THE IDENTIFICATION OF THE BOUNDARY GEOMETRY WITH CORNER POINTS IN INVERSE TWO-DIMENSIONAL POTENTIAL PROBLEMS

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(Received 28 May 2002; revised manuscript received 16 August 2002)

Abstract: The paper presents fragment of a larger study concerning the effective methods of solving the inverse boundary value problems. The boundary value problem described here is formulated as a problem of the identification of a boundary geometry with corner points. A method using a parametric integral equations system (PIES) is proposed. PIES used in the method makes the easy modelling of the geometry with corner points possible. This effect is obtained by the application of modified splines. An evolution algorithm is used for the effective control of modifications of the boundary geometry. Some experimental tests of the efficiency of the discussed method were performed for two-dimensional inverse potential problems.

Keywords: inverse boundary value problem, identification of boundary geometry, parametric integral equations system, finite element method, boundary element method, evolution algorithm, genetic algorithm

1. Introduction

The boundary value problems are frequently encountered in many areas of science and technology. Apart from the considerable practical importance, these problems are of high diversity. The following boundary value problems classification is used in the paper:

1. Forward boundary value problems – these problems consist in searching the solution of a differential or integral equation in a given domain and for given boundary conditions. The finite element method (FEM) [1] and the boundary element method (BEM) [2] are used most frequently in solving such problems.
2. Inverse boundary value problems [3, 4], usually depending on the searched quantity, refer to:
   - material constants identification,
   - boundary conditions identification,
   - boundary geometry identification.

Within the scope of this paper the inverse boundary value problem is formulated as a boundary geometry identification problem. This problem consists in searching such a shape of the domain boundary for which the desirable solution of the forward boundary value problem will be obtained. The most important thing in such problems is a way in which the boundary geometry is modelled. In [5, 6] the identification of the boundary with a smooth geometry (belonging to the continuity class C^2) is performed. In practice, however, the boundary geometry with corner points occurs very often. These points are inconvenient from the boundary integral equations application point of view. Most frequently, singularities resulting from the existence of ambiguous normal vectors occur in these points.

The goal of this paper is to propose an identification method of the boundary geometry modelled by modified splines called ν-splines. Depending on the choice of the values of the parameters, these curves enable modelling the boundary geometry with the continuity class C^2 or the reduction of the continuity class in a selected point of the geometry. The proposed method consists of two fundamental elements. The first component is a parametric integral equations system (PIES). The geometry modelled by ν-splines is taken into consideration in kernels of PIES. This fact makes the boundary geometry easy to model. The second part of the proposed method, an evolution algorithm, is used in order to control effectively geometry modifications during the identification process. Some experimental tests for the efficiency of the proposed method were performed on the ground of the identification of the boundary geometry with corner points in two-dimensional inverse potential problems.

2. Problem formulation

In this paper the inverse boundary value problem is formulated as a problem of the reconstruction of a boundary geometry with corner points (in Figure 1 points: V_0^*, V_1^*, V_2^*). The values of the solution of the forward boundary value problem (u_i^*) in some specific points of the domain are treated as given. These values will be called model values. Also Neumann $p(s)$ and Dirichlet $u(s)$ boundary conditions are given. The boundary geometry or its part (in the figure marked by the dashed line) is searched in this problem. The geometry being searched should meet the following condition: the values of the solution of the forward boundary value problem for this geometry should be equal to the model values. Therefore the identification algorithm may consist of three steps repeated serially:

- solving the forward boundary value problem with some found geometry,
- the comparison of the obtained solutions with the model solutions,
- the modification of the found geometry in order to improve it.

Using such an approach causes two fundamental problems. Solving these problems has a large influence on the efficiency of the applied method. The first problem concerns the manner of the boundary geometry modelling. On the one hand it should guarantee easy modifications of geometry, whereas on the other hand corner points should be respected. The corner points are singular, because there are ambiguous normal vectors in these points. The manner of the boundary geometry modelling is closely connected with the choice of the forward boundary value problem solving method. FEM or BEM may be applied here but both methods require the discretization of the considered domain or its boundary. In such a case, after every modification of the boundary geometry the discretization process has to be restarted. That is why both FEM and BEM applied to the repeated process of solving the forward boundary value problem turn out to be ineffective. In [5, 6] PIES is applied instead of FEM and BEM. It does not require either the boundary or the domain discretization, and the boundary geometry is taken into account in the kernels of PIES. PIES enables modelling the boundary by means of B-splines or Bézier curves. The boundary with a sufficiently smooth geometry is modelled by these curves very effectively, because they easily guarantee the preservation of the continuity class C^1 and C^2 in every point of the boundary, in the binding points of segments as well. But, in the considered problem, the boundary geometry includes corner points where the required continuity class is not preserved.

