NUMERICAL SIMULATION
OF THE CHAOTIC BEHAVIOUR
OF A THREE-DIMENSIONAL PENDULUM

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Abstract: A nonlinear pendulum is designed to demonstrate the chaotic instability of trajectories. Here, we present a simplified theoretical description of its dynamics. Trajectories are found by numerical integration of the Lagrange equations. The results of the simulations agree with the Poincaré-Bendixon theorem. Generic trajectories display chaotic behaviour and are similar to those obtained experimentally.

Keywords: chaos, dynamical systems

1. Introduction

The aim of this note is to provide a numerical solution of the equations of motion of a three-dimensional pendulum designed for a didactic purpose to demonstrate chaotic behaviour [1]. We have found that the solution is a transparent demonstration of the Poincaré-Bendixon theorem [2]. One of the consequences of this theorem is that chaos is not possible in two-dimensional phase space. The phase space of our pendulum is of $2N-2=4$ dimensions, where -2 comes from two constants of motion: the energy and the z-th component of the angular momentum. Below, we will show that chaos persists in the remaining four-dimensional phase space even if the energy connected with one of the degrees of freedom is very large. However, once this latter degree is finally quenched, chaos disappears.

The pendulum is constructed as shown schematically in Figure 1. A vertical bar $B3$ is placed so as to rotate freely around the vertical axis. A rigid horizontal rod $B2$ of length $L2+L3=0.16\,\mathrm{m}$ is fixed to the bar. A counterbalance of mass $M2=0.3\,\mathrm{kg}$ is placed to one end of the rod $B2$. At the other end, a pendulum is fastened so as to rotate only within the vertical plane perpendicular to the rod $B2$. The pendulum consists of an elastic rod $B1$ of length $L1=0.55\,\mathrm{m}$, with a cone C ($M1=0.47\,\mathrm{kg}$) at its lower end. The elastic rod is equivalent to a spring with the elastic coefficient k
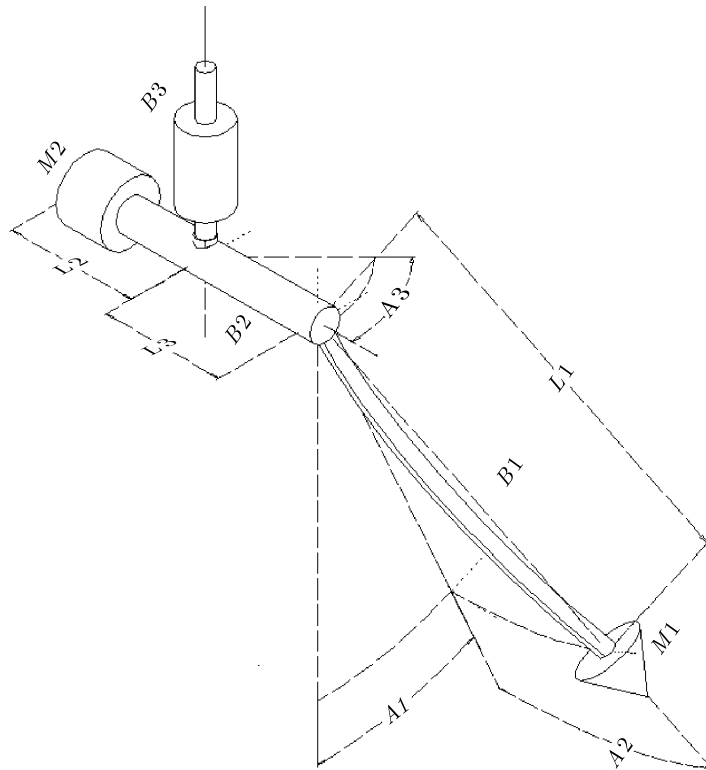
Figure 1. Schematic construction of the pendulum

about 20 N/m. The only external force is gravitation. Dissipation is reduced by means of ball bearings.

Basically, there are four degrees of freedom: the angle $A3$ of rotation within the horizontal plane around the vertical bar $B3$, the angle $A1$ of rotation of the pendulum around the horizontal axis of $B2$, and two angles of elastic deflection of the elastic rod $B1$. However, for moderate or strong stiffness constant k of this rod one of the latter angles can be omitted, because the respective deflection is within the plane of rotation of the pendulum, and the energy of the elastic deformation is much larger than the energy of the free oscillation of the pendulum. Then, we effectively have a three-dimensional system: $A1$, $A3$ and Δ. The third degree of freedom is the elastic deflection Δ in the vertical plane containing the bars $B2$ and $B3$.

The construction is inspired by (and somewhat similar to) the one described by Clerc *et al.* [3]. The important differences are as follows:

- in our system, the fixing point of the pendulum P is out of the vertical axis,
- our system is autonomous, and
- in the Clerc *et al.* system there is no elastic deflection.

