the Heaviside zero-order hold. The input sample-and-hold (S/H) circuit, sometimes called an aperture circuit, which is the next integrated circuit following the D/A converter and installed on the same D/A microelectronic chip, performs a hold function to buffer instability in the analog signal and correct for high-frequency roll-off. When the D/A output voltage is stable and any glitches have passed, the S/H output forms a pulse amplitude staircase signal.

The S/H circuit is essentially made of a capacitor and switch. It tracks the signal until the sample command causes the switch to open, isolating the capacitor from the signal. The capacitor holds this analog voltage during conversion. The timing of the sample command must be carefully controlled to prevent jitter, the phenomenon of imprecise sample times. Then the pulses in the output are the width of a sampling period. Reconstruction requires pulses of infinitely short duration. This is impossible to achieve because it would require infinitely large current amplitude flow. Because of the finite duration of the output samples, a filtering effect occurs in which the amplitude response declines to zero at the sampling frequency. This is beneficial because image spectra are attenuated.

To summarize the hardware design which realizes an audio PCM digitization transducer, its recording section consists of input amplifiers, a dither generator, input anti-aliasing low-pass filters, S/H circuitry, A/D converters, a multiplexer, digital processing and modulation circuitry, and a storage medium such as optical disc. On the digitization transducer's output side are demodulation and processing circuits, a demultiplexer, D/A converters, S/H aperture circuitry, output anti-imaging low-pass filters, and output amplifiers. From the mathematical point of view, the transducer's input-output reflection symmetry is of particular importance.

The essence of digital audio lies in its numerical basis. Usually, the mathematical interest of digital audio storage media such as the CD-DA format is restricted to the error correction which is performed by the cornerstones of error correction: interleaving and parity. Interleaving is employed to guard against the occurrence of burst errors. The parity bit added to every data word represents the redundancy contained in the correction codes. The parity bit is chosen so that the total numbers of ones and zeros in the data word plus parity bit is even or odd. Due to extra data created from the original data to help detect errors, the chance to correct errors is easier with digital data than with acoustic analog signals. The particular algorithm commercially used in the CD-DA information transmission channels is the Cross Interleave Reed-Solomon Code (CIRC). The CIRC circuit uses two correction codes for additional correcting capability and three interleaving stages to encode data before it is placed on the disc. Similarly, CIRC performs error correction while decoding the serial string of data during playback.

Upon playback, following demodulation, data is transmitted to a CIRC decoder for de-interleaving, error detection, and correction. The CIRC decoding process utilizes parity from two Reed-Solomon decoders and scatters consecutive errors by de-interleaving. In this way, errors become more likely random errors which are more easily corrected.

In contrast to the treatments of coding algorithms, the present paper deals with the *temporal* data readout of the CD-DA. It shows that the timing defined by the real

Heisenberg nilpotent Lie group G is behind the holographic readout procedure of the spinorial format and that the metaplectic group leaving the one-dimensional center C of G pointwise fixed dictates the optical focusing as well as the discrete time data sampling process of IT [7, 8]. Although there are, independently of the summation formulae approach, very short proofs and many *rigorous* derivations of the Shannon time sampling theorem available, the approach based on a *central* projection has to be given preference, at least from the methodological *and* the epistemological point of view, because G is a relatively elementary non-compact non-abelian Lie group which allows to define in a *conceptual* way the filter banks of IT.

Next to audio information, visual information plays a major role in human communication and orientation. It is estimated that about 80% of all information received is of visual nature. It is not surprising, therefore, that with the advent of electronic data processing the desire arose for the data acquisition, processing and analysis of pictures and sequences of images by digital computers. Since the first trials some 30 years ago, image processing has developed into a broad scientific discipline which intensively interacts with several other topics such as phase coherent optics, quantum statistics of radiation, information theory and signal processing, pattern recognition, and artificial intelligence [9–12] . As an outlook to advanced phase coherent summation imagery, the paper refers to a two-dimensional imaging implementation of this system which leads to the non-invasive diagnostic modality of clinical magnetic resonance tomography [3, 13, 14]. In fact, an understanding of the CD-DA system forms an excellent preparation for the understanding of the much more sophisticated modality of clinical magnetic resonance imaging (MRI) and synthetic aperture radar (SAR) imaging [1, 12]. Both are holographic imaging modalities which are highly demanding with respect to their conceptional as well as technical implications. Both have made substantial technical advances in the past decade.

It was the field of *optical* holography which gave birth to the CD-DA system. In 1969, Dutch physicist Klaas Compaan learned that engineers at RCA had devised an inexpansive way to manufacture optical holograms by using a master stamper with microscopic formations to press copies. After a dozen years of research and development, he believed that the filter bank technique could be used to produce optical discs holding video images. He related the brilliant idea that would result in one of the most successful consumer electronics products of all time, the CD, to his colleague Piet Kramer in 1970, and, together in Eindhoven they completed a prototype glass disc with a series of 1-mm square black and white images that could be projected onto a screen. They decided it would be more efficient to record a video signal, rather than images themselves, and to use a track of dimples to holographically encode the analog frequency modulated signal. Moreover, they found that a laser source was needed to coherently recover the signal from the hologram. In July 1972, they publicly demonstrated a color prototype. Meanwhile, in 1972 large scale integrated (LSI) circuits were introduced by various manufacturing companies such that microprocessors could evaluate the serial string data of the filter banks and drive home the point that small size is key. The develoment of very large scale integrated (VLSI) technology supported these ideas.

Whereas the RCA company indirectly helped to holographically develop the CD-DA system, with the exception of the EMI company in Britain, the first manufacturers of the X-ray computerized tomography (CT) scanner, industry took little or no notice of the clinical MRI modality. The early prototype EMI scanner, an X-ray projection-reconstruction system producing high-quality cross-sectional images, initially of the head and subsequently of the whole body, formed an invention which has revolutionized radiologic diagnosis for which Godfrey N. Hounsfield shared the 1979 Nobel prize for medicine. His co-recipient, Allan M. Cormack, received recognition for the development of accurate reconstruction algorithms for ill-posed problems, work which began in the late 1950s and came to fruition in the early 1960s. It is interesting to note that Hounsfield devoted the latter part of his Nobel prize address to a discussion of the future of nuclear magnetic resonance (NMR) rather than X-ray CT techniques. Today, MRI is a billion-dollar industry; thousands of MRI scanners are helping patients the world over. Now considered the premier diagnostic imaging modality, clinical MRI not only competes with X-ray CT scanners but many diagnosticians believe that MRI will eventually replace X-ray CT scanners in the radiological departments.

The theory of Lie groups and Lie algebras is a fundamental part of mathematics because it allows to rigorously investigate basic internal *and* external symmetry principles. According to Wolfgang Ernst Pauli (1900-1958), symmetry forms the fundamental organising principle of physics and the natural sciences. In signal processing, symmetries are used to implement *fast* processing algorithms by sophisticated special-purpose processors. Among these, spectrum analyzers implementing the FFT are particularly popular [6]. Albert Einstein's intuitive treatment of relativity was followed shortly by a more sophisticated treatment by Hermann Minkowski (1864–1909) in which Lorentz transformations were shown to constitute a Lie group of rotational collineations. Similarly, shortly after Werner Karl Heisenberg (1901–1976) introduced his famous Commutation Relations in quantum physics, which underlie his Uncertainty Principle, Hermann Weyl used the Lie commutator bracket $[.,.]$ to show that they could be interpreted as the structure relations

$$\left[\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix},\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}\right]=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}$$

for the canonical basis of the real Lie algebra $\mathrm{Lie}(G)$ of the real Heisenberg nilpotent Lie group G. Unexpectedly, the Heisenberg Lie group G is also at the basis of optical holography and thus *conclusively* explains the filter bank concepts holographically implemented by CD, MRI, and SAR.