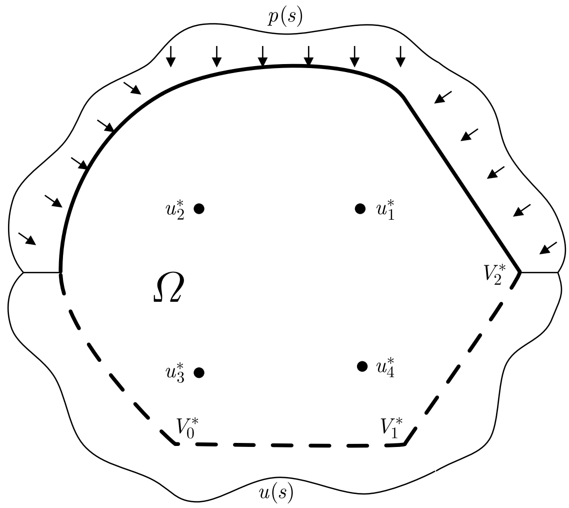

Figure 1. The identification of the boundary geometry with corner points

The complexity of the solving procedure of the forward boundary value problem is not the only element having an influence on the total computational complexity of the discussed approach. The total complexity depends on the number of calls of that procedure during the identification process and the number of calls depends on the efficiency of an algorithm controlling boundary shape modifications. Therefore, the second fundamental problem of the discussed approach application is the choice of the algorithm designed for the control of the identification process.

3. Identification method

The method arising from the combination of PIES and the evolution algorithm is applied to the problem of the identification of the boundary geometry with corner points. PIES is assigned to solve the forward boundary value problems. For the two-dimensional potential problem the equations system assumes the following form [7]:

$$0.5u_l(s_1) = \sum_{j=1}^{n} \int_{s_j-1}^{s_j} \left\{ \bar{U}_{lj}^*(s_1,s)p_j(s) - \bar{P}_{lj}^*(s_1,s)u_j(s) \right\} J_j(s)ds, \quad s_{j-1} > s, s_1 < s_j. \quad (1)$$

That equations system has the kernels $\bar{U}^*_{lj}$ and $\bar{P}^*_{lj}$ described by the following formulas:

$$\bar{U}^*_{lj}(s_1,s)=\frac{1}{2\pi}\ln\frac{1}{[\eta_1^2+\eta_2^2]^{0.5}},\qquad \bar{P}^*_{lj}(s_1,s)=\frac{1}{2\pi}\frac{\eta_1 n_1^{(j)}(s)+\eta_2 n_2^{(j)}(s)}{\eta_1^2+\eta_2^2}. \tag{2}$$

The boundary geometry is taken into account in the kernels of PIES. The geometries considered in the paper include the corner points. That is why ν-splines [8] are assigned to the geometry modelling. The geometry segments P_i described by these curves are taken into account in the kernels of PIES by the following dependence:

$$\eta_1=P_l^{(1)}(s_1)-P_j^{(1)}(s),\qquad \eta_2=P_l^{(2)}(s_1)-P_j^{(2)}(s), \tag{3}$$

where $P_i=\begin{cases}P_i^{(1)}\\ P_i^{(2)}\end{cases}$. The individual boundary segments $P_i(s)$ are given by means of the formula:

$$P_i(s)=h_{00}(s^*)P_{i-1}+h_{01}(s^*)P_i+h_{10}(s^*)d_iP'_{i-1}+h_{11}(s^*)d_iP'_i, \tag{4}$$

and the tangent vectors $P'_i(s)$ are calculated from the conditions of the continuity:

$$P''_{i+1}(s_i)=P''_i(s_i)+\nu_iP'_i(s_i),\qquad P'_{i+1}(s_i)=P'_i(s_i). \tag{5}$$

The shape of ν-splines depends on two parameters: firstly on the location of the so-called boundary points V_i. These points split the geometry into the segments P_i. The modifications of the boundary geometry are made by boundary points shifts (Figure 2).