The cone C is empty and it can be filled with fine sand or dry salt. During the experiment the sand pours out through a small hole, marking the trace of the cone. In Figure 2 we see an example of such a trace – a photograph from the experiment.
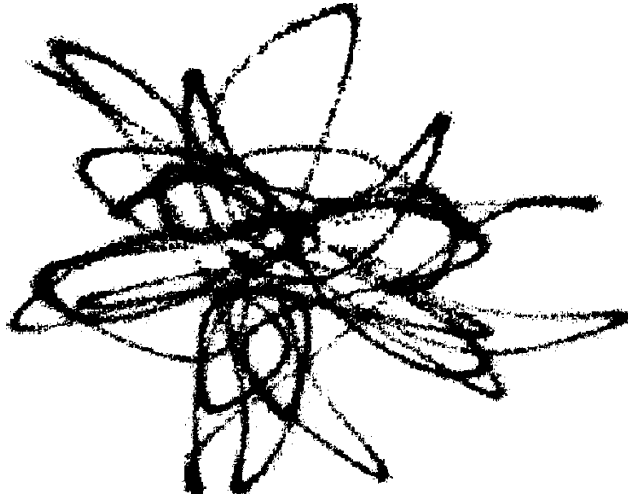
Figure 2. Horizontal projection of an experimental trajectory – the trace of sand

2. Calculations

Both the counterbalance and the cone are treated as point masses. The Lagrangian of the cone C is

$$L_1 = \frac{M1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x,y,z), \tag{1}$$

where x, y, z are the coordinates of the cone:

$$\begin{aligned}
x &= (L3/2 + \Delta)\cos(A2) + L1\sin(A1)\sin(A2), \\
y &= (L3/2 + \Delta)\sin(A2) + L1\sin(A1)\cos(A2), \\
z &= -L1\cos(A1),
\end{aligned} \tag{2}$$

and its potential energy is

$$U = mgz + \frac{1}{2}k\Delta^2. \tag{3}$$

The Lagrangian for the counterbalance is merely its kinetic energy:

$$L_2 = \frac{M2\ L2^2}{2}(\dot{A3})^2. \tag{4}$$

This approximated form neglects the gravitational energy connected to the deflection Δ. After several tests, we know that our conclusions are not influenced by this simplification.

The Lagrangian equations are transformed to six first-order equations, and they are then solved numerically with the Runge-Kutta method of the 4$^{\text{th}}$ order. The calculations are performed for several values of the coefficient k, in the range from 10^{-5} till 10^6. For completeness, we have repeated the calculations for the case of an infinitely stiff rod $B1$, *i.e.* the deflection Δ was kept to zero. In all the cases we have also calculated the largest Lyapunov coefficient by the method described in [4].

3. Results and discussion

In Figures 3a and 3b, we have shown typical time dependence of the difference between two trajectories which are initially very close. The inclination of the plot

measures the Lyapunov exponent. On the other hand, the time scale can be determined from the experiment – the period of the oscillations is about 1 second. Then we can determine the Lyapunov exponent λ in s^{-1}. It is calculated as an average over not less than 20 different initial points, selected randomly in the phase space. Although the statistical error remains at about 0.3 of the mean value, we have never encountered a negative value of the exponent. In Figure 4 we see that its mean value is in the range 0.5–$5.0\,s^{-1}$, as long as $k < 4 \cdot 10^3$. Above this value, we see that λ increases sharply. We treat this result as a numerical artifact, because for this range of k the results start to depend on the numerical accuracy. On the other hand, the Lyapunov exponent is zero for the case when the deflection Δ is quenched. This can be deduced from the plot in Figure 3b, which cannot be treated as linear.
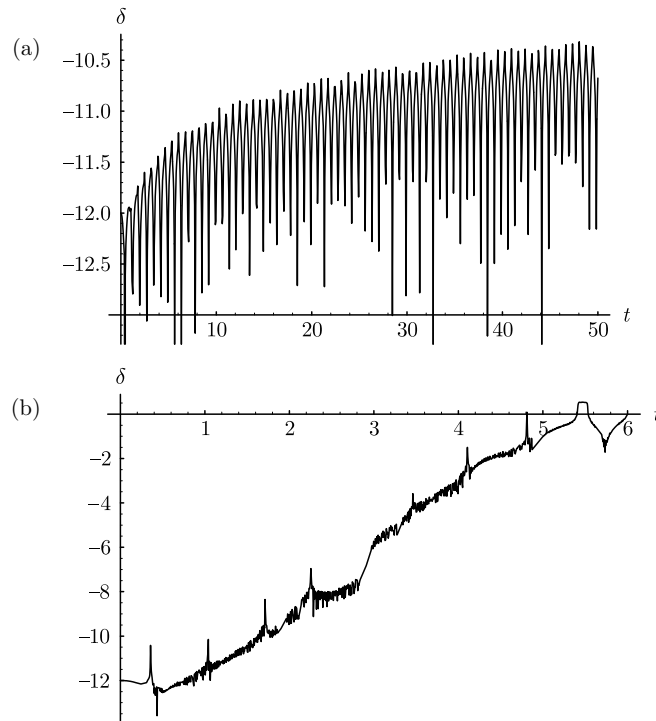

Figure 3. Time dependence of a calculated difference (δ) between two initially neighboring trajectories: (a) for the three-dimensional pendulum, (b) for the two-dimensional pendulum (Δ is quenched); the vertical axis is in logarithmic scale

The trajectory shown in Figure 5 seems to be generic for the case when $\Delta \neq 0$, in the sense that most initial conditions give results which are qualitatively similar. At the same time, the plot is also similar to the experimental trace in Figure 2. The essence of chaos is that it is not possible to reproduce the experimental plot.

