ν-SPLINE CURVES FOR BOUNDARY GEOMETRY MODELING IN TWO-DIMENSIONAL POTENTIAL BOUNDARY VALUE PROBLEMS WITH SINGULAR CORNER POINTS

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Abstract: The paper presents a new modeling method of boundary geometry in boundary value-problems by ν-spline curves. To define a smooth boundary geometry both Bézier and B-spline curves are applied. At the segment join points Bézier curves ensure continuity C^1, and B-spline curves allow us to maintain C^2 continuity. However, the curves hinder boundary modeling with corner points. In order to weaken the continuity at segment join points ν-spline curves are proposed. These curves are combined analytically with the Green formula, thus yielding the Parametric Integral Equation System (PIES). To solve the PIES a pseudospectral method is used. The results obtained for the domains with singular corner points are compared with the corresponding non-singular ones as defined by the ν-spline curves.

Keywords: parametric integral equation system, boundary integral equation, potential problem, ν-spline

1. Introduction

The solution of boundary value problems is reduced to searching for functions that meet the criteria of integral and differential equations in a given domain and under the assigned boundary conditions. Mathematical formalism of integral and differential equations does not take into account the domain geometry, therefore, its consideration while solving the equation is the greatest problem. Due to the above, the Finite Element Method (FEM) [1] and Boundary Element Methods (BEM) [2] are frequently applied to solve complicated boundary value problems. These methods allow modeling of any domain in the discrete manner only.

In our papers [3–5] we searched for such an integral equation that would take into account the domain geometry in its mathematical formalism. As a result of the modification of the classical boundary integral equations by Fourier's transformation a new Parametric Integral Equation Systems (PIES) was obtained. In contrast to the classical Boundary Integral Equations (BIE) the above system takes into account the boundary geometry in the kernels obtained from the PIES. To define a boundary geometry in the boundary value problems linear segments [3], Bézier as well as B-spline curves [5] were applied. These curves provide a continuity of C^1 and C^2 classes at the segment join points. Hence, the application of these curves is very useful only in the case of a smooth boundary geometry.

In many practical boundary value problems the boundary geometry may have corner points where the required class of continuity is not maintained. Such points are singular because they have unequivocal normal vectors at the points and they are troublesome as regards the possibility of applying classical BIE. Instead one should use the modified curves, called ν-spline curves [6], since they enable boundary modeling with singular points, maintaining at the same time the unequivocal nature of normal vector at these points. ν-spline curves are more general as compared with traditional B-spline and Bézier curves. They provide a more effective modeling of boundary geometry with or without singular points.

The aim of this paper is the application of ν-spline curves for the modeling of boundary geometry in the classical BIE. As a result of the combination of ν-spline curves and BIE a new PIES was obtained, which may be used to solve the boundary value problems with corner points. A pseudospectral method was proposed to solve the PIES numerically. A possible application of ν-spline curves is illustrated by the example presented in this paper.

2. Modeling of singular boundary geometry with ν-spline curves

The curve interpolating the boundary geometry composed of cubic segments $P_i(s)$, $i=1,2,\ldots,n$ may be created under the assumption that there are points on the boundary P_i $(i=0,1,\ldots,n)$, through which the interpolating curve must pass, and also that there exist vectors of the derivatives P_i' which are tangential to the curve at these points. The segments defined by two points and vectors tangential to them are called Hermite segments [6]. If the distance between points P_i $(i=0,1,\ldots,n)$ given on the boundary differs considerably, then the easiest way is to apply parameterization by chord $d_i=|P_i-P_{i-1}|$, $i=1,\ldots,n$ where the distance between the given points corresponds to the increments of parameter s. Particular boundary segments $P_i(s)$ may be presented in a vector form by formula [6]:

$$P_i(s)=h_{00}(s^*)P_{i-1}+h_{01}(s^*)P_i+h_{10}(s^*)d_iP_{i-1}'+h_{11}(s^*)d_iP_i', \tag{1}$$

where $s^*=(s-s_{i-1})/d_i$, $0.0\le s^*\le 1.0$, $s_{i-1}\le s\le s_i$, $s_0=0.0$, $s_i=s_{i-1}+d_i$.