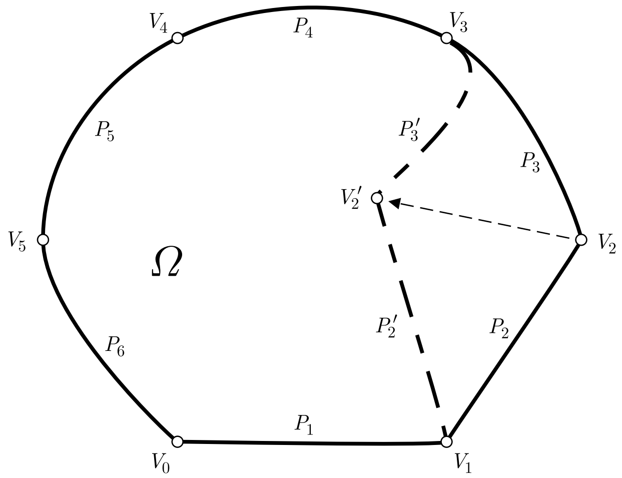

Figure 2. The influence of the location of the boundary points on the boundary geometry

The second parameter which influences the shape of the curve is the coefficient ν (formula (5)) called curvature coefficient. This coefficient enables modelling of corner points. Depending on the value of the curvature coefficient in the binding points of segments, ν-splines are similar to the conventional splines (the continuity class C^2 for $\nu=0$) or broken lines (the continuity class C^0 for $\nu\to\infty$). Values of $\nu>0$ enable to obtain shapes between broken lines and the conventional splines (Figure 3). Therefore ν-splines make it possible to receive the curves with the continuity class

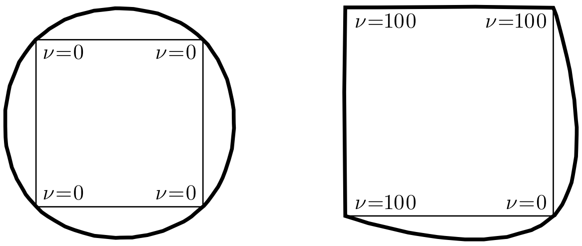

Figure 3. The influence of the coefficient ν on the boundary geometry

C^2, or to reduce the continuity class in a chosen point V_i. Thanks to that, ν-splines satisfactorily approximate the geometry with corner points (visually it seems that the boundary geometry has the continuity class C^0 in these points), on the other hand these curves guarantee the unique normal vector in every point of the boundary.

The identification of the geometry modelled by ν-splines comes down to searching the co-ordinates of the boundary points and the values of the curvature coefficients in these points. During that search the geometry is modified and the forward boundary value problems are solved repeatedly. Modifications of the boundary shape are made by the displacement of the boundary points and the change of the value of the coefficient ν_i. The second component of the method – the evolution algorithm – is responsible for the control of the geometry modifications.

The principle of the operation of the evolution algorithm [9, 10] is based on the evolution processes occurring in the natural environment and it is presented as Algorithm (see below). The population of individuals, selected from the set of potential solutions of the considered problem, is processed by the evolution algorithm. In the first step of the algorithm, the population is created randomly. That random subset of solutions is called initial population. The population consists of a given number of individuals. Each of them represents the single solution of the problem in the form of the string of genes and is assessed on the basis of the value of so-called fitness function. The definition of the fitness function is the condition of the correct run of the algorithm.

The next step of the evolution algorithm is a selection and it is performed on the basis of the values of the fitness function. The selection is a genetic operator taking advantage of the principle of the survival of the best adapted individuals. This operator consists in the random choice of solutions from the population, but with the survival probability depending on the value of the fitness function. Therefore, the best adapted individuals have a greater chance to insert one or more descendants into the next generation and the worst adapted are eliminated.

After the selection, the next operator processing the population of the solutions is a cross-over operator. The cross-over guarantees the exchange of the genetic material between the individuals from the parental pool in order to generate new, maybe better adapted, solutions. Initially, the crossed individuals are selected from the population randomly. Then the individuals are joined in pairs and the genetic material is exchanged.

A mutation is another genetic operator and consists in the accidental substitution of the value of a single gene. The mutation occurs with low probability. This operator prevents the irreparable loss of a potentially essential genetic material. This material may be eliminated from the population as the result of the selection and cross-over operations.