This paper presents an introduction of harmonic analysis on the "almost abelian" Heisenberg Lie group G to information theory with an outlook to the enchanting area of theta identities, such as the Jacobi and the Landsberg-Schaar identities, and the field of phase coherent summation imagery which is conceptually based on the filter bank concept well known from multirate signal analysis or subband coding.

2. The holographic data readout procedure of IT

Many methods of audio storage and detection have evolved since Thomas Alva Edison (1847–1931) made the first audio recording in 1877 on a cylinder covered with tin foil. Ironically, the invention of the analog phonograph was performed while he was experimenting with a device for storing digital data, a telegraphic code repeater. Early acoustical recordings were made on wax cylinder and shellac disc. Subsequently, numerous magnetic tape formats were developed. However, all of these audio systems recorded and reproduced analog signals by using a mechanical pickup. Because optical holography was far ahead, conceptionally as well as technically, time was not mature for the highly complex process of disc manufacturing.

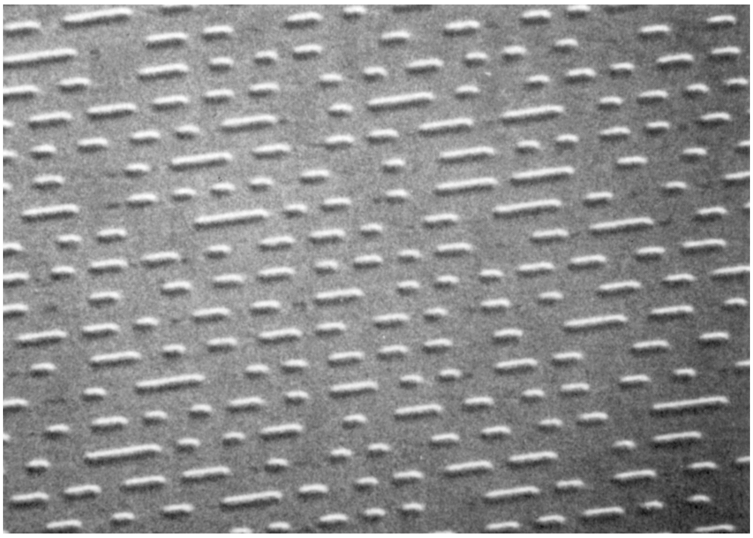

Figure 1. Visualization of the tracks of pits on the metalized CD data surface by use of a scanning electron microscope. The horizontal line of the scan indicates a scale of 5μm. The track pitch, the distance between adjacent laps of the pit spiral, is 1.6μm so that there are about 600 tracks to a mm. Each pit has a width of about 0.5μm. In comparison, the cross-section of a human hair has a width of 75μm. The minimum pit length is 0.833μm to 0.972μm, the maximum pit length is 3.05μm to 3.56μm so that a track of pits might contain about 3 billion pits precisely arranged on a spiral. Unspiraled, the track would stretch about 3.5 miles. Because each outer track revolution contains more pits than each inner track revolution, the CD must be slowed down as it plays in order to maintain a constant rate of data. Based on the timing of the master clock, the CD player automatically regulates the disc rotational speed to maintain a constant bit rate of 4.3218MHz during holographic readout

In IT the CD is certainly one of the most advanced storage media available. The CD-DA format stores its information digitally and uses a laser optoelectronic pickup. The length of its data represents the binary bits which represent the original audio signal. A laser beam of carrier frequency ν is focused to read the data stream. The data is physically contained in the disc's pits which are impressed along its top surface and are covered with a 50 to 100nm metal layer. The data storage in pits on a flat surface is not directly visible to the naked eye. A scanning electron microscope is needed to get a sufficiently good look on the track of pits (Figure 1) arranged in a continuous spiral running from the inner circumference to the outer one. Another 10μm to 30μm plastic layer protects the metalized pit surface. The laser beam is focused on the metalized data surface embedded inside the disc and passes through the transparent plastic substrate and back again.

The fact that the laser beam passes the disc substrate provides one of the significant assets of the CD system. The plastic substrate has refractive index of 1.55 whereas air has normalized refractive index 1.0. The speed of light slows from $c = 3 \cdot 10^8$ m/s to $c' = 1.9 \cdot 10^8$ m/s and changes the carrier frequency ν to the fraction νy. It is the *instantaneous* frequency νy which characterizes the frequency modulation of the carrier frequency ν induced by the substrate. Because of the refractive index, the thickness of the CD, and the numerical aperture of the objective lens, the size of the laser beam on the disc surface is approximately 2μm. Hence, the laser beam is focused to a point slightly larger than a pit width but does not overlap the tracks of pits (Figure 1).

The reflective flat surface, called land (Figure 2), causes almost ninety percent of the laser light to be reflected into the optoelectronic pickup. When considered from the laser's perspective, the pits are viewed as tracks of bumps. The height of each bump is between 0.11μm and 0.13μm. This height is slightly smaller than the semiconductor laser's wavelength $\lambda = 780$nm in air. Inside the polycarbonate substrate with a refractive index of 1.55, the laser's wavelength is about $\lambda' = 500$ nm. The height of the bumps is therefore approximately $\frac{1}{4}$ of the laser's wavelength λ' inside the disc substrate.

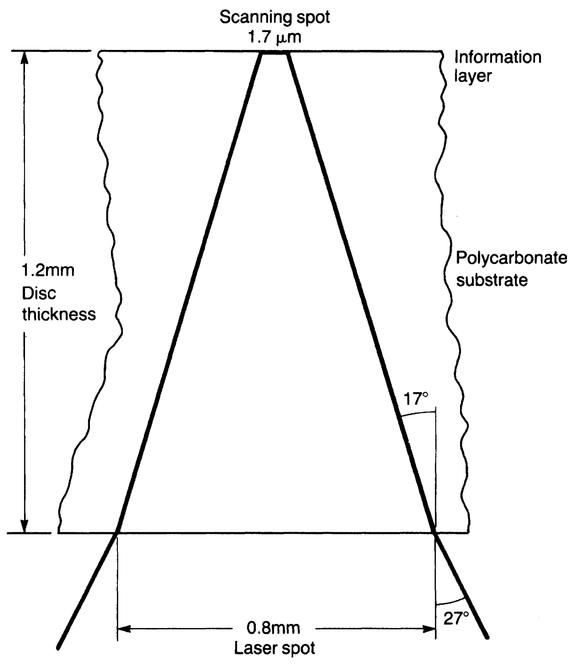

Figure 2. For holographic data readout, the semiconductor laser beam passes the disc substrate. The refractive index of the substrate contributes to the optical focusing of the laser beam. The transparent plastic substrate forms most of the CD's 1.2mm thickness

Notice that light striking land travels a distance $\frac{1}{2}\lambda$ further than light striking a bump. This creates a *phase* difference

$$x=\frac{1}{2}\lambda$$

between the part of the beam diffracted from the bump and the part reflected from the sourrounding land (Figure 3). The phase difference x causes the two parts of the beam destructively interfere with and cancel each other. Actually each pit edge, whether leading or trailing, is a one and all areas in between, whether inside or outside a pit, define zeros. This holographic technology is a more efficient storage technique than coding the binary bits directly with pits. Combinations of the varying lengths of the pits encode the binary data stream to holographically be read by the semiconductor laser beam. Because the holographic readout procedure is non-invasive, the CDs are completely immune to damage from repeated playing.

