ANALYSIS OF PRESSURE EVOLUTION IN GRANULAR MATERIALS IN CONVERGING BINS DURING FILLING AND EMPTYING PROCESSES

IRENA SIELAMOWICZ[1] AND ZENON MRÓZ[2]

[1] *Faculty of Civil and Environmental Engineering, Bialystok Technical University, Wiejska 45E, 15-351 Bialystok, Poland*
sielirena@pb.bialystok.pl

[2] *Institute of Fundamental Technological Research, Polish Academy of Sciences, Świętokrzyska 21, 00-049 Warsaw, Poland*
Zenon.Mroz@ippt.gov.pl

Abstract: A simplified analysis of deformation and stress states in converging hoppers during filling and discharge of granular material is presented. In particular we discuss a method for solving the set of differential equations governing the flow of granular material in a plane wedge hopper. The equilibrium conditions and stress-strain relations are satisfied for the planar slice elements assuming the dependence of displacement and stress on the Cartesian coordinate z. The transient flow of an icompressible, non-cohesive granular material in a two-dimensional converging hopper is considered. We assume the material to be in elastic or elasto-plastic states within the hopper satisfying the Coulomb yield condition and the non-associated flow rule. The paper presents a detailed analysis of the evolution of pressure acting on the hopper wall during the filling and emptying processes when the initial active state of pressure is transformed into a passive state. Analytical and numerical analyses are presented. It is shown that at the initial stage of the emptying process a considerable switch overpressure develops, exceeding the steady-state passive pressure several times.

Keywords: granular materials, pressure, elastic and elasto-plastic model, filling, emptying

1. Introduction

During filling, discharge and storage of granular materials in silos it is important to understand numerous problems, such as the evolution of pressures on silo walls, the modes of flow during filling and discharge of the material, particle segregation, the effect of vibration and aeration, *etc.* Various simplified material models are usually used in theoretical treatments of these problems. Here, we assume granular material to be linear, elastic and satisfying Hooke's law, or perfectly plastic and satisfying the Coulomb yield condition and the non-associated flow rule. A simplified geometry of the silo has been assumed in this theoretical analysis by considering plane converging

hoppers. Alternative numerical and analytical approaches to granular flow problems, using the finite element or the characteristics method, are also presented in the literature [1–12]. There is a large group of papers related to flow in converging channels based on the assumption of radial flow velocity and steady or transient state of flow [13–18]. By accounting for gravity and inertial forces, the rate of discharge of the material can be determined [19–25] assuming a rigid, perfectly plastic material model. Similarly, the stress state and material pressure on hopper walls can be determined [26–38]. Following the work of Jenike, where the density hardening model was introduced [39–41], it has become possible to predict the required outlet area for continuing flow and to more realistically characterize important material parameters, in particular cohesion varying with material density and the critical state parameters.

The present paper presents a simplified analysis of material pressure acting on the hopper wall during the filling and emptying processes in converging channels. Some simplifying assumptions have been made for the sake of this analysis. Vertical velocity and displacement fields have been assumed, so that the stress field depends only on the vertical coordinate, and the material-wall interaction is treated by introducing tractions at the interface into the equilibrium conditions. The elastic and elasto-plastic material model is assumed, satisfying the Coulomb yield condition, and the non-associated flow rule, satisfying the incompressibility condition. Under these assumptions, the state of material after filling and the transient states during emptying are analysed in detail. Two stress states are shown: the initial stress state after filling, which varies considerably, from the so-called "active pressure state" to the "passive pressure state", with the transition of the major principal stress in the vertical direction to the major stress in the horizontal direction during the emptying process. The transition from the filling stage to the emptying stage is achieved by assuming a variation of boundary conditions at the bottom boundary, namely from the vanishing vertical displacement to imposed displacement, usually dependent on a controlled emptying procedure. The analysis, though based on simplified assumptions, describes the evolution of stress and wall pressure as a function of the material parameters of granular materials.

2. Fundamental equations of the problem

2.1. Constitutive equations

The constitutive relations and equilibrium or strain-displacement equations will be formulated in the Cartesian x, y, z system. The small strain theory is used with the usual linear relations between strain and displacements. Compressive stresses and contractive strains are assumed to be positive, as is usually assumed in soil mechanics. The stress and strain tensors are decomposed into deviatoric and spherical parts, as follows:

$$\sigma_{ij} = s_{ij} + p\delta_{ij}, \qquad \varepsilon_{ij} = e_{ij} + \frac{1}{3}\varepsilon_\nu\delta_{ij}, \tag{1}$$

where $p = \frac{1}{3}\sigma_{kk}$, $\varepsilon_\nu = \varepsilon_{kk}$, and δ_{ij} is the Kronecker delta.

Let us consider an elastic-perfectly, plastic model of the material (Figure 1). Here, we neglect the effect of density hardening and softening and the critical state regime, typical for granular materials and powders.
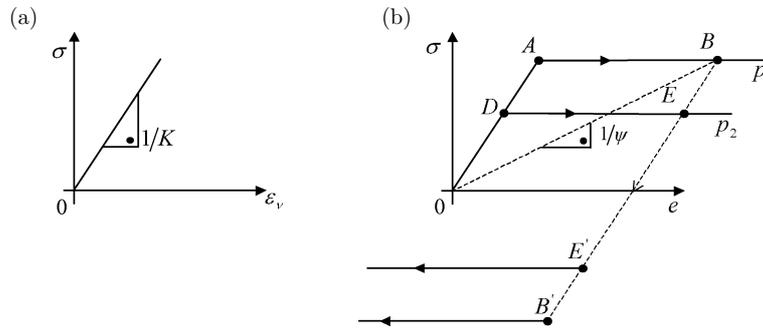
Figure 1. Material models: (a) linear elastic and (b) elasto-plastic

In the elastic state Hooke's law applies, so that, for an isotropic material, we have:

$$\varepsilon_{ij} = K\sigma\delta_{ij} + \frac{1}{2G}s_{ij}, \qquad i,j = 1,2,3, \tag{2}$$

where K is the bulk compliance modulus and G is the shear stiffness modulus of the material. We also have the familiar relations:

$$K = \frac{1-2\nu}{E}, \qquad G = \frac{E}{2(1+\nu)}, \tag{3}$$

where E and ν are the Young modulus and the Poisson ratio, respectively.

In the plastic regime, the Coulomb yield condition for a non-cohesive material is assumed to be valid; thus:

$$f(\underline{\sigma}) = (\sigma_1 - \sigma_2) - (\sigma_1 + \sigma_3)\sin\varphi \leq 0, \tag{4}$$

where $\sigma_1 \geq \sigma_2 \geq \sigma_3 > 0$ are the principal stresses, and φ denotes the friction angle, which is assumed to be constant. The deformation theory is applied, so that the finite stress-strain relations can be assumed in the elasto-plastic regime:

$$\varepsilon_{ij} = K\sigma\delta_{ij} + \psi s_{ij}, \tag{5}$$

where $\psi > 0$, a secant compliance modulus dependent on the plastic strain value. In the elastic regime $\psi = 1/2G$, and in the plastic regime $\psi = ||\underline{e}||/||\underline{s}||$.

