A CONSTRUCTIVE APPROACH TO MANAGING FUZZY SUBSETS OF TYPE 2 IN DECISION MAKING

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Abstract: The aim of this paper is to present a constructive methodology and algorithms for operations with fuzzy sets of type 2. The need to elaborate this methodology came from practical problems of Decision Making. To realize the methodology, some simplifications of the problem have been introduced. Particularly, only the trapezium form of membership functions was used. To highlight the difference between the proposed approach and the classical theory of fuzzy sets of type 2, the terms "hyperfuzzy set" and "hyperfuzzy function" have been introduced. Some base situations of hyperfuzzy functions with real arguments and real functions of hyperfuzzy arguments are performed.

Keywords: fuzzy sets of type 2, hyperfuzzy sets, decision making

1. Introduction

Experience in the field of decision making based on the fuzzy set theory shows that in many practically important cases it is necessary to apply more complex performance of data given than the usual fuzzy sets or fuzzy numbers. In many real-life situations, we are confronted with the necessity to take into account opinions of several experts participating in the formalization of local criteria. Such problems are mostly typical for the tasks of Multicriteria Multiperson Decision Making [1, 2] and Group Decision Making [3, 4]. Nevertheless, they may also occur in other areas when we must build a membership function on the base of opinions of several persons. Obviously, in these cases, averaging of experts' estimations leads to the loss of important initial information. Therefore, a more realistic approach should be used *viz.* methods based on fuzzy sets of type 2.

Fuzzy sets of type 2 were originally proposed by L. Zadeh [5] for the mathematical formalization of linguistic terms. In essence, these sets are an extension of usual fuzzy sets (type 1) to the case when the membership function of a fuzzy subset is performed by another fuzzy subset.

More strictly, let A be a fuzzy set of type 2 in the universe of discourse X. Then, for any $x \in X$, the membership function $\mu_A(x)$ of A is the fuzzy set with the membership function $f_x(y)$, where $y \in Y \subset [0,1]$. As a result, for the continuous set we get:

$$\mu_A(x) = \int_Y f_x(y)/y, \tag{1}$$

and for the discrete set:

$$\mu_A(x) = \left\{ \frac{f_x(y_i)}{y_i} \right\}, \quad i = 1,\ldots,n,$$

where n is the number of elements of the set Y.

Further development of the theory of fuzzy sets of type 2 is presented in articles [6–8], where main mathematical operations on such sets are defined. The authors of [6] have proved that using the extension principle introduced by L. Zadeh, it is possible to build fuzzy sets of types 3, 4 and so on.

Nevertheless, the existing, rather general definition of fuzzy sets of type 2 in many cases cannot be directly used in practice. Therefore, additional simplifications and specific definitions (depending on the situation observed) of the base theory are needed to construct an effective methodology and algorithms for its realization.

In order to do so, this paper is set out as follows. Section 2 presents some base cases of realization of hyperfuzzy functions with real arguments. Section 3 describes the main features of the case when we must deal with real functions with hyperfuzzy arguments.

2. Definition of hyperfuzzy sets

In practice of decision making, the need to introduce of fuzzy sets of type 2 often arises when we must define a membership function describing a local criterion on the basis of expert opinions. Since, generally, experts in different situations give various estimations, determined by their experience and intuition, certain objectiveness in the mathematical formalization of the appropriate membership function can be achieved by aggregation of experts' opinions.

Let us assume that the basic membership function characterizing a local criterion has the form of a trapeze. It is well known that such a function may be completely determined by the quadruple (x_1,x_2,x_3,x_4), where the intervals $[x_1,x_4]$ and $[x_2,x_3]$ are the support (bottom) and core (top) of the trapeze, respectively. Usually, each expert can present his own quantitative estimations of x_1, x_2, x_3, x_4. The averaging of experts' assessments for each x_i, for $i = 1,\ldots,4$, lead inevitably to the loss of initial information obtained from experts. Therefore, an alternative approach for aggregation of experts' opinions has been elaborated to represent the parameters x_1, x_2, x_3, x_4 in the form of fuzzy numbers. So, we change each x_i by the corresponding trapezoidal fuzzy interval X_i. The core of X_i is the crisp interval which corresponds to the majority of experts' estimations, and the support of X_i is the interval of all estimations. As the result, we obtain a complicated structure, the left-hand part of which is shown in Figure 1.