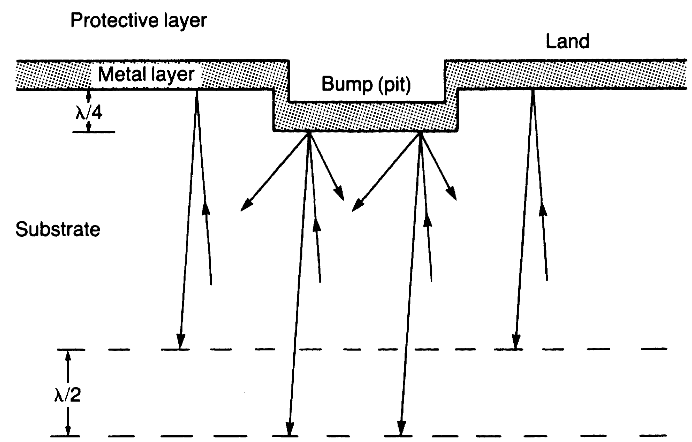

Figure 3. Data is physically contained in pit tracks which are impressed along the CD top surface and are covered with a thin 50 to 100nm metal layer. Another thin 10 to 30μm plastic layer protects the metalized pit surface. A semiconductor laser beam is used to read the data. In the laser illumination, a pit height causes a wavelength path difference of $x=\frac{1}{2}\lambda$ relative to surrounding land. The laser optoelectronics, servo system for automatic focusing and tracking, control microprocessors, and integrated output D/A circuitry are all high, high tech

The pits on the CD data surface are the physical manifestation of data encoding, including multiplexing, interleaving, parity, error protection, and EFM encoding, all of which take place at the lathe when the master disc is cut. A CD might contain 3 billion pits precisely arranged on a spiral track. The optoelectronic pickup has to focus on, track, and read that data spiral. To achieve sharp focus within a $\pm$ 2μm tolerance on the data surface, and proper frequency modulation for the definition of the disc data y, a monochromatic illumination of the CD data surface is required. Otherwise the phase interference between the direct and reflected laser light is lost along with the audio data, as well as the tracking information, and, ironically, the focusing information itself. The objective lens must therefore be able to refocus as the disc surface deviates vertically. A servo-driven auto-focus system manages this control problem of the spinorial readout procedure.

Auto-focus control is an *absolute* prerequisite in a laser optoelectronic pickup system. Disc warpage and other irregularities would place the data out of the pickup's depth of focus, making it impossible to holographically create the necessary phase interference pattern with the pit height and land. Specifically, phase coherence is vital to implement phase cancellation in the near-infrared light beam produced by disc pits so that disc data x can be read. Both specifications can be accomplished by a laser diode placed at the focus of the collimator lens with a long focal length. A monitor diode stabilizes the semiconductor laser's output.

Any optoelectronic pickup system must, of course, control both tracking and focusing simultaneously. When the auto-focus is not operative, the system pulls the objective lens back to prevent damage to the lens or CD.

Because the parameters x and y have timelike character, a spindle motor is used to rotate the CD with constant linear velocity, a phase coherency condition in which a uniform relative velocity is maintained between the disc and the pickup. To achieve this, the rotation speed of a CD has to vary depending on the position of the pickup underneath the surface. Because each outer track revolution contains more pits than each inner track revolution, the CD must be slowed down as it plays in order to maintain a constant rate of data.

When the laser beam is reflected at the revolving disc surface during playback, the response is detected by a photodiode sensor. The optoelectronic pickup's servo loops use electric signals to control motors to mechanically adjust the pickup's position horizontally and vertically, relative to the disc surface. In another servo loop, information from the data itself is used to determine the disc's precise rotational speed, and maintain the proper data stream rate. It is the voltage stemming from the sensor which is ultimately transformed into the analog audio signal output from the CD player. The encoded data from the pickup must first be decoded.

In the CD-DA player, the numerical aperture of the objective lens, wavelength of the semiconductor laser, thickness and refractive index of the disc, and size and height of the pits all work together to allow data to holographically be read from the disc. The various subsystems in a CD player are closely interrelated with a tightly interlocked timing relationship: The audio data rate is 176.4 kbytes per second. Because there are 24 audio bytes in a frame, the frame rate is 7350 Hz and the master clock 4.3218 MHz. A single master clock is employed for all the signal processing circuitry, including the *oversampling* filter bank and D/A converters. The master clock establishes that the CD forms a *temporal* device. As a result, the data stream of the CD-DA player is *synchronous*, preventing any internal beating.

While CD-ROM uses a data format similar to that of the CD-DA format, the players are not compatible. A CD-ROM player contains laser optics, modulation, and error correction, but D/A conversion and audio output sections are replaced with a computer interface to output the ROM data to a host computer. Data is transmitted to the host computer in blocks of 2 kbytes. Because they are not tied to one specific operating system or data processor, CD-ROM devices can be interfaced with all existing computer systems. CD-ROM is limited only by the capabilities of the operating system and microprocessor of the host computer.

A DVI (Digital Video Interactive) all-digital optical disc is a CD-ROM format containing DVI specific data of reproducing full-motion, full-screen video, computer generated video graphics, and digital audio via a CD-ROM drive [15]. Although data on DVI discs is formatted to CD-ROM specifications and can be played on a CD-ROM drive, special DVI decoding technology is required. The DVI format is incompatible with the CD-I format and diverse other CD-ROM implementations. Interestingly, CD-I was the first large volume application of the Moving Picture Experts Group (MPEG) international standard for coded representation of moving pictures, associated audio, and their combinations when used for storage and signal retrieval on digital storage media. The MPEG audio and video coding algorithms allow use of video sequences coded with a variety of CD-I picture formats, as well as a variety of CD-I audio formats.

As a subset of the CD-ROM data format, the CD-I system offers five levels of audio quality, to be selected according to the need for fidelity, and calls for a total storage capacity of approximately 650 megabytes. Because a CD-I disc is recorded with constant linear velocity, a constant readout rate of 75 frames per second is achieved. The adaptive delta pulse code modulation (ADPCM) is employed in the CD-I audio format as a hybrid encoding technique that correlates successive data samples to adapt to changes in the signal. It uses 4- or 8-bit words, depending on the level of sound quality required, and can be considered as a bit rate reduction technique. Since each of the quantization levels is assigned a step-size scale factor, step sizes may be adapted with greater accuracy, because more step-size information is available. The scale factors are based on the statistics of the signal itself. For instance, scale factors for an ADPCM circuit designed to process speech would be selected differently from those for a system designed to process music. Decoding is accomplished in the CD-I player. EFM demodulation and error detection/correction decoding are effected. Interpolation is performed if error flags are present, and ADPCM audio data is block-decoded and expanded to linear 16-bit form. Digital-to-analog (D/A) conversion, low-pass filtering, and de-emphasis complete the audio processing of the CD-I audio format. As implemented in the CD format, ADPCM's fidelity is lower than the PCM signal on an audio CD, but it is suitable for many CD-I applications, particularly when speech composes the audio program.

