A PROBABILISTIC APPROACH TO FUZZY AND CRISP INTERVAL ORDERING

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Abstract: The paper presents a new method of crisp and fuzzy interval comparison (ordering). The method is based on the probabilistic approach and the representation of fuzzy numbers as ordered α-level sets. It allows all the cases of interval location and overlapping to be taken into account, including the ordering of intervals and real numbers. Additionally, the method implicitly allows the widths of intervals to be used in ordering procedures. It should be noted that the probabilistic approach was employed only to infer the set of formulas needed to estimate quantitatively the degree to which one interval is less than or equal to another interval. However, the measure of this value may be treated as probability. Some simple examples are also presented to illustrate the technique's practical efficiency.

Keywords: probabilistic approach, interval ordering, fuzzy interval ordering

1. Introduction

The problem of crisp and fuzzy interval (number) ordering is of perennial interest, because of its direct relevance in practical modeling and optimization of real world processes.

Theoretically, fuzzy numbers can only be partially ordered and hence cannot be compared. However, when fuzzy numbers are used in practical applications or when a decision has to be made among alternatives, a comparison of fuzzy numbers becomes necessary.

There are numerous definitions of the ordering relation between fuzzy quantities (as well as crisp intervals) [1–13]. In most cases, the authors use quantitative indices. The values of such indices represent the degree to which one interval (fuzzy or crisp) is larger/smaller than another interval. In some cases, even several indices are used simultaneously, for example, in [7] four indices of inequality and three of equality are proposed. Reviews of the best known approaches [12, 13] note that although some of these methods have shown more consistency and better performance in difficult cases, no single method of fuzzy interval comparison may be put forward as the best. The existing approaches to fuzzy interval comparison may be divided into three groups. There are qualitative methods [2–5] and quantitative methods, which use indices obtained from the base definitions of the fuzzy set theory [1, 6, 7] in the ordering procedure. There is also a third group of methods, based on the representation of fuzzy numbers as α-level sets [8–11]. It should be noted that the last group of methods is especially advantageous. Moreover, they can be used with all types of membership functions, with no restrictions. This feature is of great practical importance, especially in the case of numerical computation. Additionally, because α-levels are in essence a set of regular crisp intervals, the powerful tools of interval arithmetic can be employed to solve the problem of fuzzy interval ordering.

The widest review of the problem of fuzzy quantities ordering based on more than 35 literature indices has been performed in [14], where the a new, interesting classification of methods and reasonable properties was proposed for fuzzy values ordering.

In this article, we present a further development of such methods. The approach proposed is based on α-level representation of fuzzy intervals and the probability estimation of the fact that a certain interval is larger than/equal to another interval. It should be noted that the probabilistic approach was used only to infer a set of formulas needed for the deterministic quantitative estimation of the inequality/equality of intervals. This value may be formally treated as probability, since it is in accordance with main rules of the theory of probability. The method allows intervals and real numbers to be compared and takes into account (implicitly) the widths of ordered intervals.

2. Crisp interval relation expressions

Since the proposed method is based on the representation of fuzzy numbers as α-level sets, the main problem is to compare crisp intervals. Let $A=[a_1,a_2]$ and $B=[b_1,b_2]$ be independent crisp intervals, and $a\in[a_1,a_2]$, $b\in[b_1,b_2]$ – independent random variables allocated on these intervals. Since we are dealing with crisp (non-fuzzy) intervals, it is natural to assume that the values of the random variables a and b are uniformly distributed. In the case of overlapping intervals, there are some subintervals which play an important role in our analysis. For example (see Figure 1), the fall of random variables $a\in[a_1,a_2]$, $b\in[b_1,b_2]$ in the subintervals $[a_1,b_1]$, $[b_1,a_2]$, $[a_2,b_2]$ may be treated as a set of independent random events.

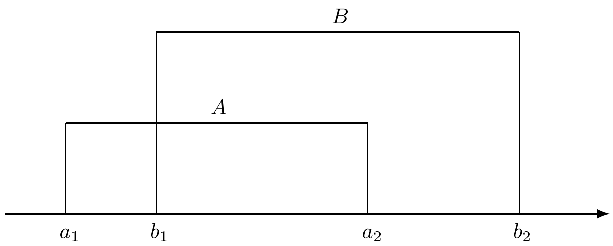

Figure 1. Example of overlapping intervals

Let us define the events $H_k : a \in A_i, b \in B_j$, for $k=1$ to n, where A_i and B_j are certain subintervals of intervals A and B in accordance with $A = \bigcup_i A_i$, $B = \bigcup_j B_j$. In the case considered in Figure 1, n=4. It is easy to see that events H_k form an entire group of events, describing all the cases of variables a and b in the various subintervals A_i and B_j, respectively.