The analysis will be performed for plane strain and strain states with stress components $[\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz} = 0]$. For the plane case it is assumed that $\sigma_z = \sigma_3$ is the intermediate principal stress, $\sigma_1 \geq \sigma_z = \sigma_3 \geq \sigma_2$; the Coulomb yield condition in the general stress state can now be written as follows:

$$F_1(\underline{\sigma}) = (\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 - \sin^2\varphi(\sigma_x + \sigma_y)^2 \leq 0. \tag{6}$$

Let us consider a physical plane or surface Π with unit normal and tangent vectors $\underline{n}$ and $\underline{t}$. Let us assume that the traction is specified on Π, so that the stress components σ_n, τ_{nt} are given. From the yield condition, written in the local reference system n, t,

$$F_3(\underline{\sigma}) = (\sigma_n - \sigma_t)^2 + 4\tau_{nt}^2 - \sin^2\varphi(\sigma_n + \sigma_t)^2 \leq 0 \tag{7}$$

the value of σ_t can be obtained, provided that $|\tau_{nt}| < \sigma_n \tan\varphi$. There are two solutions for σ_t in the plastic state, namely:

$$\sigma_t = \frac{(1+\sin^2\varphi)\sigma_n \mp 2\sqrt{\sigma_n^2\sin^2\varphi - \tau_{nt}^2\cos^2\varphi}}{\cos^2\varphi}, \tag{8}$$

where $\sigma_t^{(+)}$ is the maximal state and $\sigma_t^{(-)}$ – the minimal state. They are illustrated by the Mohr circle in Figure 2. In this figure, the stress pole position is at P, and two stress circles tangent to the Coulomb envelope specify the stress state at the surface Π. If the stress circles are not tangent to the envelope, the elastic or rigid state occurs at the surface Π. If Π separates two material domains and is the interface separating maximal and minimal plastic states or both sides, then it is the stress discontinuity surface. The components σ_n and τ_{nt} are continuous on Π, but σ_t suffers discontinuity, so we have:

$$[\sigma_n] = 0, \qquad [\tau_{nt}] = 0, \qquad [\sigma_t] = \frac{4\sqrt{\sigma_n^2 \sin^2\varphi - \tau_{nt}^2 \cos^2\varphi}}{\cos^2\varphi}, \qquad (9)$$

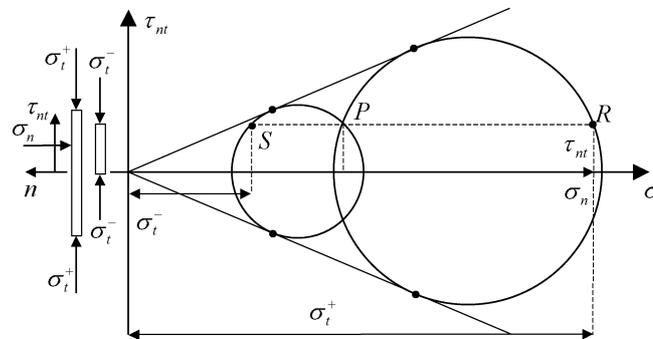where [ ] denote the discontinuity of the enclosed symbols.

Figure 2. Stress state on both sides of the stress discontinuity line

When considering the pressure of granular material on silo walls, we distinguish passive and active regimes. For a passive regime, the normal pressure σ_n is greater than σ_t (the minimal state), and for an active regime σ_n is smaller than σ_t. In fact, the active pressure on the wall is usually several times smaller than the pressure in the passive regime. Active states are usually generated during filling the hopper while passive states develop during emptying. However, there is a transient state when the emptying process starts and the upper part of the hopper is still in the active state, with the passive regime progressing towards the upper boundary. This transient state is most dangerous to the structure containing the granular material, as a travelling pressure peak develops at the interface Π between the active and passive regimes and moves towards the upper boundary.

The present analysis is aimed at describing this pressure redistribution during the transient stage. For a rigid-plastic material, the interface Π constitutes the stress discontinuity surface and the effect of pressure peak can be easily traced.

2.2. Static and kinematic equations

Referring to Figure 3, let us consider a plane wedge-shaped hopper. Let us introduce the Cartesian x, y, z reference system with the z-axis coinciding with the symmetry axis of the hopper and the y-axis perpendicular to the plane of flow. The displacement field is assumed to be as follows:

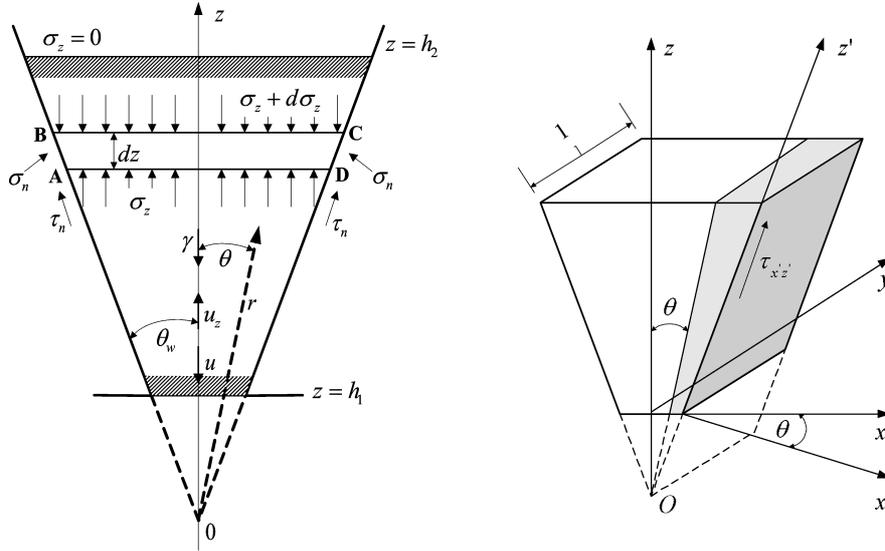$$u_z = -u(z), \qquad u_x = 0, \qquad u_y = 0. \qquad (10)$$

Figure 3. Vertical flow in the hopper: stresses acting on element ABCD

Now, the strain components are:

$$\varepsilon_z = \frac{1-\nu^2}{E}\left[\sigma_z - \frac{\nu}{1-\nu}\sigma_x\right] = \frac{du_z}{dz},$$

$$\varepsilon_x = \frac{1-\nu^2}{E}\left[\sigma_x - \frac{\nu}{1-\nu}\sigma_z\right] = \frac{du_x}{dx} = -\frac{u_z}{z} = \frac{u}{z}. \tag{11}$$

The constitutive relations for the elasto-plastic regime now have the following form:

$$\varepsilon_z = K\sigma + \psi s_z, \qquad \varepsilon_x = K\sigma + \psi s_x, \qquad \varepsilon_y = K\sigma + \psi s_y,$$

$$\gamma_{x'z'} = 0 = \psi\tau_{z'x'} \qquad \gamma_{z'x'} = \gamma_{xy} = \gamma_{yz} = 0, \tag{12}$$

and for the elastic case there is $\psi = 1/2G$. The yield condition is now as follows:

$$f_3(\underline{\sigma}) \equiv (\sigma_z - \sigma_x)^2 - \sin^2\varphi(\sigma_z + \sigma_x)^2 \le 0, \tag{13}$$

since σ_z, σ_x and σ_y are assumed to be principal stresses, with σ_y being the intermediate stress, so that $\sigma_z \le \sigma_y \le \sigma_x$ or $\sigma_x \le \sigma_y \le \sigma_z$. The assumption $\tau_{x'z'} = 0$, $\tau_{xy} = 0$, $\tau_{yz} = 0$ results from the assumption of vertical flow, as then the shear strain component is $\gamma_{z'x'} = 0$. This assumption is certainly not valid near the hopper wall, where frictional stress induces considerable shear, but is strictly valid at the hopper axis, $\theta_w = 0$. The yield condition (13) can also be stated as:

$$\frac{1}{k}\sigma_z \le \sigma_x \le k\sigma_z, \qquad k = \frac{1+\sin\varphi}{1-\sin\varphi}, \tag{14}$$

and the equality sign occurs in the plastic state.