3. Harmonic analysis on the real Heisenberg Lie group

Taking into account the aforementioned parameters detected by the laser optoelectronic pickup, the real Heisenberg group G collects the phase difference x, the local frequency νy, and the real variable z dual to the carrier frequency ν into an upper triangular matrix with real entries:

$$g=\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}.$$

The transversal trace of the element $g\in G$ is given by:

$$g_0=\begin{pmatrix}1 & x & 0\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix} \qquad (x\in\mathbf{R}, y\in\mathbf{R}).$$

The group law of G is matrix multiplication:

$$g_1 \cdot g_2 = \begin{pmatrix} 1 & x_1 & z_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & x_2 & z_2 \\ 0 & 1 & y_2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x_1+x_2 & z_1+z_2+x_1y_2 \\ 0 & 1 & y_1+y_2 \\ 0 & 0 & 1 \end{pmatrix},$$

so that G forms a non-commutative real Lie group of dimension 3. The inverse of the element $g \in G$ is given by:

$$g^{-1} = \begin{pmatrix} 1 & -x & -z+xy \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix}.$$

The center $C \hookrightarrow G$ is formed by the subgroup of matrices

$$\begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (z \in \mathbf{R}),$$

and is therefore isomorphic to the real line $\mathbf{R}$. The matrix exponential, which maps the Lie algebra Lie(G) onto G, projects the one-dimensional center of Lie(G) of all matrices

$$\begin{pmatrix} 0 & 0 & \zeta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (\zeta \in \mathbf{R})$$

onto the real line C. If the elements $g \in G$ are written as triples of real numbers (x,y,z), then the elements of C are identified with the real numbers $z \in \mathbf{R}$. The non-trivial unitary characters of C are then given by the functions

$$z \rightsquigarrow e^{2\pi i \nu z} \qquad (\nu \neq 0).$$

For the next step of reasoning it is important to note that the physical meaning of the elements $g \in G$ and its applications to IT becomes apparent only by unitarily *representing* the Lie group G.

Let $\mathcal{S}_{\mathbf{C}}(\mathbf{R})$ denote the Schwartz space of infinitely differentiable complex-valued functions ψ on the real line $\mathbf{R}$ which are, as well as all their derivatives, rapidly decreasing at infinity. For each real number $\nu \neq 0$, the Lie group G acts time generating on the waveform $\psi \in \mathcal{S}_{\mathbf{C}}(\mathbf{R})$ according to the temporal rule

$$\rho_\nu\Bigg(\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \Bigg)\psi(t) = e^{2\pi i \nu (z+yt)}\psi(t+x) \qquad (t \in \mathbf{R}).$$

Then there is a unique unitary linear extension of ρ_ν from $\mathcal{S}_{\mathbf{C}}(\mathbf{R})$ to the standard complex Hilbert space $L^2_{\mathbf{C}}(\mathbf{R})$. The unitary linear representation ρ_ν of G in $L^2_{\mathbf{C}}(\mathbf{R})$ is *irreducible*. It is called the linear Schrödinger representation of G [7] associated to the carrier frequency $\nu \neq 0$. Because the irreducible unitary linear representation ρ_1 is *square* integrable mod C, it admits a reproducing kernel K allied to the global Frobenius reciprocity theorem [16, 17]. In terms of multirate signal analysis or subband coding, the transversal linear mappings

$$\rho_1\Bigg(\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \Bigg) : L^2_{\mathbf{C}}(\mathbf{R}) \longrightarrow L^2_{\mathbf{C}}(\mathbf{R}) \qquad (x \in \mathbf{R}, y \in \mathbf{R})$$

defined by the transversal traces $g_0 \in G$ are called filter bank operators associated to the linear Schrödinger representation ρ_1 of G.

Often the linear Schrödinger representation of G is confused with the Schrödinger equation of quantum mechanics. Whereas the Schrödinger equation is related to the probabilistic *detection* of signals and therefore not invariant under the action of the Lorentz group, the irreducible linear Schrödinger representation of G describes the time *modi* of information *transmission* by phase coherent signal processing. An important consequence of the irreducibility is that the unitary linear representations ρ_ν and $\rho_{\nu'}$ of G are inequivalent for $\nu \neq \nu'$ [7]. As a consequence, the associated diffraction patterns do not interfere so that audio signals can be conveyed from one device to another with a minimum amount of confusion. Specifically in the DVD-ROM format this consequence of the fundamental Stone-von Neumann theorem [7] is used to increase the information content by stacking several transparent slices of pit tracks. The transmission of digital data streams, however, is a great deal more complicated on account of the potential disagreements of the sampling frequency, the synchronization method used, and the block length. In this case, a time-sharing multiplexing transmission channel transmits or receives frames, each containing left and right channel data alternatively. The transmission rate corresponds exactly to the source sampling frequency. When the time sampling frequency is 44.1kHz, the CD-DA format allows to transmit 44100 frames per second. One frame consists of two subframes, labelled left and right stereo channel, each containing 32 bits of audio information.

In the case of a unique slice as in the CD-DA format it is convenient to normalize the carrier frequency scale such that $\nu = 1$ holds. Then the Levi-Cività mapping

$$J : \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \qquad (z \in \mathbf{R})$$

forms an automorphism of period 4 of G which leaves the center $C \hookrightarrow G$ pointwise fixed. Of course, the elements

$$\begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in C$$

are the only fixed points of the mapping J. It is not difficult to establish that the Fourier cotransform $\bar{\mathcal{F}} : \mathcal{S}_{\mathbf{C}}(\mathbf{R}) \longrightarrow \mathcal{S}_{\mathbf{C}}(\mathbf{R})$ which is defined by the assignment

$$\bar{\mathcal{F}}\psi(s) = \int_{\mathbf{R}} e^{2\pi i s t} \psi(t) \mathrm{d}t \qquad (s \in \mathbf{R})$$

satisfies the intertwining identity

$$\bar{\mathcal{F}}^{-1} \circ \rho_1(g) \circ \bar{\mathcal{F}} = \rho_1\big(J(g)\big)$$

for all matrices $g \in G$. It follows that the linear Schrödinger representation ρ_1 is isomorphic to the unitary linear representation $\rho_1 \circ J$ of G. The unitary isomorphism is given by the Fourier cotransform $\bar{\mathcal{F}}$. It is immediate that its inverse $\bar{\mathcal{F}}^{-1}$ is associated to the automorphism

$$J^{-1} : \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & y & z \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{pmatrix} \qquad (z \in \mathbf{R})$$

of G, and the Fourier transform $\mathcal{F}:\mathcal{S}_{\mathbf{C}}(\mathbf{R})\longrightarrow\mathcal{S}_{\mathbf{C}}(\mathbf{R})$, where

$$\mathcal{F}\psi(t)=\int_{\mathbf{R}}e^{-2\pi its}\psi(s)\mathrm{d}s \qquad (t\in\mathbf{R}).$$

It is immediate that both $\bar{\mathcal{F}}$ and $\mathcal{F}$ admit unique extensions to $L^2_{\mathbf{C}}(\mathbf{R})$. The mappings J and J^{-1} are *entangled* elements of the metaplectic group and can be associated with the entangled symplectic matrices

$$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

and

$$\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},$$

respectively. It makes sense to denote them also by J and $J^{-1}=-J$, respectively.