Let $P(H_k)$ be the probability of event H_k, and $P(B > A/H_k)$ be the conditional probability of $B > A$ given H_k. Hence, the composite probability may be expressed as follows:

$$P(A > B) = \sum_k^n P(H_k) P(B > A/H_k). \tag{1}$$

Since we are dealing with uniform distributions of the random variables a and b in the given subintervals, the probabilities $P(H_k)$ can be easily obtained geometrically.

To illustrate the procedure of inferring the resulting formulas, let us consider the case presented in Figure 1. There is a set of four events:

$$\begin{aligned} &H_1: a \in [a_1, b_1] \wedge b \in [b_1, a_2], \\ &H_2: a \in [a_1, b_1] \wedge b \in [a_2, b_2], \\ &H_3: a \in [b_1, a_2] \wedge b \in [b_1, a_2], \\ &H_4: a \in [b_1, a_2] \wedge b \in [a_2, b_2]. \end{aligned} \tag{2}$$

Since events $a \in [a_1, b_1]$, $b \in [b_1, a_2], \ldots$ are independent, we obtain the following probabilities:

$$\begin{aligned} P(H_1) &= \frac{b_1 - a_1}{a_2 - a_1} \frac{a_2 - b_1}{b_2 - b_1}, \\ P(H_2) &= \frac{b_1 - a_1}{a_2 - a_1} \frac{b_2 - a_2}{b_2 - b_1}, \\ P(H_3) &= \frac{a_2 - b_1}{a_2 - a_1} \frac{a_2 - b_1}{b_2 - b_1}, \\ P(H_4) &= \frac{a_2 - b_1}{a_2 - a_1} \frac{b_2 - a_2}{b_2 - b_1}. \end{aligned} \tag{3}$$

It is easy to notice from Figure 1 and Equations (2) that the conditional probabilities equal:

$$\begin{aligned} P(B > A/H_1) &= 1, \\ P(B > A/H_2) &= 1, \\ P(B > A/H_3) &= \frac{1}{2}, \\ P(B > A/H_4) &= 1. \end{aligned} \tag{4}$$

Some comments about event H_3 may help to explain the obtained results. From Equations (2) we see that event H_3 is simultaneously evidence of events $a \in [b_1, a_2]$ and $b \in [b_1, a_2]$. Hence $P(B > A/H_3) = P(A > B/H_3) = 0.5$. Substituting Equations (2)–(4) to Equation (1) we have:

$$P(B > A) = 1 - \frac{1}{2} \frac{(a_2 - b_1)^2}{(a_2 - a_1)(b_2 - b_1)}.$$

The probabilities $P(B > A)$ for all possible cases of interval overlapping, as well as comparison of intervals and real numbers, have been inferred similarly. To complete the set of expressions for interval relations, formulas for the probabilities of intervals equality $P(A=B)$ have been obtained. The results are shown in Table 1, where obvious cases (without overlapping) are omitted.

The approach described above can be treated as a framework for elaboration of constructive methods of interval comparison in various special situations. Some aspects of the interval comparison and ordering group of intervals, based on this approach, is presented *e.g.* in [15].

We have to make some additional remarks to clarify the results. Of course, if A is an interval $[a_1,a_2]$, where $a_1<a_2$, and b is a real number, then equality expression $A=b$ is nonsensical, because of the impossibility of simultaneously performing the conditions $a_1=b$ and $a_2=b$. Thus, in such cases, we have $P(A=b)=0$. On the other hand, the inequality expression $A<b$ may be used in analyses, since in the case, for example, $a_2<b$, there is no doubt that $P(A<b)=1$. It is clear that the case $a_1\leq b\leq a_2$ is reasonable, too, and probability $P(A<b)$ may be easily calculated (see Table 1). There is an interesting situation in the case of estimation of $P(A=B)$, where A and B are intervals. The simplest way is to state a "strong" rule such as "$A=B$ only if $a_1=b_1$ and $a_2=b_2$". However, when dealing with optimization problems, we often use equality-type restrictions. Of course, interval or fuzzy extension of such tasks leads inevitably to the extension of the corresponding equality-type restrictions. It is clear that the satisfaction of the "strong" equality rules, especially when using numerical optimization methods, is rather impossible in practice. Nevertheless, in the framework of the proposed probabilistic approach, "weak" equality rules have been elaborated. So, if $a_1\approx b_1$ and $a_2\approx b_2$, then $P(A=B)\neq 0$.