Such mathematical objects may be treated as a special form of fuzzy subsets of type 2. To emphasize the specificity of the new mathematical object introduced, we

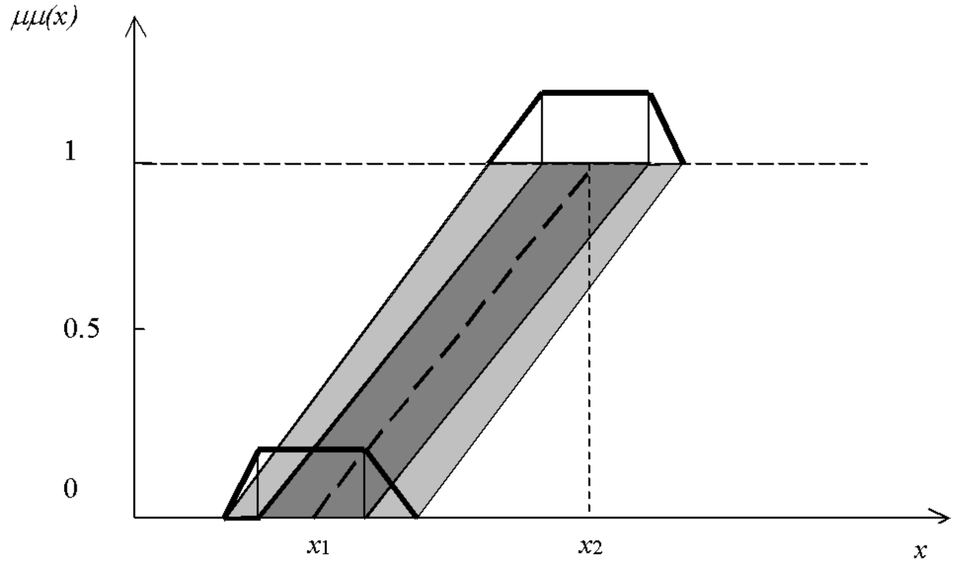

Figure 1. A graphical representation of the left-hand side of a trapezoidal hyperfuzzy number (including only two parameters, x_1 and x_2; $\mu\mu(x)$ is the membership function of the hyperfuzzy number)

apply the notation $\mu\mu(x)$ for the membership function of the hyperfuzzy set instead of the commonly used $\mu(x)$.

However, the presence of some specific features and variations from the classical definitions results in the expediency of introducing some new definitions.

Definition 1 *A hyperfuzzy set is a set described by the trapezoidal membership function characterized by fuzzy parameters X_1, X_2, X_3, X_4, which are trapezoidal fuzzy numbers.*

Obviously, this definition includes the triangular membership function as a particular case. Figure 2 shows a hyperfuzzy set on a plane. Darker areas correspond to

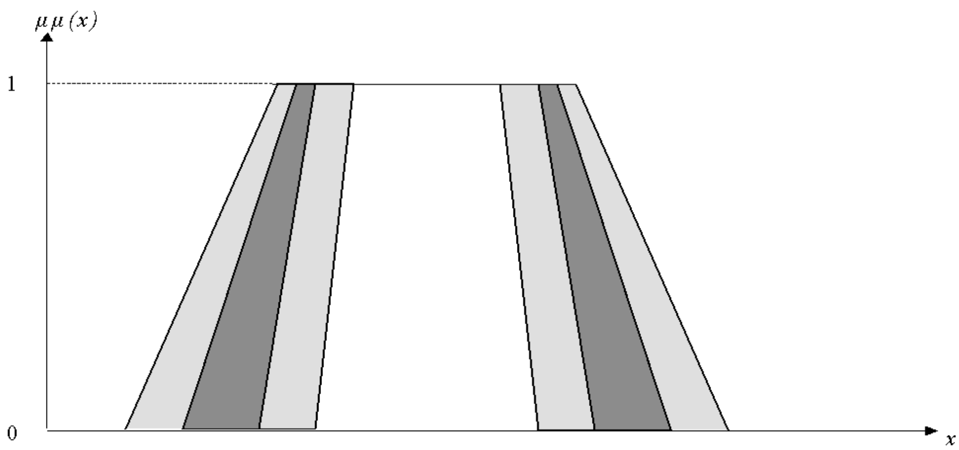

Figure 2. A hyperfuzzy set on a plane

the greatest unanimity among the experts concerning values of parameters x_1, x_2, x_3, x_4, while the lighter areas correspond to discord in experts' opinions.