The entangled symplectic matrices J and J^{-1} represent turns of 90° of opposite orientation in the affine Euclidean plane $\mathbf{R}\oplus\mathbf{R}$ (Figure 4). They suggest a complexification of the plane $\mathbf{R}\oplus\mathbf{R}$ in order to identify the matrices J and J^{-1} with i and $\bar{i}$ in $\mathbf{C}$, respectively. Because the Shannon sampling process reflects the symplectic structure of the Levi-Cività matrices, the CD-DA system is actually two-dimensional. This aspect does not only contribute to the high precision, the insight into this aspect also makes the understanding of the CD-DA system an excellent preparation of the clinical MRI modality which has enjoyed, along with the art of electronics, an explosive development in the last decades [3, 6, 18, 19].

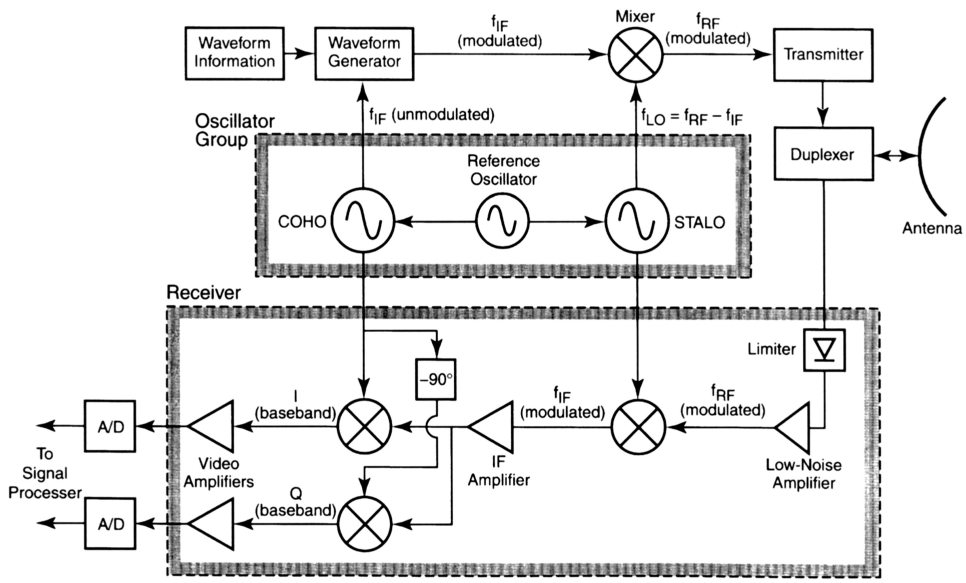

Figure 4. Simplified block diagram of a coherent radar system. The metaplectic group provides coherent signals via mixers which are indicated by the symbol $\otimes$. In the receiver, the phase delay of -90° implements the element of the metaplectic group associated to the matrix $-J$ (f_{IF} = intermediate frequency, f_{RF} = radio frequency, COHO = coherent oscillator, LO = local oscillator, STALO = stable local oscillator). The two output channels of the receiver I, Q feed A/D converters which are the most critical and costly components in a reliable audio digitization transducer

4. The Shannon time sampling process

Sampling means dividing a signal into evenly spaced discrete points in time and smoothing by linear convolution. If the two-dimensional lattice $\mathbf{Z}\oplus\mathbf{Z}$ is considered as a subgroup of $\mathbf{R}\oplus\mathbf{R}$, periodization of $\rho_1 \bmod \mathbf{Z}\oplus\mathbf{Z}$ transforms:

$$G \ni \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \rho_1\Big(\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}\Big)\psi(t) \in \mathbf{C} \qquad \big(\psi \in \mathcal{S}_{\mathbf{C}}(\mathbf{R})\big),$$

where $t \in \mathbf{R}$, into the *equivalent* mapping

$$G \ni \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \sum_{n\in\mathbf{Z}} e^{2\pi i(z+ny)}\psi(x+n) \in \mathbf{C}.$$

It is actually sufficient to assume that the function ψ is continuous on an interval of periodicity and to understand convergence in the distributional sense. The central projection $z=0$ provides the periodized filter bank operator, and subsequently an application of the mapping J under which the two-dimensional digitization lattice $\mathbf{Z}\oplus\mathbf{Z}$ is invariant provides the Poisson summation formula in its *symmetrized* form originally due to G. H. Hardy [20]:

$$e^{\pi ixy}\sum_{n\in\mathbf{Z}} e^{2\pi iny}\bar{\mathcal{F}}\psi(n+x) = e^{-\pi ixy}\sum_{n\in\mathbf{Z}} e^{2\pi inx}\psi(n-y) \qquad \big((x,y)\in\mathbf{R}\oplus\mathbf{R}\big).$$

The projection $y=0$ of the two-dimensional digitization lattice $\mathbf{Z}\oplus\mathbf{Z}$ yields the less symmetric Poisson summation identity [8]:

$$\sum_{n\in\mathbf{Z}} \bar{\mathcal{F}}\psi(n+x) = \sum_{n\in\mathbf{Z}} \psi(n)e^{2\pi inx} \qquad (x\in\mathbf{R}).$$

Suppose that $\bar{\mathcal{F}}\psi \in L^1_{\mathbf{C}}(\mathbf{R})$ holds. Since

$$\int_{\mathbf{R}} |\bar{\mathcal{F}}\psi(s)|\mathrm{d}s = \sum_{n\in\mathbf{Z}}\int_n^{n+1} |\bar{\mathcal{F}}\psi(x)|\mathrm{d}x = \sum_{n\in\mathbf{Z}}\int_0^1 |\bar{\mathcal{F}}\psi(n+x)|\mathrm{d}x$$

is finite, the series on the left hand side of the Poisson summation identity converges for almost all $x\in\mathbf{R}$ to a periodic integrable function. Invoking a mathematical principle first explicitly enunciated and systematically exploited by Erich Hecke: "A periodic function should always be expanded in a Fourier series", the k^{th} Fourier coefficient is given by the resonance identity:

$$\sum_{n\in\mathbf{Z}}\int_0^1 e^{-2\pi ikx}\bar{\mathcal{F}}\psi(n+x)\mathrm{d}x = \int_{\mathbf{R}} e^{-2\pi ikx}\bar{\mathcal{F}}\psi(x)\mathrm{d}x = \psi(k) \qquad (k\in\mathbf{Z}),$$

the term-by-term integration of the Laurent joined series being justified by the dominated convergence theorem. Hence the Laurent joined series on the right hand side of the Poisson summation identity is the Fourier series of the function on the left hand side. Multiplication by the character

$$s \rightsquigarrow e^{-2\pi ixs} \qquad (x\in\mathbf{R})$$

of the additive group $\mathbf{R}$, and integration over the symmetric unit interval $\left[-\frac{1}{2},+\frac{1}{2}\right]$ of low frequencies s gives:

$$\sum_{n\in\mathbf{Z}}\int_{-\frac{1}{2}}^{+\frac{1}{2}} e^{-2\pi ixs}\bar{\mathcal{F}}\psi(n+s)\mathrm{d}s = \sum_{n\in\mathbf{Z}}\psi(n)\int_{-\frac{1}{2}}^{+\frac{1}{2}} e^{2\pi i(n-x)s}\mathrm{d}s = \sum_{n\in\mathbf{Z}}\psi(n)\frac{\sin\pi(x-n)}{\pi(x-n)}$$

and so

$$\sum_{n\in\mathbf{Z}}\psi(n)\frac{\sin\pi(x-n)}{\pi(x-n)}=\int_{-\frac{1}{2}}^{+\frac{1}{2}}e^{-2\pi ixs}\bar{\mathcal{F}}\psi(s)\mathrm{d}s \qquad (x\in\mathbf{R}).$$