The coefficients $h_{00}(s)$, $h_{01}(s)$, $h_{10}(s)$, $h_{11}(s)$ with parameter s defined for $0\le s\le 1.0$ are Hermite [5, 6] parametric basis functions. In practice, the vectors of derivatives at given points are frequently unknown. These vectors are determined on the basis of the known points and additional conditions.

The ν-spline curves are obtained as a result of satisfying the condition of C^2 continuity at the segment join points. Since the geometric properties of the curve are essential, the continuity condition may be weakened in such a way that the curve be of C^2 class in the vicinity of the join points, and that it should have the continuity of a curve vector at those points. The C^2 continuity condition of the curve at the join points of the two segments has the following form [6]:

$$P''_{i+1}(s_i)=\beta_1^2 P''_i(s_i)+\beta_2 P'_i(s_i), \qquad P'_{i+1}(s_i)=\beta_1 P'_i(s_i); \qquad \beta_1>0.0. \tag{2}$$

Additionally, if we assume that the curve also has the continuity of the derivative vector ($\beta_1=1.0$), then we have only one coefficient $\nu=\beta_2$. This coefficient may have different values at different join points. The interpolating curves composed of such segments are called ν-spline curves. If the coefficient $\nu_i=0$ for $i=0,\ldots,n-1$ then it follows from Equation (2) that the resulting curves is a traditional spline curve. If, however $v_i\to\infty$ for $i=0,\ldots,n-1$, then the curve tends to assume a broken form connecting the points given. For value $\nu_i>0$ we obtain curves of intermediate forms between broken and spline curves. The examples of curves for various coefficients ν_i are shown in Figure 1.

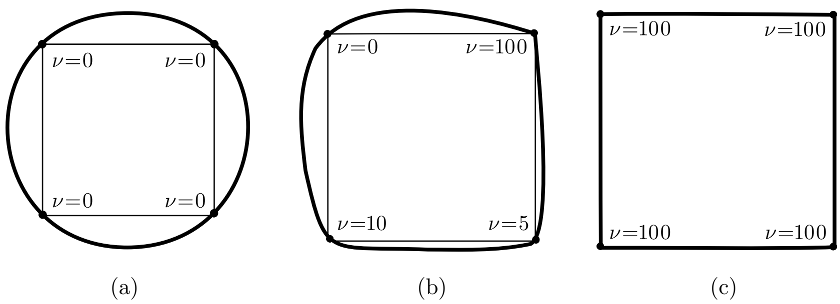

Figure 1. Closed ν-spline curves for various values of coefficients ν_i:
(a) for coefficients $\nu=0$ curve without corner points,
(b) effect of different coefficient values ν on curve form,
(c) broken curve with corner points but without singular points

The ν-spline curves may be determined using Equation (1) after previous determination of the tangential vectors P'_i. We determine these vectors on the basis of continuity condition (2). After taking second derivatives into account, we obtain the following relationship:

$$d_{i+1}P'_{i-1}+\{2(d_i+d_{i+1})+0.5\nu_i d_i d_{i+1}\}P'_i+d_iP'_{i+1}=F_i, \tag{3}$$
$$2P'_0(d_1+d_n)+P'_1d_n+P'_{n-1}d_i=F_0,$$

where

$$F_0=\left\{3\left[d_n^2(P_1-P_0)+d_1^2(P_0-P_{n-1})\right]\right\}/d_1d_n,$$
$$F_i=3d_{i+1}(P_i-P_{i-1})/d_i+3d_i(P_{i+1}-P_i)/d_{i+1}, \qquad i=1,\ldots,n-1.$$

Expression (3) is a system of algebraic equations in respect of the first derivative vectors. Generally, this system may be represented in the following form:

$$[A]\{P'_i\}=\{F_i\}, \qquad i=0,\ldots,n-1. \tag{4}$$

It is an equation system with a cyclic matrix that is strongly and diagonally dominating and having a single solution. To solve it, we may use the Gauss' elimination method.

To define boundary geometry by means of segments (1) the solution of algebraic equation system is required (4). After solving the above system we obtain the indispensable vectors (first derivatives) at segment join points.