Let us now consider the equilibrium conditions. As it is assumed that $\tau_{x'z'} = 0$, there is $\sigma_z = \sigma_z(z)$, $\sigma_x = \sigma_x(z)$ and $\sigma_y = \sigma_y(z)$. The friction stresses acting on the wall will be treated as reaction stresses, not included in the constitutive equations. These friction stresses satisfy the Coulomb friction condition:

$$\tau_n = \sigma_n \tan\mu, \qquad \mu = \text{const.} \tag{15}$$

Now we can consider the equilibrium of an element ABCD shown in Figure 3, bounded by circular segments z and $z + dz$ and radial lines $\theta = \mp\theta_w$. Accounting for shear

stresses at the wall and the specific gravity forces $\gamma = \gamma(z)$, the equilibrium equations for the wedge hopper take the following form:

$$\begin{cases} \sum z = 0 & \sigma_z(z\tan\theta_w) - \left(\sigma_z + \dfrac{d\sigma_z}{dz}dz\right)(z+dz)\tan\theta_w + \sigma_n\dfrac{dz}{\cos\theta_w}\sin\theta_w + \\[2mm] & \quad + \mu\sigma_n\dfrac{dz}{\cos\theta_w}\cos\theta_w - \gamma z\tan\theta_w dz = 0 \\[2mm] \sum x = 0 & \sigma_x dz - \sigma_n\dfrac{dz}{\cos\theta_w}\cos\theta_w + \tau_n\dfrac{dz}{\cos\theta_w}\sin\theta_w = 0 \end{cases} \tag{16}$$

These two equations can now be written as

$$\frac{d\sigma_z}{dz} + \frac{1}{z}(\sigma_z - \alpha_1\sigma_x) + \gamma = 0, \tag{17}$$

where

$$\alpha_1 = \frac{(\mu + \tan\theta_w)}{(1 - \mu\tan\theta_w)\tan\theta_w}. \tag{18}$$

It follows from the strain-displacement relations (11) that the compatibility conditions take the form:

$$\frac{d}{dz}(z\varepsilon_x) - \varepsilon_z = 0. \tag{19}$$

Using the constitutive Equations (12), the stress components in Equation (19) can be substituted to obtain:

$$\frac{d}{dz}[(K-\psi)\sigma + \psi\sigma_x] + \frac{1}{z}\psi(\sigma_x - \sigma_z) = 0, \quad (K-\psi)\sigma + \psi\sigma_y = 0. \tag{20}$$

Pressure σ and stress σ_y can be expressed in terms of σ_z, σ_x and ψ thus:

$$\sigma = \frac{\psi(\sigma_z + \sigma_x)}{2\psi + K}, \qquad \sigma_y = \frac{(\psi - K)(\sigma_z + \sigma_x)}{2\psi + K}, \tag{21}$$

and the first expression of Equation (20) can be written as:

$$\frac{d}{dz}\left[\psi\frac{(K-\psi)\sigma_z + (\psi + 2K)\sigma_x}{2\psi + K}\right] + \frac{1}{z}\psi(\sigma_x - \sigma_z) = 0. \tag{22}$$

The constitutive equations are based on the assumption that plastic strain is incompressible, so that there is no significant density variation in the material. It is therefore assumed that $\gamma = $ const during the processes of filling and emptying. Actually, this is a physically inaccurate assumption, since the material is subject to compaction during filling and exhibits dilatancy during emptying.

The displacement field can be expressed in terms of the stress components by using the constitutive equations. Then, in view of Equation (21), we have:

$$u = z\varepsilon_x = z(K\sigma + \psi s_x) = z[(K-\psi)\sigma + \psi\sigma_x]. \tag{23}$$

Let us now discuss the boundary conditions. It has been assumed that the upper surface, $z = z_2$, is free, so the vertical stress vanishes, $\sigma_z(z_2) = 0$.

The bottom surface, $z = z_1$ is rigidly supported during the filling process, but during the emptying process displacement is induced, thus controlling the intensity of discharge, so that we have:

$$\begin{aligned} u(z_1) &= 0 \quad - \text{filling stage,} \\ u(z_1) &= u_1 \quad - \text{discharge stage.} \end{aligned} \tag{24}$$

The bottom surface, $z = z_1$, is assumed to be fixed during both stages.

At the hopper walls, for $\theta = \mp\theta_w$ there is friction stress, $\tau_n = \sigma_x(z)\tan\mu$, but it is included in the equilibrium equations, so there are no boundary conditions stated for $\theta = \mp\theta_w$. In fact, radial flow satisfies the kinematic constraints of the walls.

As elastic and plastic zones co-exist the hopper, the usual continuity conditions are specified at the interfaces:

$$z = \eta, \qquad [\sigma_z] = 0, \qquad [u_z] = 0. \tag{25}$$

2.3. Non-dimensional form of equations

By introducing non-dimensional variables $\bar{z} = \frac{z}{z_1}$, $\bar{\eta} = \frac{\eta}{z_1}$, the stress and displacement components can be reduced to a non-dimensional form:

$$\bar{\sigma}_z = \frac{\sigma_z}{\gamma h_1}, \qquad \bar{\sigma}_x = \frac{\sigma_x}{\gamma h_1}, \qquad \bar{\sigma}_y = \frac{\sigma_y}{\gamma h_1}, \qquad \bar{\tau}_n = \frac{\tau_n}{\gamma h_1}, \qquad (\gamma = \text{const})$$

$$\bar{u} = \frac{E}{\gamma h_1} \cdot \frac{u}{h_1}, \qquad \bar{\psi} = 2G\psi = \frac{E}{1+\nu}\psi, \tag{26}$$

and the fundamental Equations (16), (19)–(23) are now as follows:

$$\frac{d\bar{\sigma}_z}{d\bar{z}} - \frac{\alpha_1 \bar{\sigma}_x - \bar{\sigma}_z}{\bar{z}} + 1 = 0,$$

$$\frac{d}{d\bar{z}}\left[\bar{\psi}\frac{\bar{\psi}(\bar{\sigma}_x - \bar{\sigma}_z) + \nu_1(2\bar{\sigma}_x + \bar{\sigma}_z)}{2\bar{\psi} + \nu_1}\right] + \frac{1}{\bar{z}}\bar{\psi}(\bar{\sigma}_x - \bar{\sigma}_z) = 0,$$

$$\bar{\sigma}_y = \frac{(\bar{\psi} - \nu_1)(\bar{\sigma}_z + \bar{\sigma}_x)}{2\bar{\psi} + \nu_1}, \tag{27}$$

$$\bar{r}_n = \bar{\sigma}_x \tan\mu,$$

$$\bar{u} = \bar{z}(1+\nu)\bar{\psi}\frac{\bar{\psi}(\bar{\sigma}_x - \bar{\sigma}_z) + \nu_1(2\bar{\sigma}_x + \bar{\sigma}_z)}{2\bar{\psi} + \nu_1},$$

where $\alpha_1 = \frac{(\mu + \tan\theta_w)}{(1 - \mu\tan\theta_w)\tan\theta_w}$, $\nu_1 = \frac{1-2\nu}{1+\nu}$, and ν – the Poisson ratio.

In Equations (27) we have for the elastic region $\bar{\psi} = 1$ and $\bar{\psi} > 1$ for the elasto-plastic state.