In this article, a technique for mathematical formalization and operation on such objects is proposed.

Let us use the notation of a hyperfuzzy set, similar to the classical fuzzy sets $A=\{x,\mu_A(x)\}$, in the form $G=\{x,\mu\mu_G(x)\}$. The problem is to create a constructive method for evaluation of $\mu\mu(x)$ in all possible situations.

Definition 2

$$GX=(G1,G2,G3,G4)$$

denotes a hyperfuzzy interval, where GI for $I=1$ to 4 is the trapezoidal fuzzy number characterized by parameters

$$GI_1,GI_2,GI_3,GI_4,$$

where GI_i, for $i=1,\ldots,4$, are non-fuzzy parameters of the trapezoidal fuzzy number GI.

In our considerations, hyperfuzzy and fuzzy numbers are confined to their trapezoidal representation. It is easy to give a more general definition. However, the level of abstraction of such a definition may create difficulties in understanding the essence of operations on hyperfuzzy numbers. Moreover, practice shows that the trapezoidal form of membership functions is quite sufficient a level of abstraction for the formalization of fuzzy uncertainties in the majority of real situations.

Let us assume that there is a local criterion described by the membership function represented by the trapezoidal hyperfuzzy number GX, the left-hand side of which is shown in Figure 3. Let $x_* \in X_{GX}$ be a real number that corresponds to a certain value of the analyzed parameter; X_{GX} is the support of the trapezoidal hyperfuzzy number GX. Then, $G(x_*)$ is the usual trapezoidal fuzzy number, which is the result of mapping x_* to the hyperfuzzy number GX:

$$G(x_*)=\{(g_1(x_*),g_2(x_*),g_3(x_*),g_4(x_*)),x_* \in X_{GX}\},$$

where g_1,g_2,g_3,g_4 are the parameters of trapezoidal fuzzy numbers. Figure 3 illustrates the presented formal mathematical operations. It is clear that the resulting trapezoidal fuzzy number G may be considered as a mapping of real number $x_* \in X_{GX}$ to the hyperfuzzy number representing the description of the local criterion (see Figure 3).

Generally, the trapezoidal hyperfuzzy number has not only a left-hand part (Figure 3), but also a right-hand part (Figure 4). Thus, according to Figures 3 and 4, we must consider two base cases of mapping. For the first case, the left-hand side (Figure 3), when $G1_1 < x_* < G2_4$, the parameters of the resulting trapezoidal fuzzy number are defined as follows:

$$\begin{aligned}
g_4&=(x_*-G1_1)/(G2_1-G1_1);\\
g_3&=(x_*-G1_2)/(G2_2-G1_2);\\
g_2&=(x_*-G1_3)/(G2_3-G1_3);\\
g_1&=(x_*-G1_4)/(G2_4-G1_4).
\end{aligned} \tag{2}$$

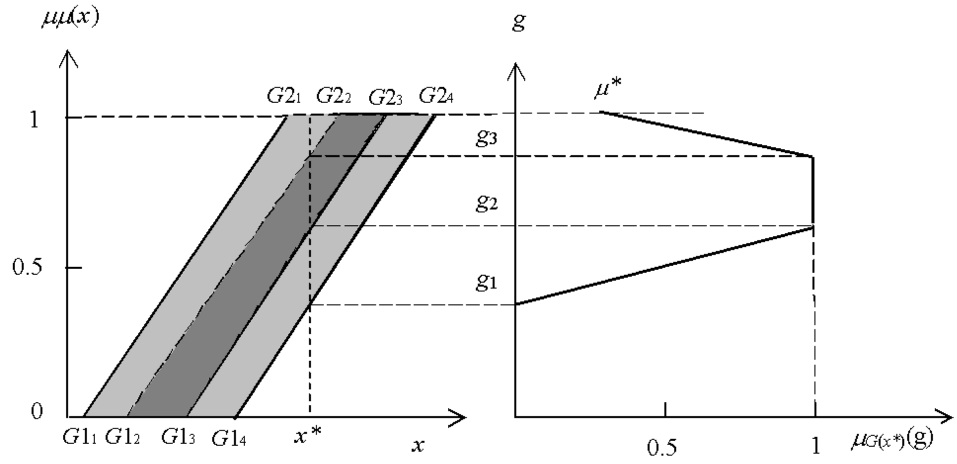

Figure 3. Mapping of real number x_* to the left-hand side of a trapezoidal hyperfuzzy number