3. Analytical compilation of the Green formula and ν-spline curves

The boundary integral identity may be presented by means of a general formula (Green's formula) in the following form [3–5]:

$$\bar{u}(\boldsymbol{x}) = \int_{\Gamma} U^*(\boldsymbol{x},\boldsymbol{y})p(\boldsymbol{y})d\Gamma(\boldsymbol{y}) - \int_{\Gamma} P^*(\boldsymbol{x},\boldsymbol{y})u(\boldsymbol{y})d\Gamma(\boldsymbol{y}), \tag{5}$$

where $\bar{u}(\boldsymbol{x}) = \begin{cases} u(\boldsymbol{x}) & \boldsymbol{x}\in\Omega \\ 0.5u(\boldsymbol{x}) \quad \text{for} & \boldsymbol{x}\in\Gamma \\ 0 & \boldsymbol{x}\notin\bar{\Omega} \end{cases}$, $p(\boldsymbol{y}) \equiv \frac{\partial u(\boldsymbol{y})}{\partial n(\boldsymbol{y})}$ and $P^*(\boldsymbol{x},\boldsymbol{y}) \equiv \frac{\partial U^*(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{n}(\boldsymbol{y})}$.

If $\boldsymbol{x}\in\Gamma$ then the Green's formula (5) is the boundary integral equation (BIE). In the identity (5), an integrand $U^*(\boldsymbol{x},\boldsymbol{y})$ is the classical fundamental solution, whereas $P^*(\boldsymbol{x},\boldsymbol{y})$ is the classical singular solution.

In order to modify the Green's formula the Fourier transform is applied and after its application to Equation (5) we obtain the following transform [3–5]:

$$\hat{u}(\boldsymbol{\xi}) = \Delta^{-1}(\boldsymbol{\xi})\{\tilde{p}(\boldsymbol{\xi}) + i[\xi_1\tilde{u}\tilde{n}_1(\boldsymbol{\xi}) + \xi_2\tilde{u}\tilde{n}_2(\boldsymbol{\xi})]\}, \qquad \boldsymbol{\xi}\equiv(\xi_1,\xi_2), \tag{6}$$

where $\Delta^{-1}(\boldsymbol{\xi}) = [\xi_1^2+\xi_2^2]^{-1}$.

In formula (6) the boundary is defined by means of the following boundary integrals:

$$\tilde{p}(\boldsymbol{\xi}) = \int_{\Gamma} e^{-i(\xi_1 y_1+\xi_2 y_2)}p(\boldsymbol{y})d\Gamma(\boldsymbol{y}), \tag{7}$$

$$\tilde{u}\tilde{n}_m(\boldsymbol{\xi}) = \int_{\Gamma} e^{-i(\xi_1 y_1+\xi_2 y_2)}n_m(\boldsymbol{y})u(\boldsymbol{y})d\Gamma(\boldsymbol{y}), \quad m=1,2 \quad \boldsymbol{y}\in\Gamma, \tag{8}$$

where n_m is a directional cosine of the normal vector to the boundary Γ.

We use integral (8) to define the function transform $\tilde{u}\tilde{n}_m(\boldsymbol{\xi})$ on the boundary Γ. The unknown integrand $u(\boldsymbol{y})$ in Equation (8) may be defined by means of the following Fourier formula:

$$u(\boldsymbol{y}) = \frac{1}{4\pi^2}\int_{R^2} e^{i(\omega_1 y_1+\omega_2 y_2)}\hat{u}(\boldsymbol{\omega})d\boldsymbol{\omega}, \qquad \boldsymbol{\omega}\equiv(\omega_1,\omega_2), \tag{9}$$

where the integrand $\hat{u}(\boldsymbol{\omega})$ is given by:

$$\hat{u}(\boldsymbol{\omega}) = 2\Delta^{-1}(\boldsymbol{\omega})\{\tilde{p}(\boldsymbol{\omega}) + i[\omega_1\tilde{u}\tilde{n}_1(\boldsymbol{\omega}) + \omega_2\tilde{u}\tilde{n}_2(\boldsymbol{\omega})]\}. \tag{10}$$

Formula (10) is a particular case of transform (6).