3. Solutions in the elastic and elasto-plastic regions

3.1. The elastic solution

Let us first discuss the elastic solution. After setting $\bar{\psi} = 1$ in (27) we have:

$$\frac{d\sigma_z}{dz} - \frac{\alpha_1\sigma_x - \sigma_z}{z} + 1 = 0,$$

$$\frac{d}{dz}\left[\frac{(\sigma_x - \sigma_z) + \nu_1(2\sigma_x + \sigma_z)}{2 + \nu_1}\right] + \frac{1}{z}(\sigma_x - \sigma_z) = 0. \tag{28}$$

Non-dimensional quantities are used and the dash over the symbols is omitted. The set (28) can be rewritten as follows:

$$\frac{d\sigma_z}{dz} - \frac{\alpha_1\sigma_x - \sigma_z}{z} + 1 = 0,$$

$$\frac{d\sigma_x}{dz} - \frac{1}{z}\sigma_z + \frac{1}{z}\frac{1 - \nu\alpha_1}{1 - \nu}\sigma_x + \frac{\nu}{1 - \nu} = 0, \tag{29}$$

while the general integrals have the following form:

$$\sigma_z^e = C_1 z^{k_1} + C_2 z^{k_2} + \frac{\nu-2}{1-\nu}\frac{1}{1+a+b}\cdot z,$$
$$\sigma_x^e = \frac{C_1(k_1+1)}{\alpha_1}z^{k_1} + \frac{C_2(k_2+1)}{\alpha_1}z^{k_2} + \frac{1}{\alpha_1}\left(\frac{\nu-2}{1-\nu}\frac{2}{1+a+b}+1\right)z,$$

(30)

for $\alpha_1 \neq \frac{\mu+\tan\theta_w}{(1-\mu\tan\theta_w)\tan\theta_w} \neq 0$ and $\alpha_1(1-\nu)(1+a+b) \neq 0$, where C_1, C_2 are the integration constants and the remaining symbols are defined as follows:

$$a = \frac{2-\nu}{1-\nu} - \frac{\nu}{1-\nu}\tan(\varphi+\theta_w)\cot\theta_w, \qquad b = \frac{1}{1-\nu}\left[1-\tan(\varphi+\theta_w)\cot\theta_w\right],$$
$$k_1 = \frac{-a-\sqrt{\Delta}}{2}, \qquad k_2 = \frac{-a+\sqrt{\Delta}}{2}, \qquad \Delta = a^2 - 4b.$$

(31)

The remaining stress functions are expressed as:

$$\sigma_y^e = \frac{1-\nu_1}{2+\nu_1}\left[C_1\left(1+\frac{k_1+1}{\alpha_1}\right)z^{k_1} + C_2\left(1+\frac{k_2+1}{\alpha_1}\right)z^{k_2} + \left(\frac{\nu-2}{1-\nu}\frac{1}{1+a+b}\left(1+\frac{2}{\alpha_1}\right)+\frac{1}{\alpha_1}\right)z\right],$$

$$\tau_n^e = \sigma_x\tan\mu,$$

(32)

$$\tau_n^e = \tan\mu\left[\frac{C_1(k_1+1)}{\alpha_1}z^{k_1} + \frac{C_2(k_2+1)}{\alpha_1}z^{k_2} + \frac{1}{\alpha_1}\left(\frac{\nu-2}{1-\nu}\frac{2}{1+a+b}+1\right)z\right].$$

and the displacement field as:

$$u^e = \frac{(1+\nu)}{2+\nu_1}z\left\{C_1\left[\frac{(1+2\nu_1)(k_1+1)}{\alpha_1}+(\nu_1-1)\right]z^{k_1} + C_2\left[\frac{(1+2\nu_1)(k_2+1)}{\alpha_1}+(\nu_1-1)\right]z^{k_2} + \right.$$

$$\left. + \left[\frac{1}{\alpha_1}\left(\frac{\nu-2}{1-\nu}\frac{2}{1+a+b}+1\right)+\frac{\nu-2}{1-\nu}\frac{\nu_1-1}{1+a+b}\right]z\right\}.$$

(33)

The symbol e denotes the solution for the elastic region. The presented solution is valid provided that:

$$\alpha_1 \neq 0 \qquad \text{or} \qquad 1+a+b \neq 0. \tag{34}$$

When $\alpha_1 = 0$ or $\alpha_1(1-\nu)(1+a+b) = 0$, we have a singular solution. The relation $1+a+b=0$ does not occur, as it is satisfied only for the value of $\nu=2$, and for the majority of granular materials ν remains in the interval $(0<\nu\leq0.5)$. In the case of:

$$\alpha_1 = 0 \quad \Longleftrightarrow \quad \frac{\mu+\tan\theta}{(1-\mu\tan\theta)\tan\theta} = 0;$$

then, the singular angle θ_w^{es} is:

$$\theta_w^{es} = \arctan\frac{3(\nu-1)\pm\sqrt{9(\nu-1)^2-8(2-\nu)(1+\nu)\tan\varphi}}{4(2-\nu)\tan\varphi}.$$

When $\alpha_1 = \alpha_1^{es}$, the singular solution is specified by the following relations:

$$\sigma_z^{es} = C_1^s z - z\ln z,$$
$$\sigma_x^{es} = \left(C_1^s - \frac{1}{(1-\nu)(2-\nu)}\right)\frac{1-\nu}{2-\nu}z - \frac{1-\nu}{2-\nu}z\ln z + C_2^s z^{-\left(\frac{1}{1-\nu}\right)},$$

(35)

where C_1^s, C_2^s are the integration constants.

The displacement field for a singular solution is:

$$u^{es} = (1+\nu)[(1-\nu)\sigma_x^{es} - \nu\sigma_z^{es}]z. \tag{36}$$

Let us consider the filling and emptying processes for an elastic material. For the filling process, the boundary conditions are:

$$\sigma_z(z)\big|_{z=\frac{1}{n}} = 0, \qquad u(z)\big|_{z=1} = 0. \tag{37}$$

Using the general solutions (30) and (33) for the stress and displacement fields, these conditions yield equations for the C_1 and C_2 constants, namely:

$$C_1\left(\frac{1}{n}\right)^{k_1} + C_2\left(\frac{1}{n}\right)^{k_2} + \frac{\nu-2}{1-\nu}\frac{1}{1+a+b}\left(\frac{1}{n}\right) = 0,$$

$$C_1\left[\frac{(1+2\nu_1)(k_1+1)}{\alpha_1} + (\nu_1-1)\right] + C_2\left[\frac{(1+2\nu_1)(k_2+1)}{\alpha_1} + (\nu_1-1)\right] + \tag{38}$$

$$+ \left[\frac{1}{\alpha_1}\left(\frac{\nu-2}{1-\nu}\frac{2}{1+a+b} + 1\right) + \frac{\nu-2}{1-\nu}\frac{\nu_1-1}{1+a+b}\right] = 0,$$

and we have:

$$C_1 = \frac{n^{-1}\beta l + n^{-k_2}\delta}{n^{-k_1}l - n^{-k_2}m}, \qquad C_2 = \frac{-n^{-k_1}\delta l + n^{-1}\beta m}{n^{-k_1}l - n^{-k_2}m},$$

where

$$\begin{aligned}
\beta &= \frac{\nu-2}{1-\nu}\frac{1}{1+a+b}, \\
\delta &= \frac{1}{\alpha_1}\left(\frac{\nu-2}{1-\nu}\frac{2}{1+a+b} + 1\right) + \frac{\nu-2}{1-\nu}\frac{\nu_1-1}{1+a+b}, \\
l &= \frac{(1+2\nu_1)(k_2+1)}{\alpha_1} + (\nu_1-1), \\
m &= \frac{(1+2\nu_1)(k_1+1)}{\alpha_1} + (\nu_1-1).
\end{aligned} \tag{39}$$

For the emptying process, it is assumed that the upper surface is free, but the bottom material surface undergoes increasing displacement, u_1, starting from an initial zero value. We therefore assume the following boundary conditions:

$$\sigma_z(z)\big|_{z=\frac{1}{n}} = 0, \qquad u(z)\big|_{z=1} = -u_1, \tag{40}$$

and thus obtain the following equations:

$$C_1\left(\frac{1}{n}\right)^{k_1} + C_2\left(\frac{1}{n}\right)^{k_2} + \frac{\nu-2}{1-\nu}\frac{1}{1+a+b}\left(\frac{1}{n}\right) = 0,$$

$$\frac{(1+\nu)}{2+\nu_1}\left\{C_1\left[\frac{(1+2\nu_1)(k_1+1)}{\alpha_1} + (\nu_1-1)\right] + C_2\left[\frac{(1+2\nu_1)(k_2+1)}{\alpha_1} + (\nu_1-1)\right] + \tag{41}\right.$$

$$\left. + \left[\frac{1}{\alpha_1}\left(\frac{\nu-2}{1-\nu}\frac{2}{1+a+b} + 1\right) + \frac{\nu-2}{1-\nu}\frac{\nu_1-1}{1+a+b}\right]\right\} = -u_1.$$

The integration constants, C_1, C_2, now depend on the emptying parameter, u_1.