It is easy to see that in all cases we have $P(B>A)+P(A>B)=1$ and $P(B=A)+P(A\neq B)=1$.

It should be noted that values $P(B>A)$ are formally in interval $[0,1]$, but in practice it is better to use the interval $[0.5,1]$. For example, in case 5 in Table 1, we obtain $P(B>A)=1$ if $b_1=a_2$ and $P(B>A)=0.5$ if $b_1=a_1$, $b_2=a_2$, which means that $B=A$. In the latter case, the value 0.5 is the direct consequence of the nature of probability. Thus, if the probability $P(B>A)<0.5$, the opposite event $A>B$ is more probable, since $P(B>A)+P(A>B)=1$.

3. Fuzzy interval ordering

Let A and B be fuzzy intervals (numbers), and $A_\alpha=\{x/\mu_A(x)\geq\alpha\}$ and $B_\alpha=\{y/\mu_B(y)\geq\alpha\}$ be α-level sets of A and B, respectively. Since A_α and B_α are crisp intervals, probability $P_\alpha(B_\alpha>A_\alpha)$ for each pair A_α and B_α can be calculated in the way described in the previous section. The set of probabilities $P_\alpha(\alpha\in(0,1])$ may be treated as the support of the fuzzy subset:

$$P(A>B)=\{\alpha/P_\alpha(B_\alpha>A_\alpha)\};\tag{5}$$

where the values of α may be considered as grades of membership of the fuzzy interval $P(B>A)$. In this way, the fuzzy subset $P(B=A)$ may also be easily created.

As it can be seen in Figure 2, we have $P_\alpha(B_\alpha>A_\alpha)=1$ for all $\alpha>0.9$, since there is no overlapping of B_α and A_α. The broad case studies which we have carried out, allow us to think that, in the case of triangular or trapezoidal fuzzy number

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Table 1. Typical cases of crisp interval comparison

Cases	$P(B>A)$	$P(A=B)$
1. $a_1>b_1 \wedge a_1<b_2 \wedge a_1=a_2$	$\dfrac{b_2-a_1}{b_2-b_1}$	0
2. $b_1>a_1 \wedge b_1<a_2 \wedge b_1=b_2$	$\dfrac{b_1-a_1}{a_2-a_1}$	0
3. $b_1 \geq a_1 \wedge b_2 \leq a_2$	$\dfrac{b_1-a_1}{a_2-a_1}+\dfrac{1}{2}\dfrac{a_2-a_1}{b_2-b_1}$	$\dfrac{b_2-b_1}{a_2-a_1}$
4. $a_1 \geq b_1 \wedge a_2 \leq b_2$	$\dfrac{b_2-a_2}{b_2-b_1}+\dfrac{1}{2}\dfrac{a_2-a_1}{b_2-b_1}$	$\dfrac{a_2-a_1}{b_2-b_1}$
5. $b_1 \geq a_1 \wedge b_2 \geq a_2 \wedge b_1 \leq a_2$	$1-\dfrac{1}{2}\dfrac{(a_2-b_1)^2}{(a_2-a_1)(b_2-b_1)}$	$\dfrac{(a_2-b_1)^2}{(a_2-a_1)(b_2-b_1)}$
6. $a_1 \geq b_1 \wedge a_2 \geq b_2 \wedge a_1 \leq b_2$	$\dfrac{1}{2}\dfrac{(b_2-a_1)^2}{(a_2-a_1)(b_2-b_1)}$	$\dfrac{(b_2-a_1)^2}{(a_2-a_1)(b_2-b_1)}$