3.1. Transform of the integral equation system

After substituting formula (10) into Equation (9), and next the resulting expression into Equation (8) we get the *convolution integral equation* in the domain of Fourier transforms:

$$\tilde{u}\tilde{n}_m(\boldsymbol{\xi}) = \int_{R^2} \tilde{K}_m(\gamma_1,\gamma_2)\Delta^{-1}(\boldsymbol{\omega})\{\tilde{p}(\boldsymbol{\omega}) + i[\omega_1\tilde{u}\tilde{n}_1(\boldsymbol{\omega}) + \omega_2\tilde{u}\tilde{n}_2(\boldsymbol{\omega})]\}d\boldsymbol{\omega}, \tag{11}$$

where the kernel is

$$\tilde{K}_m(\gamma_1,\gamma_2) = \frac{1}{2\pi^2}\int_\Gamma e^{i(\gamma_1 y_1+\gamma_2 y_2)} n_m(\boldsymbol{y})d\Gamma(\boldsymbol{y}), \qquad \gamma_i \equiv \omega_i - \xi. \tag{12}$$

In our further consideration we divide the boundary Γ into n non-linear segments. After taking the segment representation of the boundary into account, Equation (11) has the following form:

$$\tilde{u}_l\tilde{n}_m^{(l)}(\boldsymbol{\xi}) = \int_{R^2} \bar{\tilde{K}}_m(\gamma_1,\gamma_2)\sum_{j=1}^{n}\Delta^{-1}(\boldsymbol{\omega})\left\{\tilde{p}_j(\boldsymbol{\omega}) + i\left[\omega_1\tilde{u}_j\tilde{n}_1^{(j)}(\boldsymbol{\omega}) + \omega_2\tilde{u}_j\tilde{n}_2^{(j)}(\boldsymbol{\omega})\right]\right\}d\boldsymbol{\omega} \tag{13}$$

where

$$\bar{\tilde{K}}_m(\gamma_1,\gamma_2) = \frac{1}{2\pi^2}\int_{\Gamma_l} e^{i(\gamma_1 y_1+\gamma_2 y_2)} n_m^{(l)}(\boldsymbol{y})d\Gamma(\boldsymbol{y}), \qquad l = 1,2,\ldots,n, \tag{14}$$

$$\tilde{u}_p\tilde{n}_m^{(p)}(\boldsymbol{\omega}) = \int_{\Gamma_p} e^{-i(\omega_1 y_1+\omega_2 y_2)} n_m^{(p)}(\boldsymbol{y})u_p(\boldsymbol{y})d\Gamma(\boldsymbol{y}), \qquad \boldsymbol{\omega} = \boldsymbol{\xi}, \quad p = l,j, \tag{15}$$

$$\tilde{p}_j(\boldsymbol{\omega}) = \int_{\Gamma_j} e^{-i(\omega_1 y_1+\omega_2 y_2)} p_j(\boldsymbol{y})d\Gamma(\boldsymbol{y}). \tag{16}$$

We define the segments ($\Gamma_p \equiv P_p$) in formulas (14)–(16) by ν-spline curves as in formula (1). These formulas for segments represented by such curves have the following form

$$\bar{\tilde{K}}_m(\gamma_1,\gamma_2) = \frac{1}{2\pi^2}\int_{s_{l-1}}^{s_l} e^{i[\gamma_1 P_l^{(1)}(s)+\gamma_2 P_l^{(2)}(s)]} J_l(s)n_m(s)ds, \quad s_{j-1} \le s \le s_l, \tag{17}$$

$$\tilde{u}_p\tilde{n}_m^{(p)}(\boldsymbol{\omega}) = \int_{s_{p-1}}^{s_p} e^{-i[\omega_1 P_p^{(1)}(s)+\omega_2 P_p^{(2)}(s)]} u_p(s)n_m^{(p)}(s)J_p(s)ds, \quad \boldsymbol{\omega} = \boldsymbol{\xi}, \quad p = l,j, \tag{18}$$

$$\tilde{p}_j(\boldsymbol{\omega}) = \int_{s_{j-1}}^{s_j} e^{-i[\omega_1 P_j^{(1)}(s)+\omega_2 P_j^{(2)}(s)]} p_j(s)J_j(s)ds, \tag{19}$$

where $J_l(s) = [(\partial y_1/\partial s)^2 + ((\partial y_2/\partial s))^2]^{\frac{1}{2}}$, $y_1 = P_l^{(1)}(s)$, $y_2 = P_l^{(2)}(s)$. Segments $P_p(s)$ are described by the ν-spline curves as in expression (1).