Similar relations are valid for the singular case, but the respective formulae are not quoted here. Illustrative solutions have been obtained for plane hoppers, for the following parameters: $\nu = \frac{1}{3}$, $\mu = 30°$, $\frac{z_2}{z_1} = \frac{1}{n} = 15$, $\theta_w = 15°$, $30°$.
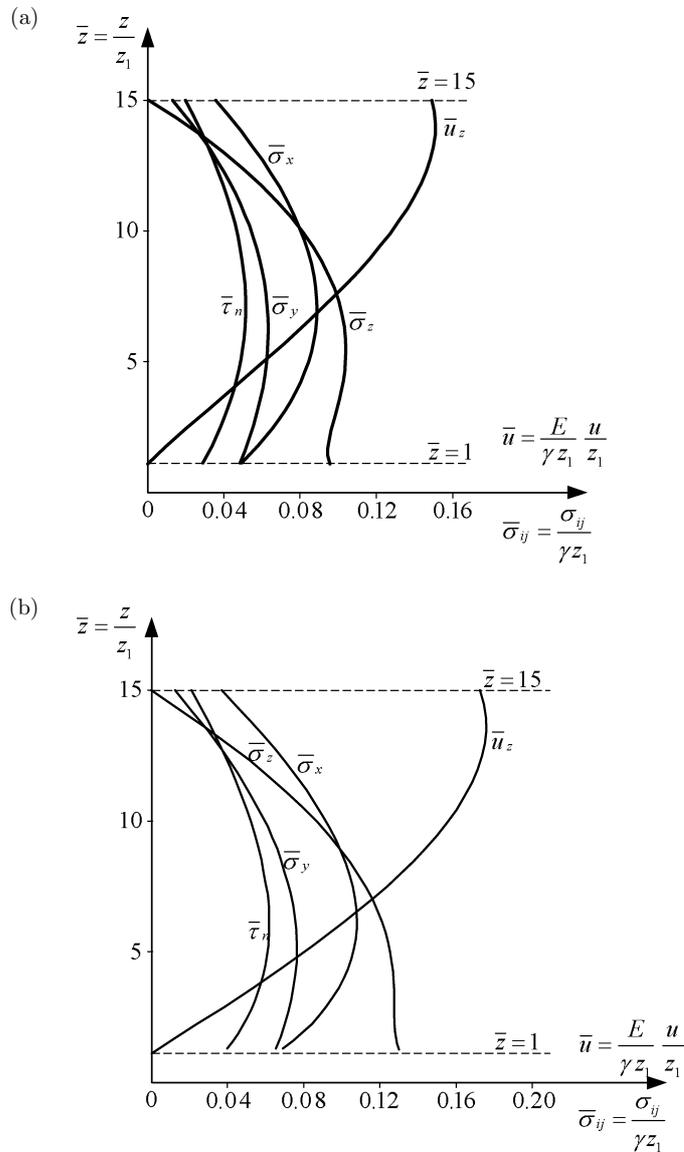
Figure 4. Stress and displacement functions in a wedge hopper after filling for two values of angle θ_w: (a) $\theta_w = 15°$, $u_z(z=1) = 0$, (b) $\theta_w = 30°$, $u_z(z=1) = 0$

Figures 4a and 4b present solutions for wedge hoppers after the filling process for two values of angle θ_w.

The diagrams show stress distributions $\sigma_z = \sigma_z(z)$, $\sigma_x = \sigma_x(z)$, $\sigma_y = \sigma_y(z)$ and $u = u(z)$ in the non-dimensional variables. The following conclusions can be drawn from the elastic solutions for the filling process. It is evident that the maximal stress values occur in the middle part of the hopper. In the lower part, an active stress state occurs, $\sigma_z^e > \sigma_x^e$, while in the upper part a passive state, $\sigma_z^e < \sigma_x^e$, develops. This character of stress distribution changes during the emptying process, when a passive state develops at the bottom of the hopper. This observation does not support the

view of Jenike and Johanson [42, 43] that an active stress state develops in the hopper after filling.

Furthermore, it is noticeable that the elastic solution cannot be valid for the whole material domain, as in the upper region near the free boundary the stress path exceeds the elastic domain. Thus, plastic deformation should be considered in the upper region of a non-cohesive material. However, when the material is cohesive, $c \neq 0$, then the elastic solution is applicable for the whole domain. In fact, one can translate the limit lines to new positions resulting from the value of cohesion to ensure that the stress profile lies within the elastic domain.

3.2. The elasto-plastic solution

Let us now discuss the elasto-plastic solution, assuming the perfectly-plastic material model. Depending on the value of material parameters, there can be two different stress profiles. These profiles result from the boundary conditions:

$$\sigma_z(z)\big|_{z=\frac{1}{n}} = 0, \qquad u(z)\big|_{z=1} = 0. \tag{42}$$

The vanishing displacement at the bottom boundary, $z = 1$, provides the condition resulting from Equation (27), namely:

$$\psi(\sigma_x - \sigma_z) + \nu_1(2\sigma_x + \sigma_z)\big|_{z=1} = 0, \tag{43}$$

which yields

$$\sigma_x\big|_{z=1} = \frac{\psi - \nu_1}{\psi + 2\nu_1}\sigma_z\bigg|_{z=1}. \tag{44}$$

In view of the yield condition $\frac{1}{k}\sigma_z \leq \sigma_x \leq k\sigma_z$, we obtain the inequalities specifying the elastic state:

$$\frac{1}{k} \leq \frac{\psi - \nu_1}{\psi + 2\nu_1} \leq k. \tag{45}$$

Since

$$0 \leq \nu \leq \frac{1}{2}, \qquad 0 \leq \nu_1 = \frac{1 - 2\nu}{1 + \nu} \leq 1, \qquad k = \frac{1 + \sin\varphi}{1 - \sin\varphi} > 1,$$

inequalities (45) are satisfied when

$$\psi\big|_{z=1} = 1, \quad k \geq \frac{1 - \nu}{\nu}, \quad \sigma_x\big|_{z=1} = \frac{\nu}{1 - \nu}\sigma_z\big|_{z=1} \quad \text{– elastic state or} \tag{46}$$

$$\psi\big|_{z=1} = \nu_1\frac{k + 2}{k + 1} > 1, \quad k < \frac{1 - \nu}{\nu}, \quad \sigma_x\big|_{z=1} = \frac{1}{k}\sigma_z\bigg|_{z=1} \quad \text{– plastic state.}$$

The expected stress profiles are shown in Figure 5.

