ON NUMERICAL CALCULATION OF ACOUSTIC FIELD OF PLANE SOURCE

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Abstract: The harmonics of an acoustic beam have been evaluated numerically by means of an integral representation according to the perturbation theory. A method for evaluation of the integrals of a rapidly oscillating function has been applied. Special attention has been paid to the accuracy of calculations and an algorithm for error estimation has been proposed. A computer program has been realized, and the profiles of the first and the second harmonics have been plotted. The used methods can be applied for the problem with dissipation or dispersion.

Keywords: Khokhlov-Zabolotskaya-Kuznetsov equation, numerical calculation, oscillating function, nonlinear acoustics

1. Introduction

The problem of the propagation of a nonlinear sound has been studied for rather along time, and several modeling equations for sound beam propagation have been proposed. Among them is the Khokhlov-Zabolotskaya-Kuznetsov (KZK) equation [1]:

$$\rho_{zt} - \frac{\varepsilon}{2c_0\rho_0}\rho_{tt}^2 - \delta\rho_{ttt} = \frac{c_0}{2}\Delta_\perp\rho. \tag{1}$$

It takes into consideration weak nonlinearity and weak absorption. The KZK equation has been widely used to describe nonlinear fields of sound and acoustic pulses. It provides an excellent model for the sound field of a plane source of an arbitrary shape when the ratio of source dimension to wavelength is large.

Like all other modelling equations for this problem, the KZK equation cannot be solved exactly for the general case and has been investigated with various numerical schemes and asymptotic methods (for examples see [2, 3]). The solution of this equation by the perturbation method has been proposed by Kunitsyn and Rudenko [4].

We propose a computer code for numerical evaluation formulas from [4]. As experimental measurements are usually made for pressure, the main equation will be rewritten in terms of pressure. The problem geometry will be the same as in [4]. Cylindrical coordinates and a piston plane source with uniform amplitude distribution have been used.

For the sake of analysis, normalization has been made. Pressure has been normalized by pressure amplitude on the source, the longitudinal coordinate σ – by the Rayleigh distance, and the transversal coordinate ξ – by the diameter of source a. Additionally, we have neglected dissipation ($\delta = 0$). The equation to solve is:

$$p_{\sigma t} - \frac{\varepsilon}{2c_0} p_{tt}^2 = \frac{c_0}{2} \Delta_\perp p. \tag{2}$$

The following dimensionless variables have been introduced: p – pressure, ε – the nonlinear constant, σ – the longitudinal coordinate along the direction of beam propagation, $\Delta_\perp$ – the Laplasian over the transversal coordinate ξ, c_0 – the velocity of linear sound propagation.

The solutions of this equation by the perturbation method for the first and the second harmonics result in:

$$p(\sigma,\xi,t) = p^{(1)}(\sigma,\xi) e^{i\omega t} + p^{(2)}(\sigma,\xi) e^{2i\omega t} + \text{c.c.}, \tag{3}$$

$$p^{(1)}(\sigma,\xi) = \int\limits_0^\infty J_1(\lambda)\, J_0(\lambda\xi)\, e^{\left(\frac{1}{4} i\lambda^2\sigma\right)} d\lambda = V(\sigma,\xi), \tag{4}$$

$$p^{(2)}(\sigma,\xi) = -2\varepsilon(ka)^2 \int\limits_0^\sigma \exp\left(-\frac{2i\xi^2}{\sigma-\sigma'}\right) \frac{d\sigma'}{\sigma-\sigma'} \times$$
$$\times \int\limits_0^\infty \left(V(\sigma',\xi')\right)^2 e^{\left(-\frac{2i\xi'^2}{\sigma-\sigma'}\right)} J_0\left(\frac{4\xi\xi'}{\sigma-\sigma'}\right) \xi' d\xi' = -2\varepsilon(ka)^2\, W(\sigma,\xi), \tag{5}$$

where c.c. means complex conjugation (see [4]).

It is supposed that $p^{(1)}(\sigma,\xi) \gg p^{(2)}(\sigma,\xi)$. Equation (4) can be simplified and rewritten as an elementary function in special cases only. There are no convenient representations of expressions (4) and (5) for arbitrary arguments σ, ξ, so that special methods to calculate integrals need to be used. Because of these complications, asymptotes are used for Equations (4) and (5). We propose a method for the calculation of the integrals of the oscillating functions (4) and (5). This work pays special attention to the estimation of calculation error. This is important for the correct comparison of the results of a medium parameters extraction by measuring the amplitudes of harmonics and the results obtained from a theoretical model.