4. Final remarks

Concluding, our results provide an example of how the number of variables influences the character of the time evolution. These results are not in contradiction
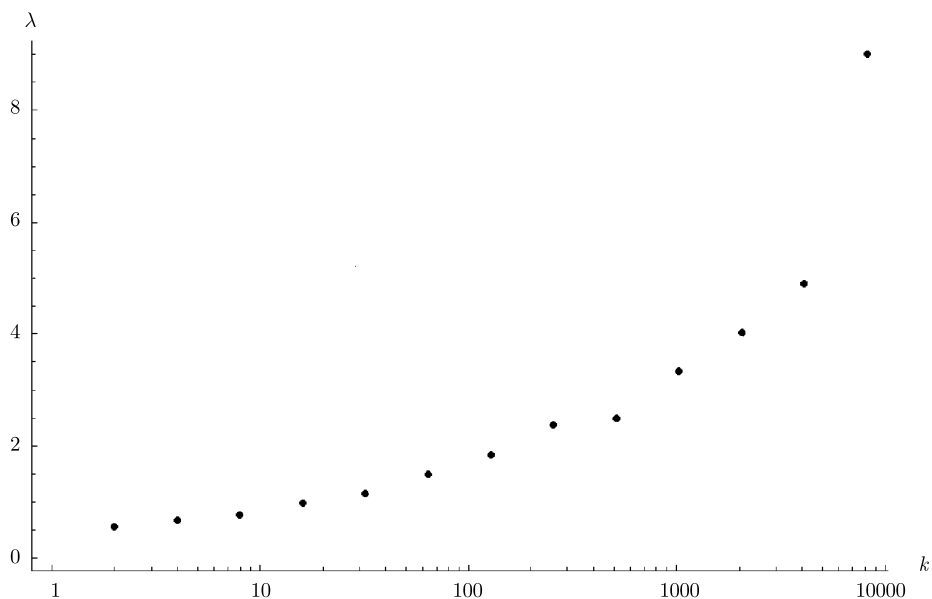
Figure 4. The largest Lyapunov exponent, calculated as dependent on the elastic coefficient k for the three-dimensional pendulum.
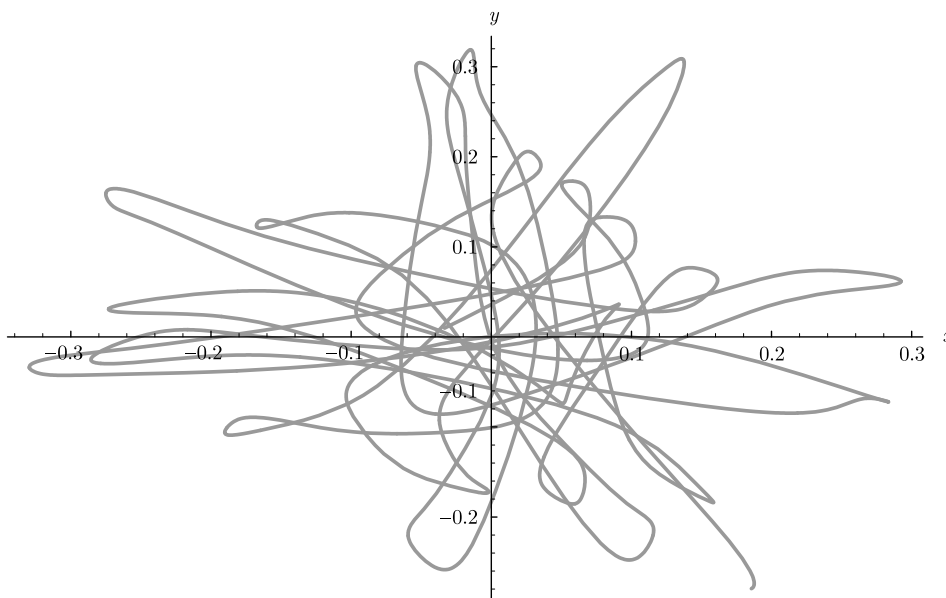

Figure 5. Horizontal projection of a calculated trajectory for the three-dimensional pendulum

with the Poincaré-Bendixon theorem. We note that although we have the experiment in hand, our demonstration is numerical and not experimental. To demonstrate the case experimentally, we should build a pendulum with a very stiff $B1$ rod. Having built it, we would likely learn that other of its parts become non-rigid: the ball bearings or the whole mounting. Contrary to the calculation, it is much easier to construct a device with many degrees of freedom.

References

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