When $k > \frac{1-\nu}{\nu}$, an active plastic zone develops in the bottom domain $1 \leq z \leq \xi$, and a passive zone near the free boundary, $\eta \leq z \leq 1/n$.

3.3. The elasto-plastic solution: emptying

The emptying process will be treated as a consecutive phase of the evolution of stress and displacement fields by assuming that at the bottom surface $z = 1$ an increasing displacement field, $u_1 = u(z)\big|_{z=1}$, is induced, with its upper surface remaining free, so that $\sigma_z(z)\big|_{z=1/n} = 0$. For an increasing u_1, the passive stress regime will develop from the bottom surface and propagate upwards in the course of discharge. We shall discuss the first stage of the emptying process by neglecting
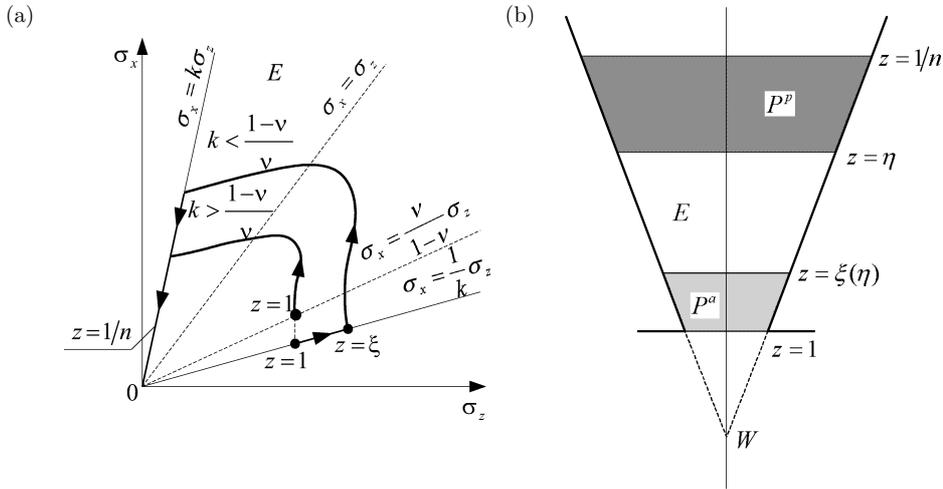
Figure 5. Stress profiles for the elasto-plastic model: (a) stress paths in the plane σ_z, σ_x, (b) plastic and elastic zones in the hopper

the configuration change of the upper surface. In fact, the stress evolution process develops for very small values of u_1 and the subsequent phase occurs within the passive stress regime. We shall consider two phases of the emptying process: the first, when there is an elastic zone in the lower part and a plastic zone in the upper part, and the second, when a passive plastic zone develops at the bottom of the hopper.

3.3.1. Initial stage of emptying: elastic and plastic zones

Let us assume boundary conditions as follows:

$$\sigma_z(z)\big|_{z=\frac{1}{n}} = 0, \qquad u(z)\big|_{z=1} = -u_1, \qquad 1/n = z_2/z_1, \tag{47}$$

and that the position of the elasto-plastic interface $z = \eta$ depends on u_1, and thus $\eta = \eta(u_1)$ or $u_1 = u_1(\eta)$. The stress and displacement continuity conditions are satisfied at the interface, therefore:

$$[\sigma_z] = [u] = 0, \qquad \sigma_x = k\sigma_z, \qquad \text{for } z = \eta. \tag{48}$$

Taking the interface position $z = \eta$ as the process progression parameter, it can be noted that for some value of $\eta(u_1) = \eta''$ the yield condition at the bottom surface is satisfied, so that:

$$F_1(z, \eta(u_1))\big|_{z=1} = (k\sigma_z - \sigma_x)\big|_{z=1} = 0, \tag{49}$$

and for increasing values of u_1 or $\eta(u_1)$ a passive plastic zone starts to develop near the bottom surface.

Let us now discuss the details of the solution when elastic and plastic zones E and P' exist within the solution's domain. The plastic stress state within P' is specified by Equation (49) and the elastic state within E is specified by Equation (30). Integration constans C, $C_1(\eta)$ and $C_2(\eta)$ are obtained from the boundary condition at $z = 1/n$ and the continuity conditions for $z = \eta$. Within zone P' we obtain:

$$\sigma_z^{P'} = \gamma_m' z \frac{1 - (nz)^{A\cot\theta_w + 1}}{(nz)^{A\cot\theta_w + 1}}, \quad \sigma_x^{P'} = k\sigma_z^{P'}, \quad \tau_n^{P'} = \sigma_z^{P'}\tan\mu \quad \text{for } \alpha_1 \neq 2k, \tag{50}$$
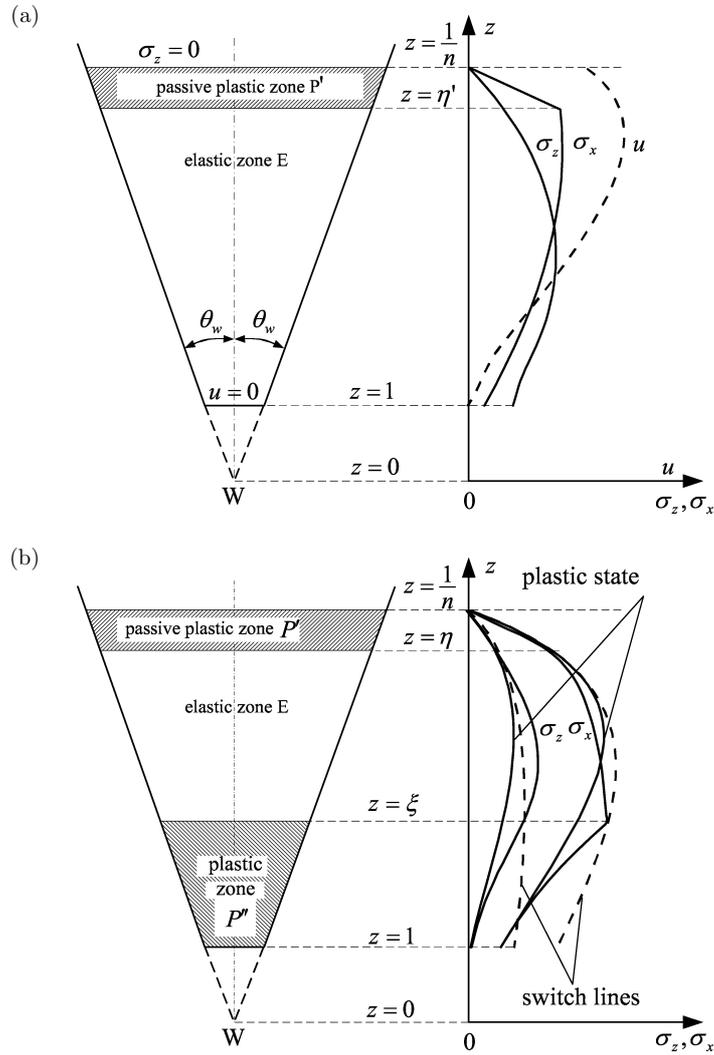
Figure 6. Elastic and plastic zones in: (a) initial and (b) advanced stages of emptying

and in zone E:

$$\sigma_z^e = C_1(\eta) z^{k_1} + C_2(\eta) z^{k_2} + \beta z,$$

$$\sigma_x^e = \frac{1}{\alpha_1} \left[C_1(\eta)(k_1+1) z^{k_1} + C_2(\eta)(k_2+1) z^{k_2} + \delta_1 z \right], \tag{51}$$

$$\delta_1 = \frac{1}{\alpha_1} \left(\frac{\nu-2}{1-\nu} \frac{2}{a+b+1} + 1 \right),$$

where

$$C_1(\eta) = \frac{T_1(\eta)\gamma_m'\eta(k_2 - \alpha_1 k + 1) - (k_2+1)\delta_2 - \eta(2\delta_2+1)}{\eta^{k_1}(k_2-k_1)},$$

$$C_2(\eta) = \frac{T_1(\eta)\gamma_m'\eta(\alpha_1 k - k_1 - 1) + (k_1+1)\delta_2 - \eta(\delta_2+1)}{\eta^{k_2}(k_2-k_1)}, \tag{52}$$

$$\delta_2 = \frac{(\nu-2)\eta}{(1-\nu)(a+b+1)}.$$

The emptying process is initiated at $u_1(\eta) = 0$ and proceeds for $u_1 > 0$. The value of η' is obtained from condition (41), as follows:

$$F_1^*(\eta) = mC_1(\eta) + lC_2(\eta) + \frac{1}{\alpha_1}\delta = 0. \tag{53}$$