2. Calculation of the first harmonic

For $\xi = 0$, integral (4) can be transformed into a simple analytical formula [4]. The latter will be used for the calculation of a field of the first harmonic on longitudinal axis:

$$V(\sigma,0) = 1 - \exp(i/\sigma). \tag{6}$$

The calculation according to this formula does not involve any numerical method, let us consider therefore the error of the calculation to be negligibly small.

Equation (4) will be used to evaluate the field for $\xi > 0$ (the non-axial case). For the convenience of presentation of the formulas, new notations have been introduced:

$$f(\lambda) = J_1(\lambda)J_0(\lambda\xi), \quad g(\lambda) = \frac{i\lambda^2\sigma}{4}, \quad V(\sigma,\xi) = \int_0^\infty f(\lambda)e^{g(\lambda)}d\lambda. \tag{7}$$

In the area $\xi > 0$, it is reasonable to present the second expression in Equation (7) as a sum of two integrals calculated by different methods:

$$V(\sigma,\xi) = \int_0^L f(\lambda)e^{g(\lambda)}d\lambda + \int_L^\infty f(\lambda)e^{g(\lambda)}d\lambda = I_1(0,L) + I_2(L,\infty). \tag{8}$$

On the interval $[0,L]$, the integral has been calculated by the method of trapezoids with a constant step h_λ:

$$I_1(0,L) = \sum_{i=1}^{N} \int_{\lambda_i}^{\lambda_i+h_\lambda} f(\lambda)e^{g(\lambda)}d\lambda \approx \sum_{i=1}^{N} \frac{h_\lambda}{2}(f(\lambda_i)e^{g(\lambda_i)} + f(\lambda_i+h_\lambda)e^{g(\lambda_i+h_\lambda)}). \tag{9}$$

For the second interval, an asymptotic formula constructed by repeated integrations by parts has been used:

$$\begin{aligned} I_2(L,\infty) &= \int_L^\infty f(\lambda)e^{g(\lambda)}d\lambda = \int_L^\infty \frac{f(\lambda)}{g'(\lambda)}d(e^{g(\lambda)}) = f(\lambda)\frac{e^{g(\lambda)}}{g'(\lambda)}\bigg|_L^\infty - \int_L^\infty e^{g(\lambda)}d\left(\frac{f(\lambda)}{g'(\lambda)}\right) = \\ &= f(\lambda)\frac{e^{g(\lambda)}}{g'(\lambda)}\bigg|_L^\infty - \left(\frac{f(\lambda)}{g'(\lambda)}\right)'\frac{e^{g(\lambda)}}{g'(\lambda)}\bigg|_L^\infty + \int_L^\infty e^{g(\lambda)}d\left(\left(\frac{f(\lambda)}{g'(\lambda)}\right)'\frac{1}{g'(\lambda)}\right) + \ldots \end{aligned} \tag{10}$$

It may be noted, that the results of the subsequent integrations differ from the previous ones by multiplier $1/g'(\lambda)$. For $\sigma\lambda \gg 1$, each subsequent term will be less then the previous one. This enables us to obtain the necessary accuracy of calculation depending on the number of terms and the value of L.

The choice of the value of L in Equation (10) and the size of step h_λ in Equation (9) are closely connected with the problem of the accuracy and time of the calculation. On the one hand, a reduction of L reduces the time of the calculations. On the other hand, the accuracy of the results thus decreases. We have used the following procedure.

Let the accuracy of integration in Equation (8), $\Delta V(\sigma,\xi) = n$, be given and suppose that:

$$\Delta V(\sigma,\xi) = \Delta I_1(0,L) + \Delta I_2(L,\infty) \quad \Delta I_1(0,L) = \frac{1}{2}n, \quad \Delta I_2(L,\infty) = \frac{1}{2}n. \tag{11}$$

Using the second term in Equation (10) as an error estimation of the asymptotic formula, let us find a minimal value of L for the given value of error $\Delta I_2(L,\infty)$:

$$\Delta I_2(L,\infty) = \frac{4J_1(L\xi)J_1(L)L\xi - 4J_0(L\xi)J_0(L)L + 8J_0(L\xi)J_1(L)}{L^3\sigma^2}e^{\frac{1}{4}i\sigma L^2}. \tag{12}$$

As the above equation is transcendent and not solvable analytically with respect to L, an estimation obtained by the following simplifications [5] has been used:

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}}\cos(x - \frac{1}{2}\nu\pi - \frac{1}{4}\pi), \quad |a+b| \le |a| + |b|. \tag{13}$$

The usage of such asymptotes for cylindrical functions essentially simplifies Equation (12). However, it simultaneously results in a drastic decrease of in accuracy for small ξ. Therefore, it is necessary to use Equation (12) to obtain a more realistic value of the accuracy of calculation (not estimation).