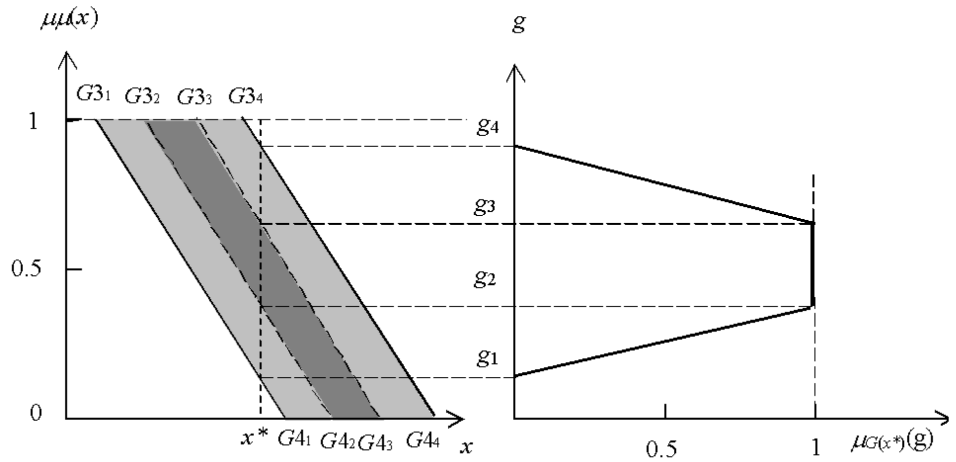

Figure 4. Mapping of real number x_* to the right-hand side of a trapezoidal hyperfuzzy number

For the second case, the right-hand side (Figure 4), when $G3_1 \geq x_* \geq G4_4$), we get:

$$g_k = (G4_k - x_*)/(G4_k - GX_k), \quad k = (1, \ldots, 4). \tag{3}$$

In the intermediate case, when $G2_4 \geq x_* \geq G3_1$, the usual real numbers, equal to 1, are the results of the mapping. It is easy to see that there are only few special cases of mapping real numbers to the trapezoidal hyperfuzzy intervals for which the usual fuzzy numbers may be obtained as the results. Some of them are presented in Figures 5–7.

The most complex situations are shown in Figures 5 and 7. For the case presented in Figure 5, using the geometrical representation shown in Figure 8, we obtain:

$$g_4 = (x_* - G1_1)/(G2_1 - G1_1).$$

As can be seen in Figure 8, in this case $g_4 > 1$.