3.2. Parametric Integral Equation System for ν-spline curves

After applying the inverse of the Fourier transform to the expression obtained by substituting Equations (17)–(19) in Equation (13) a new parametric integral equation system (PIES) is obtained [3]:

$$0.5u_l(s_1)=\sum_{j=1}^{n}\int\limits_{s_{j-1}}^{s_j}\left\{\bar{U}^*_{lj}(s_1,s)p_j(s)-\bar{P}^*_{lj}(s_1,s)u_j(s)\right\}J_j(s)ds, \quad s_{j-1}<s_1,s<s_j. \tag{20}$$

The kernels in the PIES are functions $\bar{U}^*_{lj}(s_1,s)$ and $\bar{P}^*_{lj}(s_1,s)$ given by the following integral expression:

$$\bar{U}^*_{lj}(s_1,s)=\frac{1}{4\pi^2}\int\limits_{R^2}e^{i(\omega_1\eta_1+\omega_2\eta_2)}\Delta^{-1}(\boldsymbol{\omega})d\boldsymbol{\omega}, \tag{21}$$

$$\bar{P}^*_{lj}(s_1,s)=\frac{-i}{4\pi^2}\int\limits_{R^2}e^{i(\omega_1\eta_1+\omega_2\eta_2)}\Delta^{-1}(\boldsymbol{\omega})\left[\omega_1n_1^{(j)}(s)+\omega_2n_2^{(j)}(s)\right]d\boldsymbol{\omega}. \tag{22}$$

After calculating relatively complex integrals (21) and (22) we obtain the final expressions in the following form:

$$\bar{U}^*_{lj}(s_1,s)=\frac{1}{2\pi}\ln\frac{1}{[\eta_1^2+\eta_2^2]^{0.5}}, \qquad \bar{P}^*_{lj}(s_1,s)=\frac{1}{2\pi}\frac{\eta_1n_1^{(j)}(s)+\eta_2n_2^{(j)}(s)}{\eta_1^2+\eta_2^2}, \tag{23}$$

where $\eta_1=P_l^{(1)}(s_1)-P_j^{(1)}(s)$ and $\eta_2=P_l^{(2)}(s_1)-P_j^{(2)}(s)$.

Expression (23) is an adequately modified fundamental and singular solution for the Laplace equation with the boundary geometry defined by the ν-spline curves.

4. Numerical solution of the PIES

In PIES, *i.e.* in Equation (20), the boundary functions are: $p_j(s)$ or $u_j(s)$. To be able to solve the PIES only one of these two functions may be unknown while the other must be given. Which of these will be known or unknown depends upon the type of boundary conditions. To approximate both of these functions, the following approximating series are applied:

$$p_j(s)=\sum_{k=0}^{M}p_j^{(k)}T_j^{(k)}(s), \qquad u_j(s)=\sum_{k=0}^{M}u_j^{(k)}T_j^{(k)}(s), \tag{24}$$

where $u_j^{(k)}$, $p_j^{(k)}$ are *unknown coefficients* on segments j, k – number of coefficients, whereas $T_j^k(s)$ are the *global basis functions* on individual segments. In the pseudo-spectral method (PM) any orthogonal polynomials me be used as functions [3, 4]. In our considerations we apply the Chebyshev polynomials.

Inserting Equations (24) into integral equations systems (20) we obtain the following form:

$$\frac{1}{2}u_l(s_1)=\sum_{j=1}^{n}\sum_{k=0}^{M}\left\{p_j^{(k)}\int\limits_{s_{j-1}}^{s_j}\bar{U}^*_{lj}(s_1,s)T_j^{(k)}(s)J_j(s)-u_j^{(k)}\int\limits_{s_{j-1}}^{s_j}\bar{P}^*_{lj}(s_1,s)T_j^{(k)}(s)J_j(s)\right\}ds \tag{25}$$

Equation (25) written at collocation points $n \times M$ reduces itself to a system of algebraic equations:

$$Hu = Gp, \tag{26}$$

where the column matrices u and p contain the approximating coefficients of the boundary functions (24). After considering the boundary conditions and performing necessary transformations, Equation (26) takes the form of a linear system of algebraic equations with non symmetrical coefficient matrix:

$$AX = F, \tag{27}$$

vector X contains the unknown boundary coefficients of the approximating functions (24), vector F depends upon the given boundary conditions.

After solving Equation (27) we obtain coefficients $p_j^{(k)}$ and $u_j^{(k)}$ for the approximating expressions. Substituting the coefficients to the first or the second approximating series (24) we obtain a solution that meets the boundary conditions and integral equation system (20). The expressions thus obtained are smooth functions on individual segments.