The usage of Equation (13) results in the following inequality:

$$|\Delta I_2(L,\infty)| \leq 8\frac{L(\xi+1)+2}{\pi\sigma^2 L^4\sqrt{\xi}}. \tag{14}$$

The value of L is a solution of a set of conditions:

$$8\frac{L(\xi+1)+2}{\pi\sigma^2 L^4\sqrt{\xi}} = \frac{n}{2}, \quad L>0, \quad Im(L)=0. \tag{15}$$

The value of h_λ is calculated from an estimation of the error value of the trapezoids method, by using the values of L and n:

$$\Delta I_1(0,L) = -\frac{h_\lambda^2}{12}(f(L)e^{g(L)} - f(0)e^{g(0)})'. \tag{16}$$

It is possible to select, by an appropriate choice of the parameter n, such a value of step h_λ at which, for one period of oscillations of any integrands $f(\lambda)$ and $e^{g(\lambda)}$, there will be a sufficient number of points for a correct calculation.

As for the calculation of the parameter L, Equation (15), a number of simplifications have been made. It is necessary to apply Equations (12) and (16) directly to obtain the actual error of the calculation $\Delta V(\sigma,\xi)$.

The proposed technique allows us to preserve the accuracy of the calculations while reducing the amount of steps in the trapezoids method (9) for an increasing coordinate σ. It considerably reduces the time of calculation for $\sigma > 0.1$.

3. Calculation of the second harmonic

The field of the second harmonic is described by formula (5) from [4]. Let us consider certain features of the given formula.

Firstly, both integrands are products of functions oscillating rapidly with different speeds; therefore, a special method of numerical calculation must be applied. Secondly, in the integral over ξ', the integrand is a product of the tabulated function $p^{(1)}(\sigma,\xi)$, analytic Bessel's functions and an exponent. As the integration over ξ' and σ' has been carried out along a fixed grid, where the value of $V(\sigma,\xi)$ has been calculated, then the accuracy of calculation of $p^{(2)}(\sigma,\xi)$, along with the chosen numerical method, depends on the quality of tabulation of the function $V(\sigma,\xi)$. Thirdly, as the calculations are carried out on a finite range of coordinates, the upper limit of integration on ξ' – *viz.* infinity – creates the problem of proper selection of a higher limit of the integration.

Equation (5) includes the source and medium parameters. To exclude their influence, only the integral part denoted by $W(\sigma,\xi)$ will be calculated. It differs from the second harmonics by a multiplier, which is constant for a given system. Let us rewrite the part of the expressions from Equation (5) as a sequence in which the integration will be made:

$$I(\sigma,\xi,\sigma') = \int_0^\infty (V(\sigma',\xi'))^2 J_0\left(\frac{4\xi\xi'}{\sigma-\sigma'}\right)\xi' \exp\left(-\frac{2i\xi'^2}{\sigma-\sigma'}\right)d\xi', \tag{17}$$

$$W(\sigma,\xi)=\int\limits_0^{\sigma}\frac{I(\sigma,\xi,\sigma')}{\sigma-\sigma'}\exp\left(-\frac{2i\xi^2}{\sigma-\sigma'}\right)d\sigma'. \tag{18}$$

Using the same numerical scheme for different ranges of the variable of integration for calculating integrals (17) and (18) leads to incorrect results. Formally, there are no singularities in integrand (17), but both construction of an asymptotic series by the integration by parts and integration by the method of stationary phase have failed. Therefore, it has been decided to take advantage of the method offered in [6]. Let $f(x)$, $g(x)$ be functions of a complex argument. Then, it is possible to use a special quadrature formula for the integration:

$$\begin{gathered}I=\int\limits_a^b f(x)e^{g(x)}dx=\sum_{k=1}^{N}\int\limits_{x_k-h/2}^{x_k+h/2}f(x)e^{g(x)}dx\approx\sum_{k=1}^{N}B_k,\\ B_k=\begin{cases}(f(x_k)e^{g(x_k)}h, & |g'(x_k)h|\le 0.1,\\ f(x_k)e^{g(x_k)}\frac{2}{g'(x_k)}sh(g'(x_k)\frac{h}{2}), & |g'(x_k)h|\ge 0.1.\end{cases}\end{gathered} \tag{19}$$