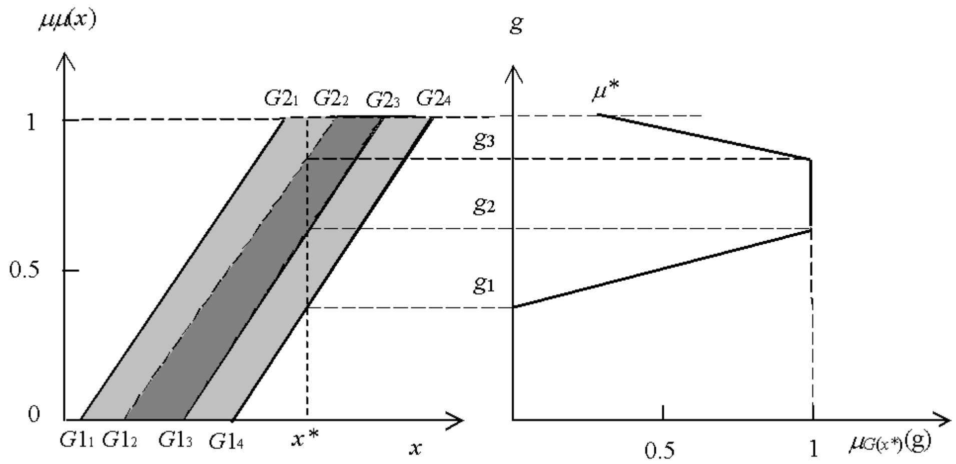

Figure 5. A result in the form of an imperfect trapeze

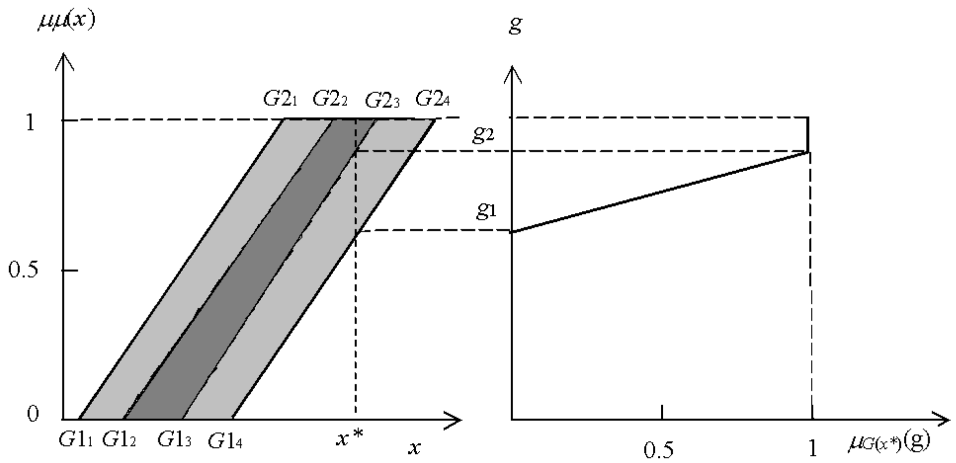

Figure 6. A result in the form of a semi-trapeze

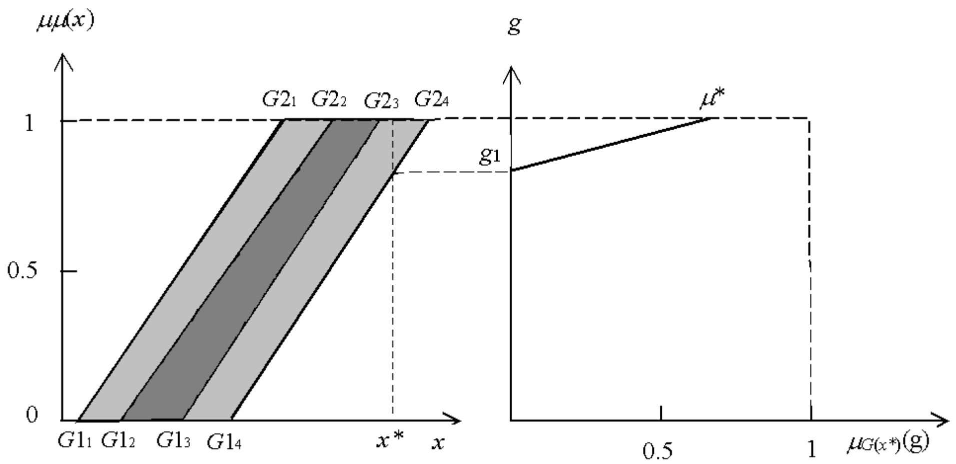

Figure 7. A resulting triangular fuzzy number

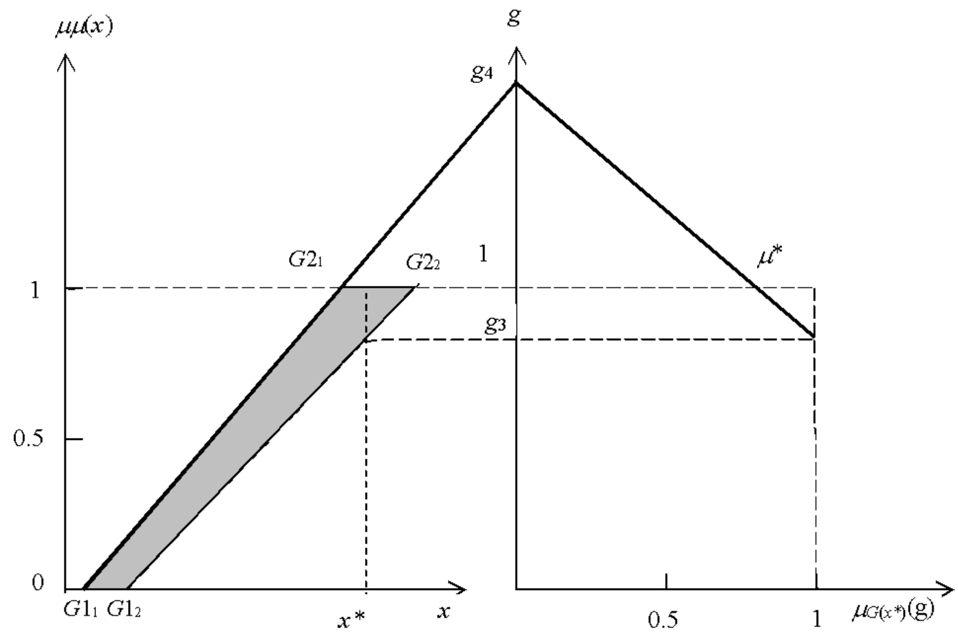

Figure 8. Graphical representation of the procedure for obtaining the parameter μ^*