The value of η'' corresponds to the onset of plasticity at $z = 1$, specified by the condition:

$$\frac{1}{\alpha_1}\left[(\alpha_1 k - k_1 - 1)C_1(\eta) + (\alpha_1 k - k_2 - 1)C_2(\eta) + \alpha_1 k\beta - \delta_1\right] = 0. \tag{54}$$

Thus, the emptying process with an elastic zone E in the lower region occurs when $\eta' \le \eta \le \eta''$.

3.3.2. Advanced stage of emptying : solution with three zones, P', E and P'' (Figure 6b)

The interface radius, $z = \xi(\eta)$, between the elastic and plastic zones P'' in the bottom part of the hopper is obtained from the yield condition:

$$F_1 = k\sigma_z^e(z,\eta) - \sigma_x^e(z,\eta) = 0, \tag{55}$$

and there should be $1 \le \xi(\eta) \le \eta$. When $\xi = \eta$, both plastic zones P' and P'' come into contact and the whole material becomes plastic. Condition (55) leads to the following equation:

$$F_1(z,\eta) \equiv \frac{1}{\alpha_1}\left[C_1(\eta)z^{k_1}(k\alpha_1 - k_1 - 1) + C_2(\eta)z^{k_2}(k\alpha_1 - k_2 - 1) + z(k\beta\alpha_1 - \delta_1)\right] = 0, \tag{56}$$

where $C_1(\eta)$ and $C_2(\eta)$ are given by Equation (52). The stress and displacement states in the plastic zone P'' are:

$$\sigma_z^{P''} = C''(\eta)z^{-A\cot\theta_w} - \frac{1}{1 + A\cot\theta_w}z, \qquad 1 \le z \le \xi(\eta),$$
$$\sigma_x^p = k\sigma_z^p, \qquad \tau_n^p = k\sigma_z^p\tan\mu, \tag{57}$$
$$u^P = (1+\nu)\psi\frac{[\psi(k-1) + \nu_1(2k+1)]}{2\psi + 1}z\sigma_z^P,$$

where

$$C'' = \left[C_1(\eta)\xi^{k_1} + C_2(\eta)\xi^{k_2} + \beta\xi + \frac{1}{1 + A\cot\theta_w}\right]\xi^{A\cot\theta_w}, \quad 1 + A\cot\theta_w \ne 0. \tag{58}$$

The stress and displacement states in the elastic zone E are:

$$\sigma_z^e = C_1(\eta)z^{k_1} + C_2(\eta)z^{k_2} + \beta z, \qquad \xi(\eta) \le z \le \eta,$$
$$\sigma_x^e = \frac{1}{\alpha_1}\left[C_1(\eta)(k_1+1)z^{k_1} + C_2(\eta)(k_2+1)z^{k_2} + \delta_1 z\right],$$
$$\sigma_\zeta^e = \nu\sigma_z^e + \nu\sigma_\theta^e, \qquad \tau_n = \sigma_x^e\tan\mu, \tag{59}$$
$$u^e = (1+\nu)[(1-\nu)\sigma_x^e - \nu\sigma_z^e],$$

while in the upper plastic zone P' we have:

$$\sigma_z^p = \gamma_m' zT_1(z),$$
$$\sigma_\zeta^p = \frac{(\psi - \nu_1)(k+1)}{2\psi - \nu_1(k+1)}\sigma_z^p, \tag{60}$$
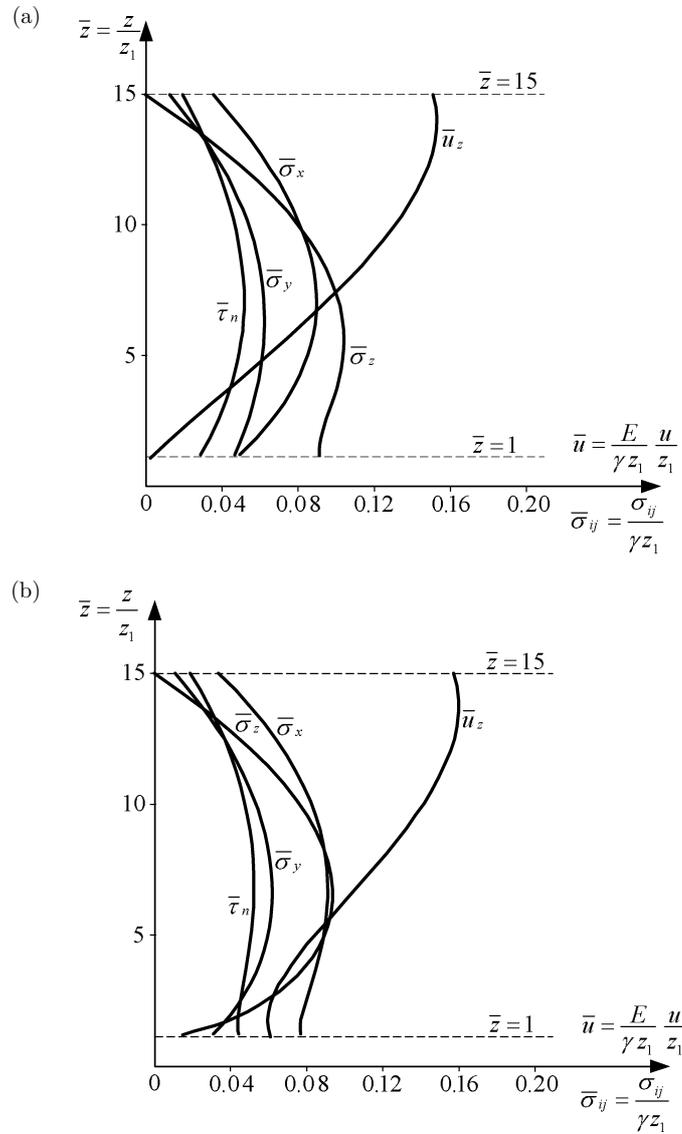
(a)

(b)

Figure 7. The elastic solution; stress and displacement functions during emptying for two values of angle θ_w ($\theta_w = 15°$ and $30°$): (a) $\theta_w = 15°$, $u_z(z=1) = \frac{1-\nu^2}{E} \cdot 0.0005$, (b) $\theta_w = 15°$, $u_z(z=1) = \frac{1-\nu^2}{E} \cdot 0.007$ (*continued on the following pages*)

where

$$\gamma'_m = \frac{1}{1 + A\cot\theta_w}, \qquad T_1(z) = (nz)^{-(A\cot\theta_w + 1)} - 1, \qquad 1 + A\cot\theta_w \neq 0,$$

$$u_p = (1+\nu)\psi\frac{\psi(k-1) + \nu_1(2k+1)}{\psi(m+2) + \nu_1}z\sigma_z^p, \qquad \eta \leq z \leq \frac{1}{n},$$

and $\psi_m(z,\eta)$ is the solution of the differential equation:

$$\frac{d\psi}{dz} = -\frac{\psi(\psi + \nu_1 m_1)\left\{\left[R_1'(z) + \frac{2}{z}\right](\psi + \nu_1 m_1) + \nu_1(k_1^p - m_1)R_1'(z)\right\}}{(\psi + \nu_1 m_1)^2 + \nu_1^2 m_1(k_1^p - m_1)}, \qquad (61)$$