In the considered problem each individual in the population describes a single boundary geometry. The computation of the value of the fitness function for the geometry described by a certain individual X requires the solution of the forward boundary value problem. The fitness function applied in the paper is described by the following formula:

$$f(X)=\sqrt{\frac{\sum_{i=1}^{m}(u_i^*-u_i^X)^2}{m}}\ , \tag{6}$$

where u_i^* – values of the model solution of the forward boundary value problem, u_i^X – values of the solution of the forward boundary value problem for a found geometry X, m – the number of points, where the values of the solution of the forward boundary value problem are calculated.

Furthermore, the ranking selection, simple cross-over, arithmetical cross-over, uniform mutation and non-uniform mutation operators [9, 10] are applied in the evolution algorithm used in the described method.

Algorithm

```
begin
    t := 0; { t – the number of the generation }
    Create the initial population P(t) randomly;
    Calculate the fitness of P(t);
    repeat
        t := t+1;
        Select individuals from P(t−1) depending on their fitness; { selection }
        Recombination of P(t); { cross-over and mutation }
        Calculate the fitness of P(t);
    until stop condition;
end.
```

4. Experimental results

The experimental study of the efficiency of the proposed method was performed for two kinds of tests. In both cases it is assumed that the values of the solution of the forward boundary value problem for the given domain and given boundary conditions are known in some specific points of the domain. These values are obtained after solving the forward boundary value problem for the given boundary geometry with corner points (further called model geometry) by using PIES. Then, the values obtained in this manner are treated as given and the boundary geometry is treated as unknown and it is identified. In the first case the identification is reduced to the determination of the values of the curvature coefficients in the binding points of segments. In the second case the coefficients ν_i as well as the boundary points coordinates are identified.

The chosen examples of the tests belonging to the first group are presented in Figures 4 and 5. Figures 4a and 5a illustrate the model geometries. Then, the

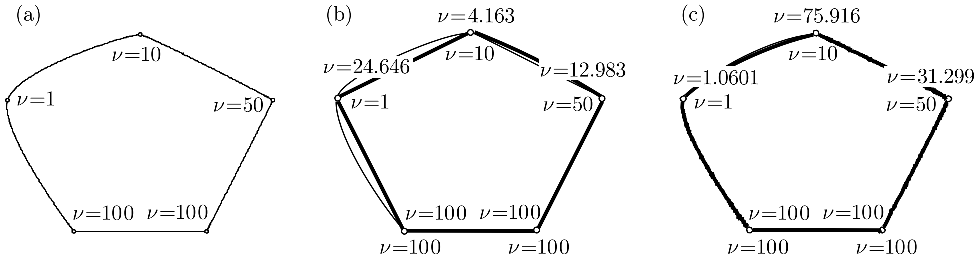

Figure 4. The identification of coefficients ν in 3 boundary points

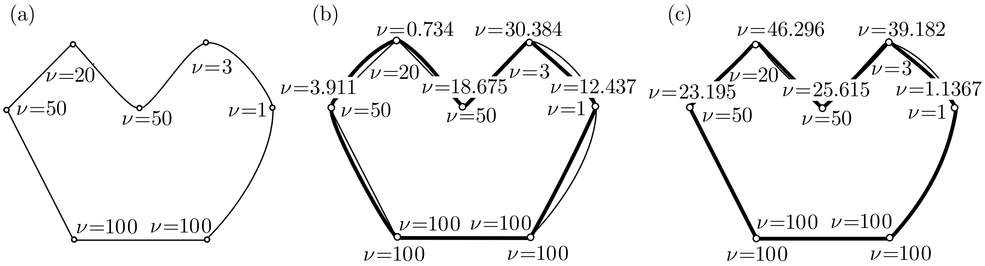

Figure 5. The Identification of coefficients ν in 5 boundary points

individual curvature coefficients are treated as unknown and they are searched for in interval $\langle 0,100 \rangle$. The examples of the geometries from the initial populations, that is the geometries with the coefficient ν values determined randomly, are presented in Figures 4b and 5b. These illustrations show that the geometries generated in such a manner are similar to the model geometries. That is why problems belonging to the first group of the tests turn out to be relatively easy. After a few generations of the evolution algorithm, the satisfied identification results presented in Figures 4c and 5c are obtained.