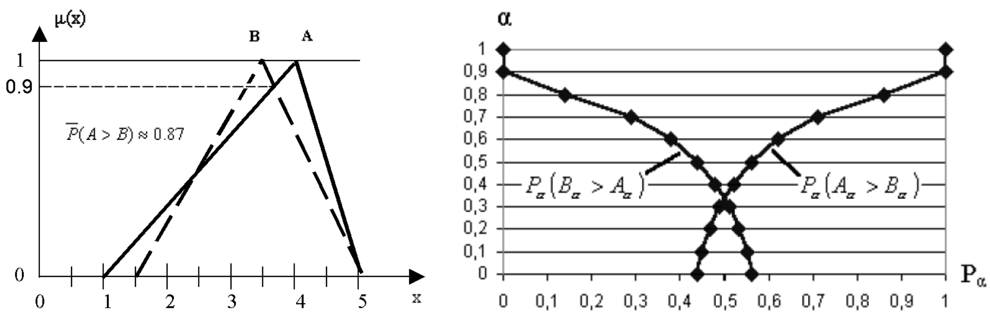

Figure 2. Typical cases of fuzzy interval comparison (I)

(interval) comparison, the results obtained may be interpreted as a fuzzy number (interval). Typical examples are presented in Figures 2–4. Moreover, as the result of comparison of fuzzy and real numbers, we also have a fuzzy number (see Figure 5).

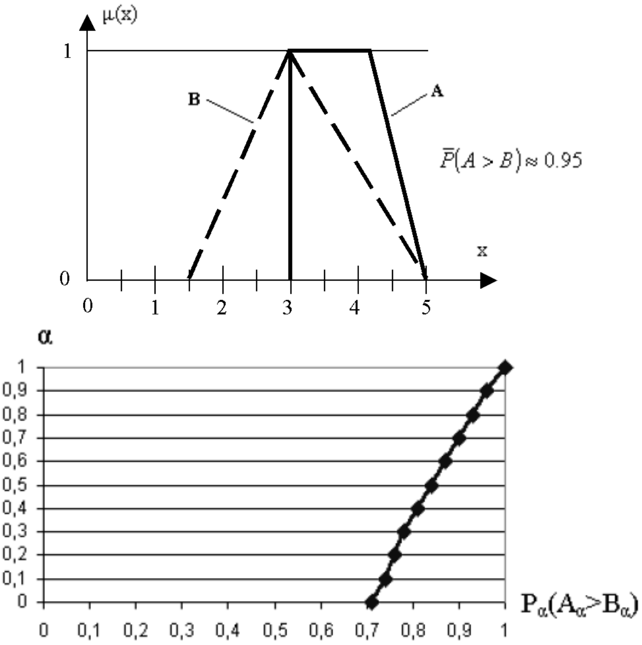

Figure 3. Typical cases of fuzzy interval comparison (II)

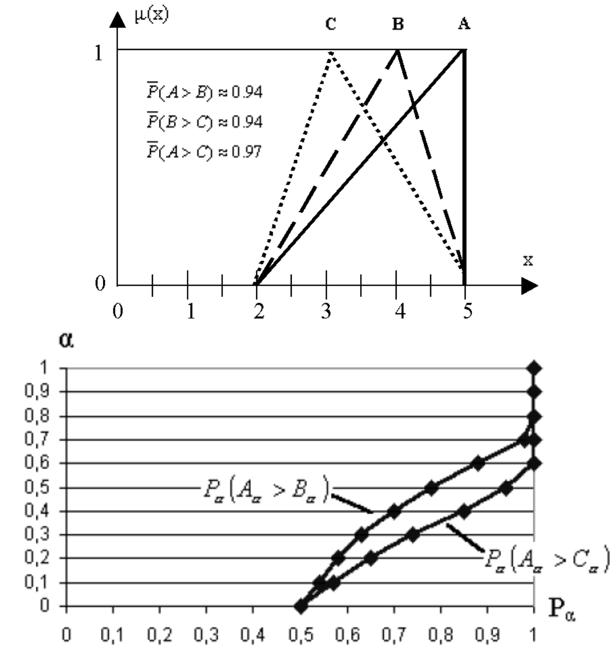

Figure 4. Typical cases of fuzzy interval comparison (III)

The result obtained is simple enough and reflects in a sense the nature of fuzzy arithmetic. The resulting "fuzzy probabilities" can be used directly. For instance, let A, B, C be fuzzy intervals and $P(A > B)$, $P(A > C)$ be fuzzy intervals expressing