4.1. Testing example

A computer program in C++ was created basing on the algorithm presented earlier in [7]. To carry out a simulation of the effect of various values of coefficients ν_i on boundary geometry, we used the example of stationary heat flow in the L-shaped living room from literature [7, 8], (Figure 2).

The living room has 6 singular corner points. Applying the proposed PIES with the ν-spline curves the same area may be defined by the curves using different values of curvature coefficients ν_i in the corner points. Their value has significant influence on approximating accuracy of the boundary geometry. Using various curvature coefficients ν_i, it is possible to describe the given boundary with different accuracy eliminating, at the same time, the singularities which occur in the corner points. Figure 2 illustrates a comparison of the boundary geometry of the living room defined by linear segments and 4 instances of ν-spline segments for different curvature coefficients ($\nu_i = 5,\ 50,\ 500,\ 5000$). It is easy to notice that only for coefficient $\nu_i = 5$ the boundary geometry is noticeably different from the boundary defined by linear segments with singular corner points. The differences are insignificant in other cases.

Applying the same boundary conditions on respective segments, calculations were performed for all boundary geometries in question. Table 1 shows the results for all the instances mentioned above. Due to the convergence of the solutions in the living room interior the results are presented at some selected points on the diagonal only. They are shown in Table 1.

Figure 3 below presents relative errors of the solutions in the area for different curvature coefficients.

Comparing the particular solutions we can see that the greater geometric similarity of the domains, the lesser the differences between solutions. It is particularly evident on the graphs of the relative errors of the solutions for different values of the coefficient of ν-spline segments in respect of the solutions of linear segments (Figure 3).

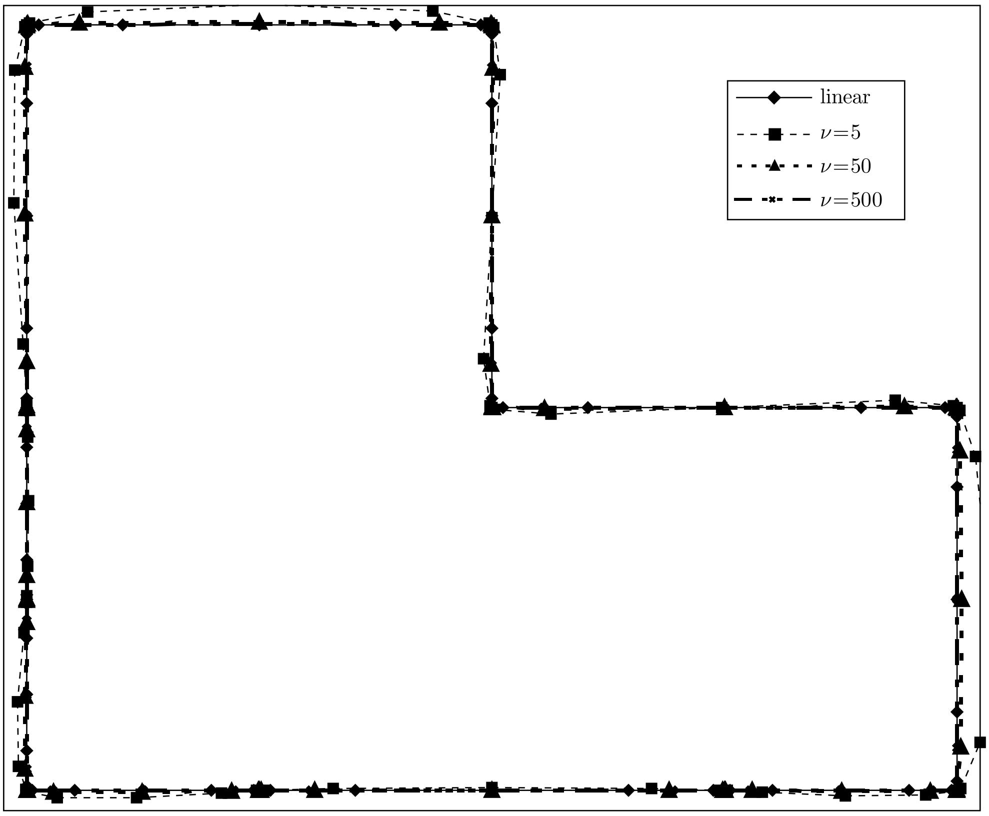

Figure 2. Living room defined by linear segments and ν-spline curves for different curvature coefficients