The second group of tests concerns the identification of the geometry reduced not only to the determination of the values of the curvature coefficients, but also the location of the boundary points. The coefficients ν_i are searched for in the interval $\langle 0,100 \rangle$, and the boundary points are identified in the given rectangular search spaces. Figure 6 presents the most elementary example of the identification of the location of one boundary point and the determination of the value of the coefficient ν in this point. Figures 7 and 8 concern the cases of the identification of 3 and 5 boundary points. Three chosen examples of the model geometry are shown in Figures 6a, 7a and 8a. Illustrations 6b and 6c, 7b and 7c, 8b and 8c present the geometries from the initial populations. The next phases of the identification are illustrated by Figures 6d and 6e, 7d and 7e, 8d and 8e. Figures 6f, 7f and 8f show the final results of the identification.

The group of problems formulated in this manner is more difficult than the previous one. First of all, in the problems from the second group more parameters are being identified, leading to a bigger search space. Therefore, there is more diversity of the geometries created as a result of the random generation of the initial population. But, as enclosed pictures show, the results of the identification are satisfying in all cases. It is necessary to affirm that the accuracy of the obtained results decreases along with the increase of the number of the identified boundary points. In the boundary

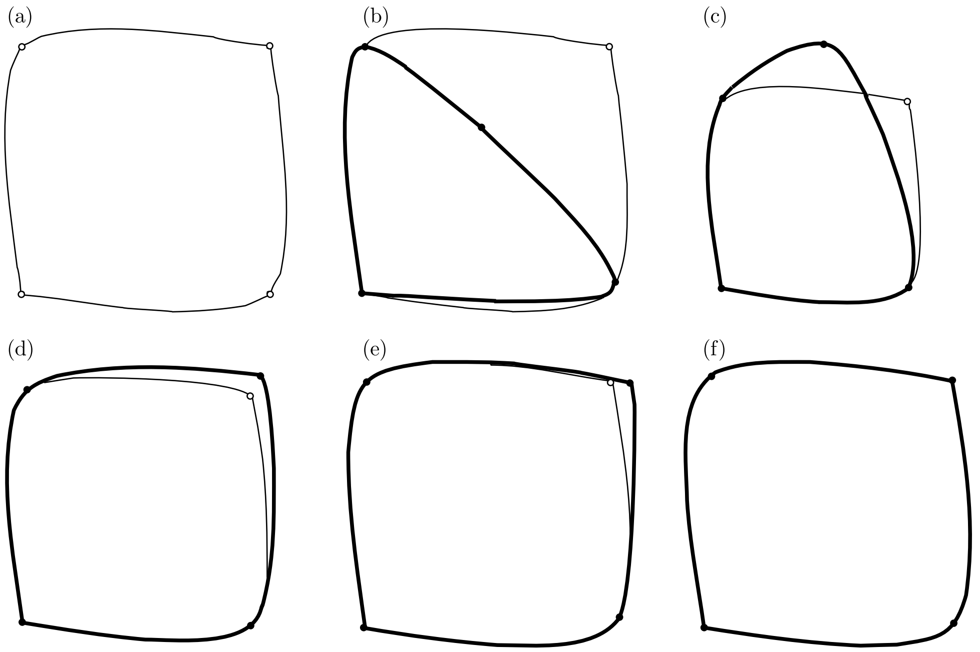

Figure 6. The identification of the location of 1 boundary point and the coefficient ν in this point