But

$$\psi(x)=\sum_{k\in\mathbf{Z}}\int_{k-\frac{1}{2}}^{k+\frac{1}{2}}e^{-2\pi ixs}\bar{\mathcal{F}}\psi(s)\mathrm{d}s=\int_{-\frac{1}{2}}^{+\frac{1}{2}}e^{-2\pi ixs}\bar{\mathcal{F}}\psi(s)\mathrm{d}s,$$

and so

$$\psi(x)-\sum_{n\in\mathbf{Z}}\psi(n)\frac{\sin\pi(x-n)}{\pi(x-n)}=\sum_{k\in\mathbf{Z}}\left(1-e^{2\pi ikx}\right)\int_{k-\frac{1}{2}}^{k+\frac{1}{2}}\bar{\mathcal{F}}\psi(s)\mathrm{d}s \qquad (x\in\mathbf{R}).$$

If ψ is a bandlimited function so that the symmetric *spectral* condition

$$\bar{\mathcal{F}}\psi(s)=0 \qquad (|s|\geq\frac{1}{2})$$

indicates the cut out procedure performed by the stop-band, the cardinal series representation of signal theory follows [21–23]:

$$\psi(x)=\sum_{n\in\mathbf{Z}}\psi(n)\frac{\sin\pi(x-n)}{\pi(x-n)}=\sum_{n\in\mathbf{Z}}\psi(n)\mathrm{sinc}(x-n) \qquad (x\in\mathbf{R}).$$

Claude Elwood Shannon himself did not prove the cardinal series representation; his derivation was purely heuristic and did not specify the sense of convergence [24]. However, "A Mathematical Theory of Communication", published by him in the same year 1948 when John Bardeen, William Bradford Shockley, and Walter Houser Brattain invented the transistor, formed a real landmark, because it contained work on information theory, including a measure of information and the capacity of a data transmission channel. In point of fact the sampling theorem had been known under the name of cardinal series at least since 1915.

To reproduce ψ from its bi-infinite sequence of samples $\big(\psi(n)\big)_{n\in\mathbf{Z}}$, the amplitude response function

$$\mathrm{sinc}\,x=\begin{cases}\dfrac{\sin\pi x}{\pi x} & \text{for } x\neq 0\\ 1 & \text{for } x=0\end{cases}$$

denotes the *sinus cardinalis* filter of spline theory. Due to the *normal* convergence of the series $\sum_{n\geq 1}\frac{z^2}{n^2}$ for $z\in\mathbf{C}$, the canonical product expansion of the window function

$$\mathrm{sinc}:\mathbf{C}\ni z\rightsquigarrow\prod_{n\geq 1}\left(1-\frac{z^2}{n^2}\right)$$

discovered by Euler in 1734 extends to an even entire holomorphic function of exponential type. Therefore the discrete time data sampling process of IT is closely allied to Carlson's theorem of complex variables which states that the trivial function is the only entire holomorphic function of exponential type $<\pi$ that vanishes at the set of integers (Section 5 below).

The cardinal series representation permits to spot the *multiresolution* flavor of IT [25]. Each sample value is multiplied by the appropriate sinc coefficient corresponding to its contribution to the overall impulse response of the filter. The products are summed to produce the output filtered sample. It thus digitally simulates the impulse response of an analog filter. The Shannon time sampling theorem dictates that the frequency content of the audio signal be less than or equal to the half-sampling frequency. The input signal may contain frequencies greater than the half-sampling frequency. A low-pass filter removes high frequencies to produce a spectrum of frequencies below the half-sampling frequency. Using the procedure repeatedly, the final approximation space obtains.

From this procedure another technique of multirate signal analysis called *oversampling* design becomes immediate. The oversampling filter bank is utilized in today's CD players in which additional sample values are computed by interpolating between original sample values on board a dual-channel linear-phase finite impulse response (FIR) digital filter chip. In view of the fact that additional samples have been generated, the sampling rate of the output signal is greater than the input signal. The spectrum of the signal is changed, with the images appearing at multiples of the oversampled sampling rate. Because the distance between the baseband and sidebands is larger, a gentle analog filter bank design can be used to remove the images without causing phase shift or other artifacts [5].

5. A reproducing kernel Hilbert space

Filtering is a fact of life for digital audio systems. An input anti-aliasing filter must precede the sampler to uphold the symmetric spectral condition for bandlimited and thus lossless sampling. Similarly, the output anti-imaging filter must filter out all frequencies above the half-sampling frequency.

The time sampling identity of the filter specified by the sinc function allows an extension to the Paley-Wiener space [26]. The entire holomorphic functions of exponential type at most π that are square integrable on the real axis forms a complex vector space $\mathcal{PW}(\mathbf{C})$. Under its natural scalar product, the complex Hilbert space $\mathcal{PW}(\mathbf{C})$ is isometrically isomorphic to the standard complex Hilbert space of low-pass filtering $L^2_{\mathbf{C}}\left(\left[-\frac{1}{2},+\frac{1}{2}\right]\right)$. Let the reflection

$$w \rightsquigarrow \bar{w}$$

denote the unique involutory automorphism of the field $\mathbf{C}$ different from $\mathrm{id}_{\mathbf{C}}$ with fixed point set $\mathbf{R}$, and therefore given by complex conjugation. The uniquely determined reproducing kernel ([17, 27, 28]) of the Paley-Wiener space $\mathcal{PW}(\mathbf{C})$ is defined by the holomorphic-antiholomorphic function of positive type

$$K:(z,w) \rightsquigarrow \mathrm{sinc}(z-\bar{w})$$

on the space $\mathbf{C}\times\mathbf{C}$. The function K reflects the global Frobenius reciprocity by incorporating production *and* reproduction simultaneously. As a consequence, the convolution representation or low-pass filter identity

$$\psi(z)=\int_{\mathbf{R}}\psi(s)\mathrm{sinc}(z-s)\mathrm{d}s \qquad (z\in\mathbf{C})$$

holds for every function $\psi\in\mathcal{PW}(\mathbf{C})$. From the convolution representation specified by the sinc filter it is immediate that the Paley-Wiener space $\mathcal{PW}(\mathbf{C})$ of reproducing kernel K is closed under the operation of differentiation.

6. Basic theta identities

A classical example of an application of the Poisson summation identity referred to above is the Jacobi identity for the theta null function:

$$\sum_{n\in\mathbf{Z}} e^{-\pi n^2\tau} = \frac{1}{\sqrt{\tau}} \sum_{n\in\mathbf{Z}} e^{-\frac{\pi n^2}{\tau}} \qquad (\tau > 0)$$

which has also the computational merits of convergence acceleration for small values of the parameter $\tau > 0$ [29]. As a matter of fact, one of the main benefits of the Poisson summation formula is the systematic supplying of analytic and arithmetic approximations.