It is reasonable to take into consideration the parameter μ^*, which plays an important role in our analyses. On the basis of the geometrical representation (see Figure 8), we infer the following simple equation for the calculation of μ^*:

$$(1-\mu^*)/\mu^* = (1-g_3)/(g_4-1).$$

In the case presented in Figure 7, for parameter μ^*, in a similar way we get:

$$(1-\mu^*)/\mu^* = (g_2-1)/(1-g_1),$$

where

$$g_2 = (x_* - G1_3)/(G2_3 - G1_3) > 1.$$

3. Hyperfuzzy functions with fuzzy arguments

The most complicated situation (Figure 9) occurs while mapping a hyperfuzzy number, described by membership function $\eta\eta$ (parameter of quality), to a fuzzy interval, characterized by membership function μ (criterion of quality).

It can be seen in Figure 9 that, in this case, two fuzzy intervals are obtained as the result of this mapping. Hence, we face the situation of mapping with ambiguity. In the situation observed, the mapping procedure may be reduced to choosing the greatest resulting fuzzy interval. Evidently, such an interval will be the best expression of the mapping result (fuzzy extension). Generally, for the trapezoidal function μ, in some cases more then two resulting fuzzy intervals may be obtained. Their overlapping may occur as well.

To order fuzzy intervals, a method was used [9] based on a synthesis of the probabilistic approach and the expression of a fuzzy interval as a set of α-levels.

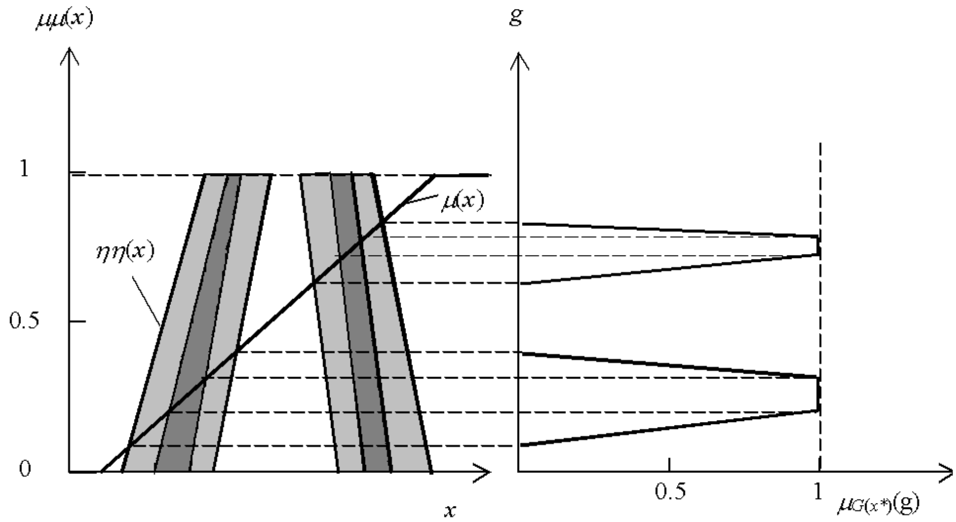

Figure 9. Mapping a hyperfuzzy number to a common fuzzy number

4. Conclusion

In this paper, we have presented a constructive approach to dealing with mathematical objects described by fuzzy subsets of type 2. A special case of its use for mathematical formalization of local criteria in decision making has been considered. To explain the technique elaborated and to emphasize the differences between the proposed approach and the classical theory of fuzzy sets of type 2, new concepts of "hyperfuzzy set" and "hyperfuzzy function" have been introduced. A hyperfuzzy set may be treated as a special case of fuzzy sets of type 2. Cases of local criterions performed by a hyperfuzzy function with real argument and of a real function with an hyperfuzzy argument have been considered.

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