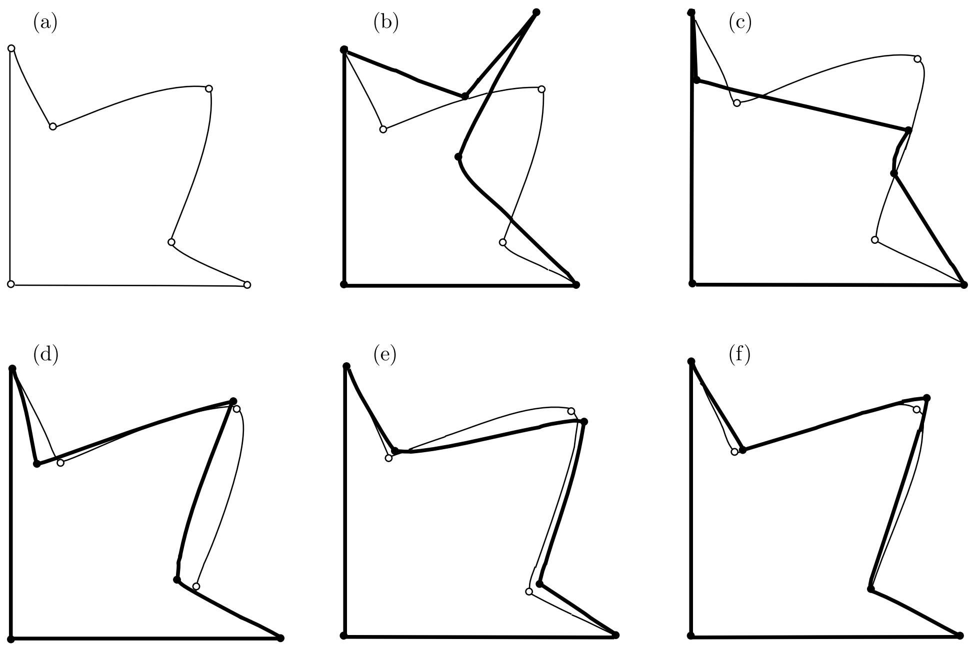

Figure 7. The identification of the location of 3 boundary points and the coefficients ν in these points

points identification case (for values of ν_i a lot more than zero) the values of the curvature coefficient, even much different than the model values, give a satisfactory approximation of the model geometry.

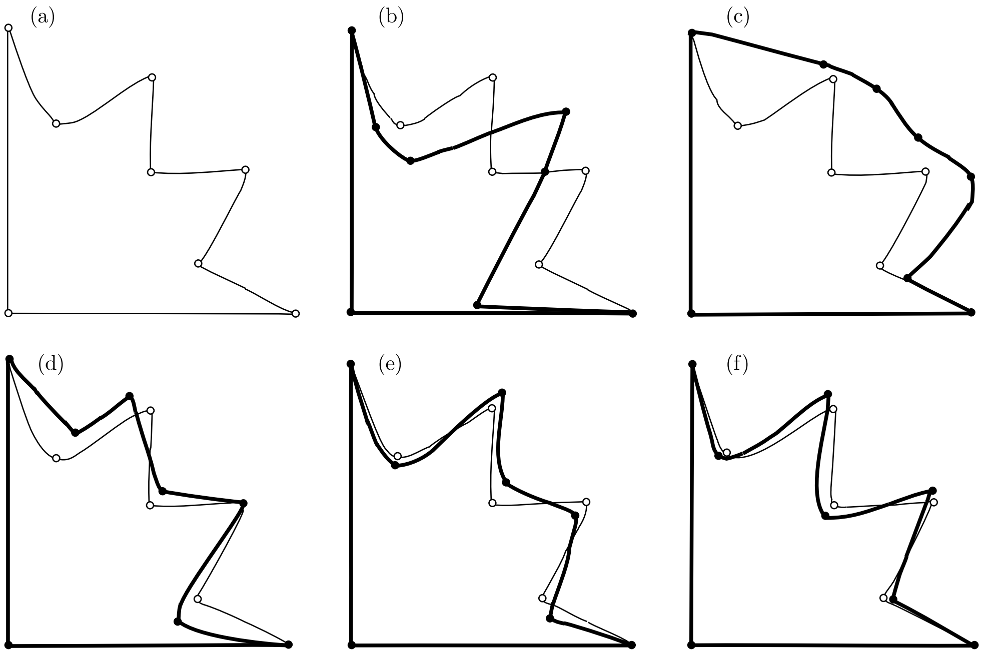

Figure 8. The identification of the location of 5 boundary points and the coefficient ν in these points

5. Conclusions

On the basis of the performed study one may say that modelling the boundary geometry with corner points without the singularities in the form of the ambiguous normal vector is possible thanks to the application of ν-splines. These curves enable modifications of the geometry amounting to the change of the location of few boundary points and the change of coefficients ν_i values in these points. Taking into consideration the boundary geometry modelled in this manner in the kernels of PIES, solving the forward boundary value problem without the necessity of the discretization of the domain or its boundary becomes possible. It is of great importance in case of solving the inverse boundary value problem, where the repeated modifications of the geometry are necessary. Therefore, the application of PIES results in the decrease of the computational complexity of the method. The efficiency of the identification method of the boundary geometry with corner points is also obtained through the evolution algorithm application to the control of the geometry modifications during the identification process. There is a potential possibility of the improvement of the control efficiency by the application of more efficient specialised genetic operators.

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