Due to its quantum mechanical background, harmonic analysis on the real Heisenberg nilpotent Lie group G is, of course, *not* restricted to the derivation of the time sampling process of IT and the reproducing kernel K of $\mathcal{PW}(\mathbf{C})$ appears as a valuable by-product for signal retrieval. An analysis based on the *dual* stochastic aspects of quantum physics, [11, 12], and the Maslov index, however, leads via the longitudinal evolution operator associated to the Schrödinger equation or the method of path integration of the longitudinally driving Lévy stochastic process to the deep Landsberg-Schaar identity for quadratic Gaussian sums [30, 31]:

$$\frac{1}{\sqrt{p}} \sum_{0\le n\le p-1} e^{2\pi i \frac{n^2 q}{p}} = \frac{1}{\sqrt{2q}} e^{\frac{\pi i}{4}} \sum_{0\le n\le 2q-1} e^{-\pi i \frac{n^2 p}{2q}} \qquad (p > 0, q > 0),$$

valid for positive integers p and q. The standard proof of the Landsberg-Schaar identity is by putting

$$\tau = 2i\frac{q}{p} + \varepsilon \qquad (\varepsilon > 0)$$

and then letting $\varepsilon \to 0+$ in the Jacobi identity. This method invokes an example of another mathematical principle first explicitly enunciated and sytematically exploited by Hecke: "Exact knowledge of the behaviour of a holomorphic function in the neighbourhood of its singularities forms a source of arithmetic theorems". In view of the Lévy-Khinchin spectral trace formula [32, 33], the Landsberg-Schaar identity may be considered, however, as *dual* to the Jacobi identity for the theta null function. The geometry of the Lévy-Khinchin formula strikingly disproves that the one-dimensional unitary representations of G are "substantially uninteresting" [34, 35]. Actually these unitary characters of G are of substantial interest for the *detection* procedure because they represent the collapsed states of phase coherent quantum field theory [12, 36].

In elementary number theory the Landsberg-Schaar identity plays a central role underpinning key results relating to the law of quadratic reciprocity in terms of Legendre symbols:

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}} \qquad (p > 0, q > 0)$$

for *odd* integers p and q, and characters. Indeed, the concept of Heisenberg group G may serve "à une démonstration de la loi de réciprocité quadratique, apparentée à celle qui figure au dernier chapitre du livre classique de Hecke sur les corps de nombres algébriques" [8]. Shor's algorithm for quantum computing suggests a close interrelation between elementary number theory and quantum information, too.

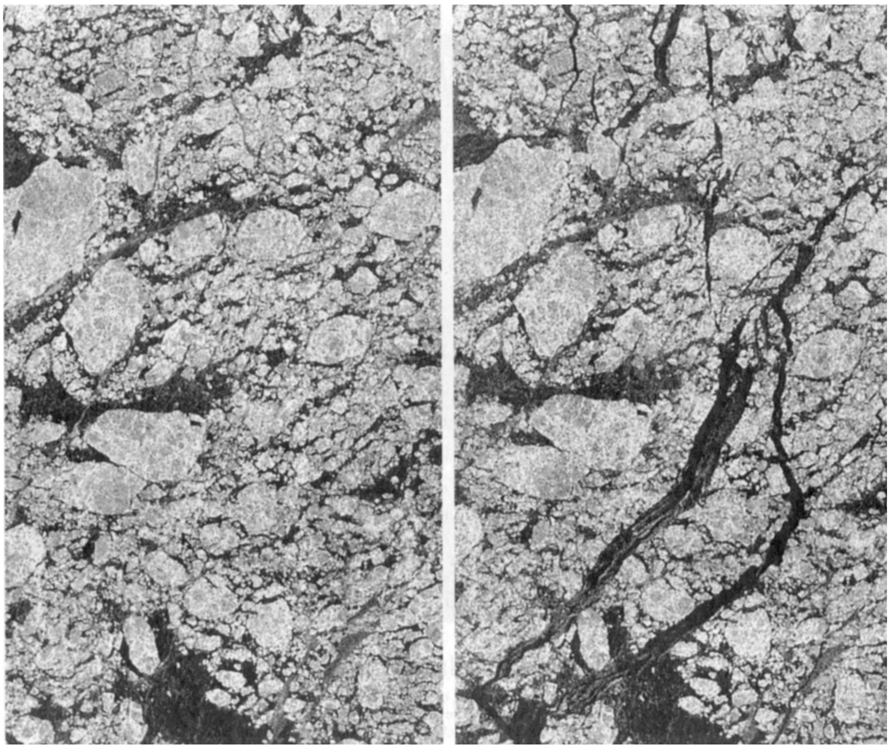

Figure 5. Radar imaging: The dynamics of pack ice in the Beaufort sea, located in the north of Canada and Alaska, has been observed and updated every three days by Radarsat. The channels, so called polynjas, have been created in the arctic ocean within nine days. Radarsat telemetry established that the lengths of some of the polynjas is up to 2000 kilometers

7. An outlook: phase coherent summation imagery

A central goal of signal processing is to describe real life signals by the concept of filter bank. Filter banks represent coherent arrangements of low-pass, band-pass, and high-pass filters used in IT for the spectral decomposition and synthesis of signals. Motivated by optical holography, they play an important role in modern signal processing applications because they easily allow the extraction of spectral components of a signal while providing very efficient implementations.

A symplectic extension of the summation formulas leads to filter banks which are at the holographic basis of clinical magnetic resonance tomography (MRI) and synthetic aperture radar (SAR) imaging (Figure 5). Both of these imaging modalities are based on the hologram idea in the radiofrequency and the microwave range, respectively [3]. In contrast to the cardinal series filter implemented by the CD-DA, the construction of a filter bank performs the recovery of the image. The output of the A/D converter installed in the CD-DA player is parallel data in which all 16 bits of the data word appear at once on 16 lines. Yet electronical storage devices permit storage only of serial string data in which the bits appear one after another. Data is therefore converted from parallel to serial format.

The symplectic extension to optical holography, however, allows the parallel processing modes of the SAR and MRI modalities by holographically implemented filter banks. The recovery of the image is then performed by an application of the symplectic Fourier transform to the modulation transfer function [1, 3]. The intrinsic symmetry of the symplectic Fourier transform allows to accelerate the image reconstruction process without severe degredation of the picture.

Clinical MRI which is based on intrinsic differences between normal and abnormal tissues, provides a multitude of image contrasts [3, 18, 19]. Due to this advantage of spin dynamics, MRI is the *non-invasive* imaging modality of choice in the majority of all cases of clinical diagnosis. Differences in longitudinal and transversal relaxation, spin density, macromolecular composition, diffusive motion, and bulk flow can be underscored by a variety of specifically designed pulse trains of suitable duration, orientation, and frequency. The subject of pulse train design is excitingly helpful and the clinical imaging results are cute (Figures 6 and 7).