To obtain an estimation of error in the case of $|g'(x_k)h|\ge 0.1$, let us turn to a method of obtaining the formula of integration. Let us expand functions $f(x)$ and $g(x)$ in a Taylor series up to the second and third terms, correspondingly, in the neighborhood of a point x_k, the middle of the interval $[x_k-h/2,x_k+h/2]$:

$$\begin{aligned}\widetilde{B_k}&=\int\limits_{x_k-h/2}^{x_k+h/2}f(x)e^{g(x)}dx\sim\\ &\sim\int\limits_{x_k-h/2}^{x_k+h/2}(f(x_k)+f'(x_k)(x-x_k))e^{g(x_k)+g'(x_k)(x-x_k)+\frac{1}{2}g''(x_k)(x-x_k)^2}dx.\end{aligned} \tag{20}$$

Subsequently, we expand factor $e^{\frac{g''(x_k)}{2}(x-x_k)^2}$ in a Taylor series. Using the first two terms of the expansion superimposes the condition $|\frac{1}{2}g''(x_k)(x-x_k)^2|\ll 1$, which is true for the variable ranges we use:

$$\begin{aligned}\widetilde{B_k}&=\int\limits_{x_k-h/2}^{x_k+h/2}(f(x_k)+f'(x_k)(x-x_k))e^{g(x_k)+g'(x_k)(x-x_k)}\left(1+\frac{g''(x_k)}{2}(x-x_k)^2\right)dx=\\ &=f(x_k)e^{g(x_k)}\int\limits_{x_k-h/2}^{x_k+h/2}e^{g'(x_k)(x-x_k)}dx+f'(x_k)e^{g(x_k)}\int\limits_{x_k-h/2}^{x_k+h/2}(x-x_k)e^{g'(x_k)(x-x_k)}dx+\\ &+f(x_k)e^{g(x_k)}\frac{g''(x_k)}{2}\int\limits_{x_k-h/2}^{x_k+h/2}(x-x_k)^2e^{g'(x_k)(x-x_k)}dx+\\ &+f'(x_k)\frac{g''(x_k)}{2}e^{g(x_k)}\int\limits_{x_k-h/2}^{x_k+h/2}(x-x_k)^3e^{g'(x_k)(x-x_k)}dx.\end{aligned} \tag{21}$$

The first term is an approximate value of the integral, the next are estimations of the error of calculation. The contribution of the last term can be neglected, when compared with the values of the second and the third ones, and $\Delta\widetilde{B_k}$ (the error of this calculation) is expressed as:

$$\Delta\widetilde{B_k} \approx f'(x_k)e^{g(x_k)}\left(\frac{h}{g'(x_k)}ch(\frac{g'(x_k)h}{2}) - \frac{2}{g'(x_k)^2}sh(\frac{g'(x_k)h}{2})\right) + \\ + f(x_k)e^{g(x_k)}\frac{g''(x_k)}{2}\left(\left(\frac{h^2}{2g'(x_k)} + \frac{4}{g'(x_k)^3}\right)sh(\frac{g'(x_k)h}{2}) - \frac{2h}{g'(x_k)^2}ch\left(\frac{g'(x_k)h}{2}\right)\right). \quad (22)$$

It is possible to simplify this formula without significantly altering its accuracy by expansion in a Taylor series on h. Thus, the formula for the error of calculation based on Equation (19) is as follows:

$$\Delta I \approx \sum_{k=1}^{N} \Delta B_k, \\ \Delta B_k = \begin{cases} \frac{1}{2}h^2(f(x_k)e^{g(x_k)})', & |g'(x_k)h| \leq 0.1\,, \\ \big(2f'(x_k)g'(x_k) + f(x_k)g''(x_k)\big)\frac{1}{24}e^{g(x_k)}h^3, & |g'(x_k)h| \geq 0.1\,. \end{cases} \quad (23)$$

While numerical integration can be performed with greater accuracy by other numerical schemes, the advantages of formulas (19) and (23) are the simplicity of program realization and the absence of calculation of higher derivatives from the tabulated function $V(\sigma,\xi)$.