Table 1. Solutions for different curvature coefficients

x	y	linear	$\nu=5$	$\nu=50$	$\nu=500$	$\nu=5000$
0.4	7.6	46.985	46.0921	46.8059	46.8643	46.8576
0.8	7.2	43.9592	43.0986	43.8056	43.9004	43.9059
1.2	6.8	40.9909	40.2381	40.8538	40.9387	40.9444
2.8	5.2	30.0274	29.7201	29.961	29.9802	29.9783
3.2	4.8	27.4154	27.1634	27.3632	27.376	27.3738
3.6	4.4	24.6291	24.3415	24.5762	24.592	24.5897
4.4	3.6	17.3833	17.0004	17.2915	17.3235	17.323
4.8	3.2	15.8591	15.5339	15.7931	15.8188	15.8179
5.2	2.8	14.7804	14.497	14.721	14.7393	14.7372
6.8	1.2	12.5753	12.402	12.5183	12.5116	12.5049
7.2	0.8	12.4392	12.2855	12.382	12.3676	12.3588
7.6	0.4	12.4083	12.2697	12.3466	12.3181	12.3035

For the highest values of the curvature coefficient ν (the greatest geometric similarity) the solutions practically, do not differ. However, in the vicinity of the corners *i.e.* singular points, the differences are somewhat greater.

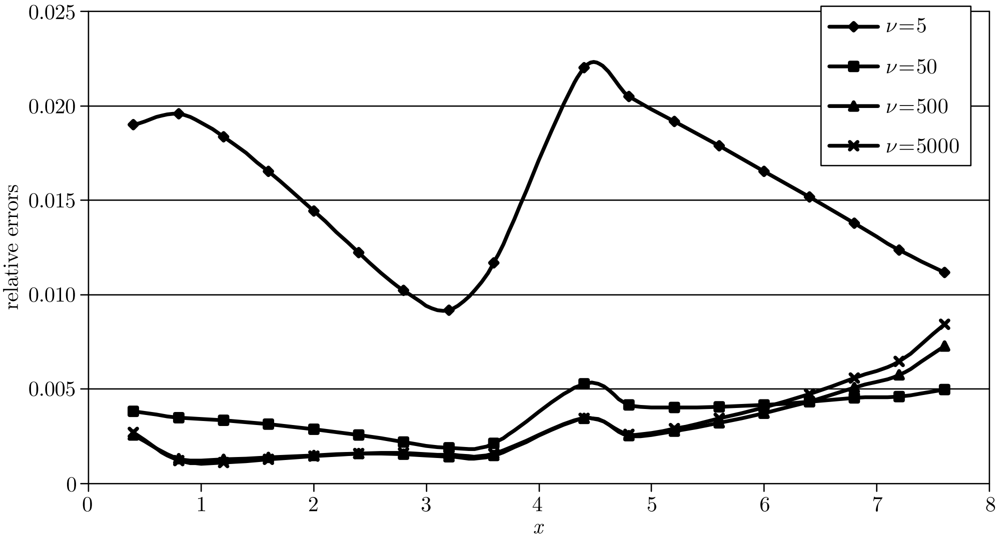

Figure 3. Relative errors for different curvature coefficients in respect of the solutions for linear segments

5. Conclusions

The paper presented an original PIES for solving potential boundary value problems with geometric singularities. For modeling singular corner points of boundary geometry ν-spline curves were used. As a result of their analytical compilation with a traditional boundary integral equation a PIES was obtained which is defined on the straight line in a parametric system of reference. The length of this line depends on the periphery of boundary geometry.

Taking into consideration the fact that the PIES in its mathematical formalism takes into account boundary geometry and is not directly defined on its boundary, the numeric solution of the PIES does not require boundary discretization contrary to the classical BIE.

This property is very important in terms of the effectiveness of the numeric solutions of the PIES.

The application of ν-spline curves makes it possible to define boundary geometry with geometric singularities. Practically, it enables modeling of the corner points as non-singular points.

The research work that has been conducted on the subject so far shows that the greater the geometric similarity of the domains, the lesser differences occur between the solutions in the domain. However, the research does not allow to draw unambiguous conclusions concerning the influence of singularities of the corner points on the accuracy of the solution in their nearest surroundings. The problem requires further investigations.

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