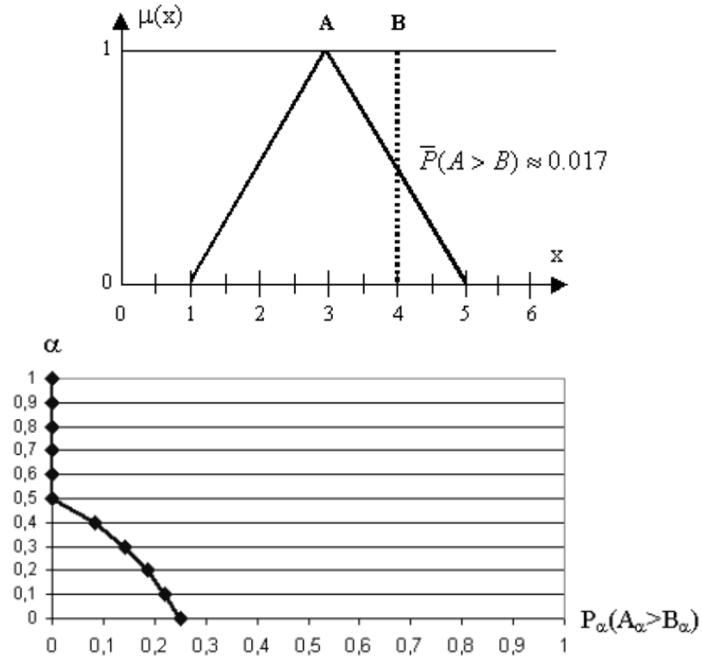

Figure 5. Comparison of a fuzzy interval and a real number

the probabilities $A > B$ and $A > C$, respectively. Hence the probability $P(P(A > B) > P(A > C))$ has the meaning of probability comparison and is expressed in the form of a fuzzy interval as well. Such fuzzy calculations may be useful at intermediate stages of analyses, since they preserve the fuzzy information available. Indeed, it can be shown that in any case $P(A > B) + P(B > A) =$ "near 1", and $P(A = B) + P(A \neq B) =$ "near 1", where "near 1" is a symmetrical relative to 1 fuzzy number. It is worth noting here that the main properties of probability are conserved in the introduced operations, though in a fuzzy sense. However, a detailed discussion of these questions is outside the scope of this article. Nevertheless, in practice, real number indices are needed for fuzzy interval ordering. For this purpose, some characteristic numbers of a fuzzy set [14] could be used. But it seems more natural to use defuzzification, which takes the following form for a discrete set of α-levels:

$$\overline{P}(A > B) = \sum_{\alpha} \alpha P_\alpha(B_\alpha > A_\alpha) / \sum_{\alpha} \alpha. \tag{6}$$

Equation (6) emphasizes that the contribution of the α-level to the overall probability estimation is increasing with an increase in its number. Of course, as proposed in [11], the set of complementary parameterized functions of α can be applied in Equation (6) instead of α. But for simplicity, only expression (6) was used to obtain the results presented below. Some typical cases of fuzzy interval comparison are shown in Figures 2–4.

It is easy to see that the resulting quantitative estimations are in accordance with our intuition.

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4. Illustrative examples

To illustrate our approach to crisp and fuzzy interval comparison, two examples are considered in this section. In both cases, we use the simple Rosenbrock's function

$$f = c(x_2 - x_1^2)^2 + (1 - x_1)^2 \tag{7}$$

as the base for interval and fuzzy interval extensions, since it is the most commonly used test for numerical methods of optimization. In practical optimization, we often deal with an interval or fuzzy interval target function. It must be emphasized that these functions have non-interval (real number) arguments. In such cases, the problem is to find the minimum/maximum of the interval (fuzzy interval) function. So if $F(\overrightarrow{x}) = [\underline{F}(\overrightarrow{x}), \overline{F}(\overrightarrow{x})]$ is an interval function, the aim is to find the real vector $\overrightarrow{x}$ directly delivering an extreme of the interval function $[\underline{F}(\overrightarrow{x}), \overline{F}(\overrightarrow{x})]$. To avoid the problems of interval and, especially, fuzzy function derivation, the interval and fuzzy generalizations of one of the direct search methods [16] had been elaborated. To obtain the test interval function, the initial Rosenbrock's function (7) was extended and expressed in the following interval form:

$$\begin{aligned} F(x_1,x_2) &= [\underline{F}(x_1,x_2), \overline{F}(x_1,x_2)] = \\ &= [c - c\alpha, c + c\alpha](x_2 - x_1^2)^2 + (1 - x_1)^2, \end{aligned} \tag{8}$$

where α is a real number parameter determining the width of an interval. The curves of equal values of Equation (8) are represented in Figure 6.