As the calculation of the first harmonic has been made with some error, the latter involves an additional error in the calculation of the field of the second harmonic, which is necessary to take into account when calculating Equations (17) and (18). Let function $\varphi(x) = f(x)e^{g(x)}$ be preset with accuracy $\Delta\varphi(x)$. After numerical integration, it results in:

$$\int(\varphi(x) + \Delta\varphi(x))dx = \int\varphi(x)dx + \int\Delta\varphi(x)dx \simeq \\ \simeq \sum_k B_k(\varphi(x)) + \sum_k B_k(\Delta\varphi(x)) + \sum_k \Delta B_k(\varphi(x)) + \sum_k \Delta B_k(\Delta\varphi(x)). \quad (24)$$

The value of error is determined by the last three terms: the error of the numerical method of calculation of an integral from function $\varphi(x)$, the value of an integral from function $\Delta\varphi(x)$, and the error of the method of calculation of this integral.

Equations (19), (23) and (24) represent a set of formulas for evaluation of the field of the second harmonic (5) and an estimation of the accuracy of results.

4. Grid parameters

The selection of the range of coordinates σ and ξ is determined by several factors. As in Equation (4), the coordinates are involved as parameters, the value of $V(\sigma,\xi)$ can be calculated at any point (σ,ξ), *i.e.* calculation on grid (σ,ξ) of any size with arbitrary values of the size of steps h_σ, h_ξ is possible. The only limitation for the interval on σ is applicability of the perturbation theory. Equations (4) and (5) are correct at the length about the Rayleigh distance:

$$r_0=\frac{1}{2}ka^2, \quad k=\frac{\omega}{c_0}. \tag{25}$$

The variable σ in Equations (4) and (5) is already normalized by this value, hence the upper boundary $\sigma_{\max}$ should be of the order of 1. The lower boundary $\sigma_{\min}$ can be determined as the minimal distance for the fixed grid pitch h_σ at which the values in two adjacent points correctly interpolate the behavior of function (4). As an estimation, we will use an absolute value of the expression (6):

$$V(\sigma,0)=2\left|\sin((2\sigma)^{-1})\right|. \tag{26}$$

The frequency of oscillations increases with the argument approaching zero and, at some range of coordinates σ, the values of the function in adjacent points of a grid do not reflect the actual behavior of the function. The criterion for calculating the minimum value of the coordinate, $\sigma_{\min}$, at fixed pitch of a grid h_σ will be the difference of arguments of a function sine in two adjacent points of the grid:

$$\sigma_{\min}>-\frac{h_\sigma}{2}\left(1-\sqrt{1+\frac{2}{h_\sigma\pi}}\right). \tag{27}$$

Due to the limitation of the area of the applicability of the KZK Equation [7], there is a minimal distance at which this equation becomes correct. It depends on the wave length and the size of the source, so that calculations at $\sigma_{KZK}<(ka)^{(1/3)}a/r_0$ are not physically correct. For a typical ultrasonic source at values of h_σ of the order of 10^{-2}, the inequality $\sigma_{\min}<\sigma_{KZK}$ is true, and our algorithm mathematically correctly covers the region where the basic Equation (2) fails. It is necessary to note that Equation (2) is not only unusable near the source, but experimental measurements also are not carried out in this area.

The selection of an interval for the construction of a grid on the transversal coordinate, ξ, is rather easy. From the slow change of the profile of a beam in the longitudinal direction follows its low transversal divergence: it is thus necessary to consider distances not greatly exceeding the size of the source. Taking into account the normalization of ξ by the size of the source, a, in Equations (4) and (5), we obtain the following: $\xi_{\min}=0$ and $1<\xi_{\max}<10$. Function (4) is also oscillating on the coordinate ξ, but there are no convenient formulas to estimate the speed of these oscillations, and the size of step h_ξ has been selected from the results of calculations. It will be shown in the discussion section that the speed of oscillation decreases with increasing σ or ξ.

The results of the first harmonic calculation are used to evaluate the second one. The accuracy of the calculation of $W(\sigma,\xi)$ depends not only on the numerical method used, but also on the quality of tabulation of $V(\sigma,\xi)$, the ranges of the grid and its pitch. As shown above, the calculation of $V(\sigma,\xi)$ is possible at any values of σ, ξ and any size of steps h_σ, h_ξ. In the case of the calculation by Equation (18), the numerical integration over σ', ξ' has been made on the same grid that the values of $V(\sigma,\xi)$ were obtained on. Therefore, for the calculation of integral (17), the upper limit is replaced by $\xi_{\max}$. Because of low beam divergence, it is possible to consider that $V=0$ at