Magnetic resonance angiography represents a category of non-invasive imaging techniques that seek to define the anatomy and morphology of vascular structures using the methods of clinical MRI. It consists of a spectrum of techniques using magnetic resonance pulse trains specifically devised to provide angiographic contrast, allowing depiction and characterization of blood vessels. Broadly speaking, these techniques

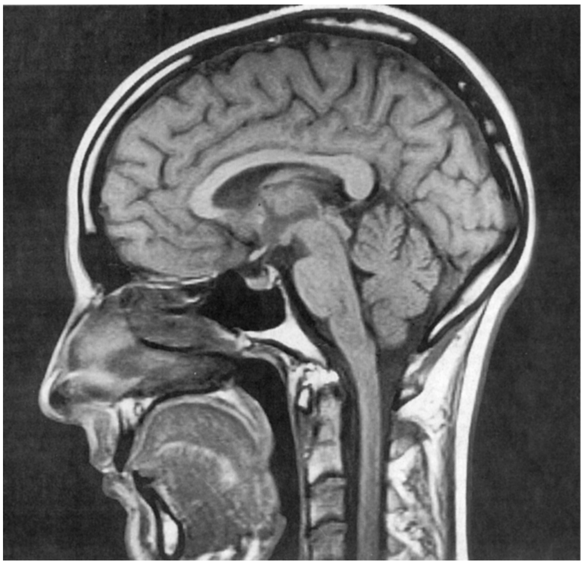

Figure 6. High resolution clinical magnetic resonance tomography: Sagittal cross-section of the neurocranium along the falx cerebri within the longitudinal interhemispheric fissure demonstrating midline sagittal neuroanatomy of the outwardly rounded gyri and inwardly invaginating fissures and sulci of the human brain. The various portions of the corpus callosum shown include the rostrum, genu, body and splenium, pineal gland, quadrigeminal plate, infundibulum, third and fourth ventricle, pituitary gland, cerebellar vermis, pons, aqueduct of Sylvius prepontine space, and craniocervical junction. High resolution MRI scans approximate the same level of detail as cut specimens to non-invasively depict neuroanatomy even in the deepest recesses of the brain. State-of-the-art magnetic resonance angiography for neuroimaging allows visualization of the blood flow without catheterization, without an external contrast agent, and with high resolution in all three dimensions. The ability to image intravascular space even for very slow blood flow velocities and to obtain adequate signal-to-noise ratio in a short data acquisition time is very appealing for the diagnostician

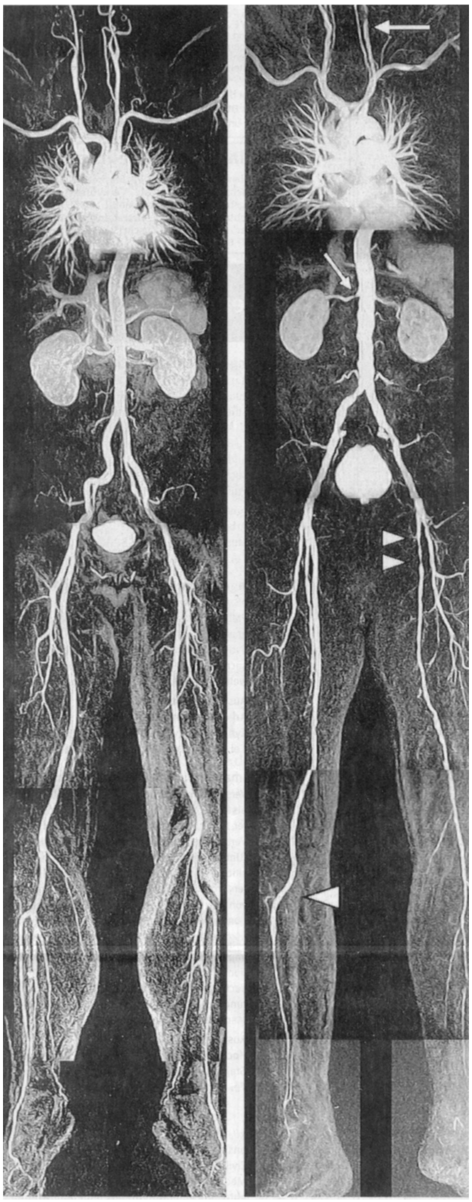

Figure 7. Angiography of the whole body performed by clinical magnetic resonance imaging within two minutes of measure time. The comparison of a healthy subject (left image) with the vessels of a 66 years old patient demonstrates various lesions (arrows) due to arteriosclerosis

may be divided into time-of-flight sequences, which take advantage of intravascular signal associated with inflow, phase-contrast pulse trains, which make use of the phase shifts associated with blood flow; black blood methods, which selectively saturate flow; and contrast-enhanced angiography, which relies on intravascular relaxation time shortening to provide blood vessel contrast.

Magnetic resonance angiography has undergone significant development over the past decade and is now widely used in many clinical applications. It has gone from being a novelty application of MRI with limited clinical use to replacing catheter angiography in some clinical application. Substantial technical advances in MRI scanner hardware including higher performance gradients with faster rise times, and greater maximum amplitudes have been very beneficial for magnetic resonance angiography when pushed to the limits of higher resolution and short data acquisition time. Improvement in pulse train design combined with improved coil design also have greatly improved the quality of magnetic resonance angiography possible today, even when compared to only a few years ago. Clinical MRI with magnetic field strengths > 1.5 T implied a significant improvement in the signal-to-noise ratio. Recently, high resolution magnetic resonance angiographic techniques have been developed that allow magnetic resonance angiography to replace conventional catheter angiography for preoperative evaluation of patients before endarterectomy.

8. Conclusion

Godfrey Harold Hardy (1877–1947), "the purest of the pure" mathematicians, thought that the existence of mathematics could only be justified as *art* if it could be justified at all [37, 38]. Although he did not specify his understanding of the metaphysics of art, he insisted on the fact that his own mathematical achievements have been based on pure thoughts and did *not* admit any real life application. However, the grand master's elegant symmetric version of the Poisson summation formula which has been put, independently, in its general group theoretical context by André Weil (1906–1998), projects from the two-dimensional digitization lattice $\mathbf{Z} \oplus \mathbf{Z}$ onto the Shannon time sampling theorem which, ironically enough, forms the base of the most successful consumer electronics products, the CD-DA, ever introduced. The CD-DA player forms the most sophisticated piece of audio electronics to reach the home. Because all CDs and players offer considerable advantage over other audio media, the audio CD has proved to be a technological *wunderkind* in the highly sophisticated and competitive field of music and data storage. Due to their versatility which has quickly become apparent to manufacturers and users alike, more than a billion audio CDs are sold every year. In the extremely storage-hungry market of IT, which forced the MD ATRAC 3 format to increase the maximum playing time of a conventional CD-DA from 74 minutes to 320 minutes, the annual worldwide demand for CD-DAs, CD-ROMs, DVIs, CD-Is, and DVD-ROMs is still rapidly climbing.

A central extension of the symplectic structure hidden by the *projected* summation formulae allows a powerful application to phase coherent summation imaging modalities of IT such as SAR and clinical MRI. Different from Hardy's and Weil's view of the Poisson summation formula, the *temporal* approach justifies the projection approach to the sampling processes and quantization modes of signal theory and opens a new perspective to the innovative field of IT. In this context, basic theta identities and the law of quadratic reciprocity are valuable by-products of harmonic analysis on the Heisenberg nilpotent Lie group G which reveals itself as the *universal* mathematico-temporal structure of multirate signal analysis governing the mathematically rigorous information theoretic approach to spinorial quantum physics [11].

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