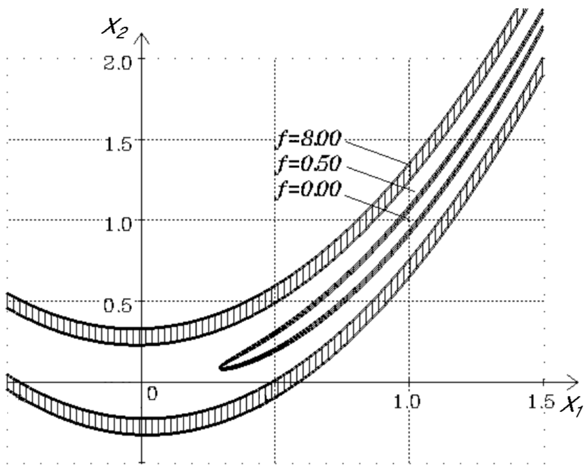

Figure 6. Interval extension of Rosenbrock's function (8) for $c = 100$, $\alpha = 0.05$

The results of the tests are summarized in Table 2. It is interesting to note that the same level of accuracy as in the case of a real number function was achieved using the same number of steps of our algorithm (3900 in our case). It is easy to see that the level of accuracy does not depend on the width of interval representing the minimized function.

Table 2. Results of tests (interval function (8) with $c=100$)

	α		
	0.005	0.01	0.05
Initial (starting) point: $x_1^{\text{in}}=-0.5$, $x_2^{\text{in}}=-0.5$			
$[\underline{F},\overline{F}]_{\text{in}}$	[58.2,58.8]	[57.9,59.0]	[55.7, 61.3]
$[\underline{F}+\overline{F}]_{\text{in}}/2$	58.5	58.5	58.5
$[\underline{F}-\overline{F}]_{\text{in}}$	0.563	1.125	5.625
The point of minimum			
$[\underline{F},\overline{F}]_{\min}$	$[5.43,5.44]\cdot 10^{-3}$	$[4.67,4.72]\cdot 0^{-3}$	$[3.31,3.46]\cdot 10^{-3}$
$[\underline{F}+\overline{F}]_{\min}/2$	$5.44\cdot 10^{-3}$	$4.69\cdot 10^{-3}$	$3.39\cdot 10^{-3}$
$[\underline{F}-\overline{F}]_{\min}$	$0.01\cdot 10^{-3}$	$0.05\cdot 10^{-3}$	$0.15\cdot 10^{-3}$
$x_1^{\min}$	1.022	1.015	0.986
$x_2^{\min}$	1.044	1.029	0.971

The next example is a minimization of a fuzzy interval function. For simplicity, the case of a trapezoidal fuzzy function is considered. Let us represent our base function (7) in a fuzzy, extended form:

$$\begin{aligned} F(x_1,x_2) &= [F_1(x_1,x_2),F_2(x_1,x_2),F_3(x_1,x_2),F_4(x_1,x_2)] = \\ &= [c-c\alpha,c-c\alpha/2,c+c\alpha/2,c+c\alpha](x_2-x_1^2)^2+(1-x_1)^2, \end{aligned} \tag{9}$$

where F_1, F_2, F_3, F_4 are the left support, left core, right core, right support of a trapezoidal fuzzy interval (number), respectively, and α is the real parameter determining the form of a fuzzy interval.

Fuzzy generalization of the direct search method was used to minimize function (9). The same values of parameter α and the initial point as in the case of interval function (8) were used. After the same number of steps of the algorithm as in the last case (3900), the results represented in Table 3 were obtained.

Table 3. Results of tests (fuzzy function (9) with $c=100$)

	α		
	0.005	0.01	0.05
$x_1^{\min}$	0.992	0.989	0.997
$x_2^{\min}$	0.982	0.977	0.992

It can be seen that the results are also good in the fuzzy case. These simple examples prove the practical validity of the presented crisp and fuzzy interval ordering method and its ability to generate effective numerical realizations.

5. Conclusions

In this article, only some of the theoretical results have been presented. Our experience has shown that a highly useful feature of the method is the possibility of a flexible approach to restrictions in optimization tasks. In practice, some restrictions can require a high level of accuracy. In these cases, we require the execution of the corresponding interval relations with the probability of 0.95–0.99. For other, not so rigid (important) restrictions, the probabilities of only slightly more than 0.5 may be applied. It must be emphasized that in order to apply the proposed method, software based on C++ was elaborated and successfully used for optimization of power units [17], in simulation [18] and logistics [19].

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