(c)

(d)
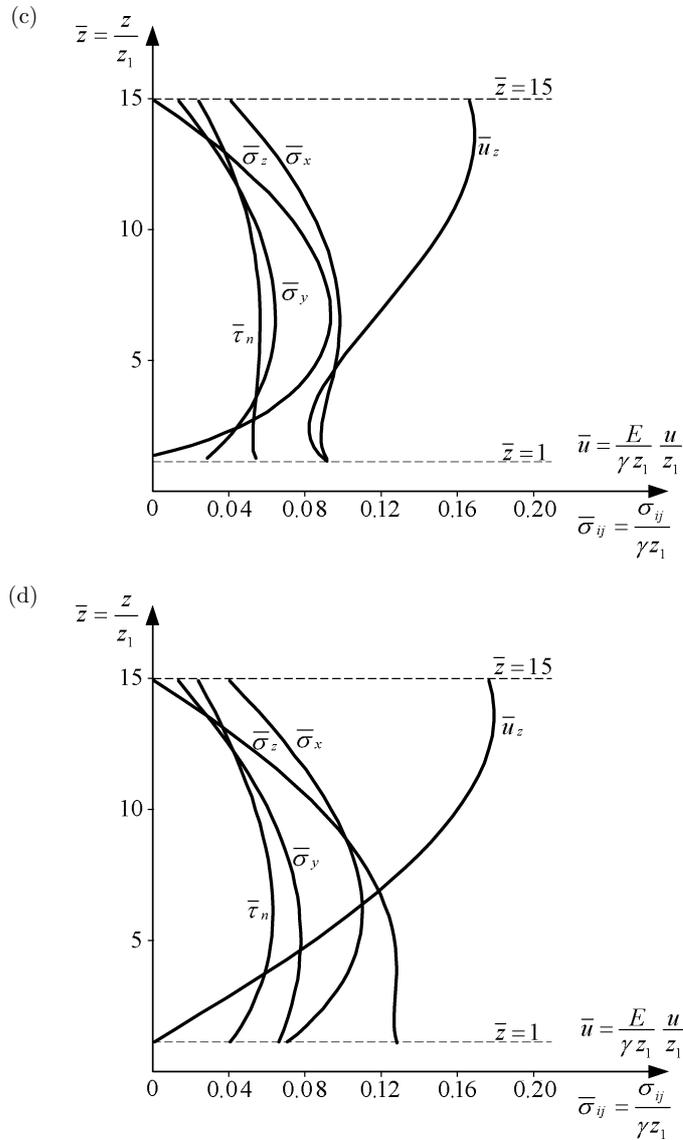
Figure 7 – continued. (c) $\theta_w = 15°$, $u_z(z=1) = \frac{1-\nu^2}{E} \cdot 0.01$, (d) $\theta_w = 30°$, $u_z(z=1) = \frac{1-\nu^2}{E} \cdot 0.0005$

where

$$R_1'(z) \equiv \frac{1}{\sigma_z^{p'}(z)} \frac{d\sigma_z^{p'}}{dz} = \frac{-A\cot\theta_w (nz)^{-A\cot\theta_w - 1} - 1}{z\left[\frac{1}{1+A\cot\theta_w}(nz)^{-A\cot\theta_w - 1} - 1\right]}. \tag{62}$$

4. Illustrative numerical solutions

Numerical calculations have been performed using the *Derive* programme. Numerical solutions were generated for plane hoppers, for two values of the hopper angle: $\theta_w = 15°$ and $\theta_w = 30°$, and for three values of the position of the upper surface: $\frac{1}{n} = \frac{z_2}{z_1} = 5$, 10, 15. The numerical results are illustrated in Figures 7–11, presenting
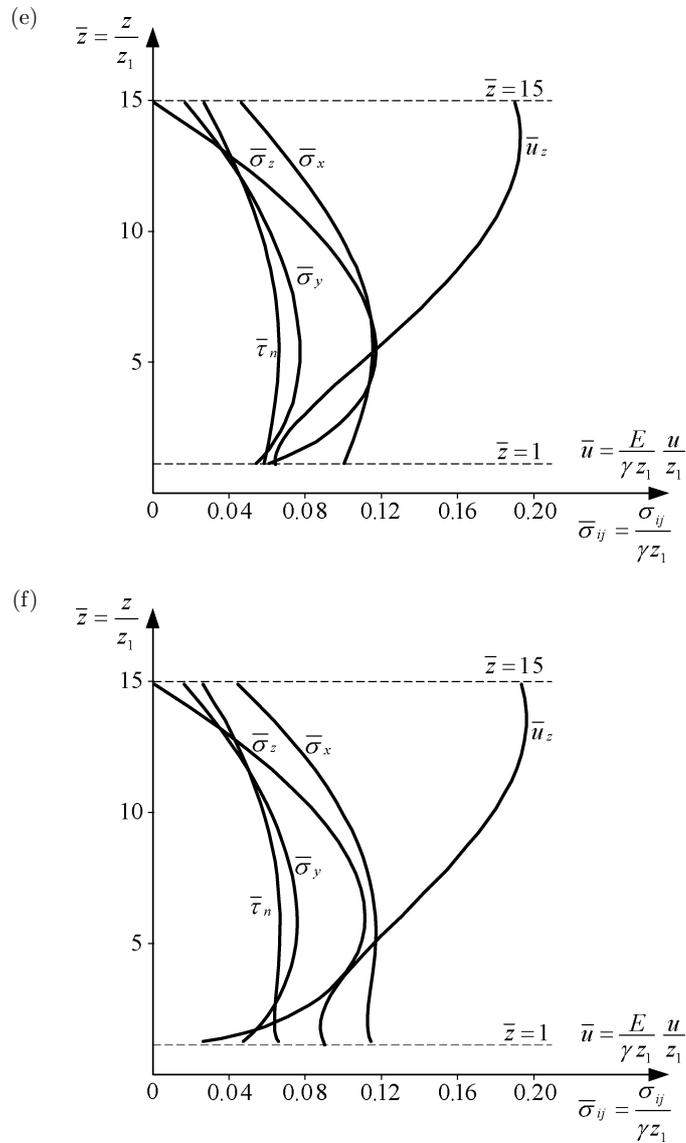
(e)

(f)

Figure 7 – continued. (e) $\theta_w = 30°$, $u_z(z = 1) = \frac{1-\nu^2}{E} \cdot 0.007$, (f) $\theta_w = 30°$, $u_z(z = 1) = \frac{1-\nu^2}{E} \cdot 0.01$

the stress and displacement functions in the consecutive stages of the filling and discharge process.

Figures 10a and 10b present the stress and displacement functions during the consecutive stages of emptying of a wedge hopper in an advanced stage of the process. It is noticeable that, when the plastic passive zone P'' propagates upwards, the wall pressure reaches its maximum at the instantaneous interface position $z = \xi$ between P'' and E. There is a moving "switch" between the passive and elastic states with the pressure peak at $z = \xi$ moving toward the upper plastic zone P'. It can be seen that for higher hoppers the moving interface $z = \xi$ generates higher switching pressures with respect to initial pressures at the filling stage or ultimate pressures for the total
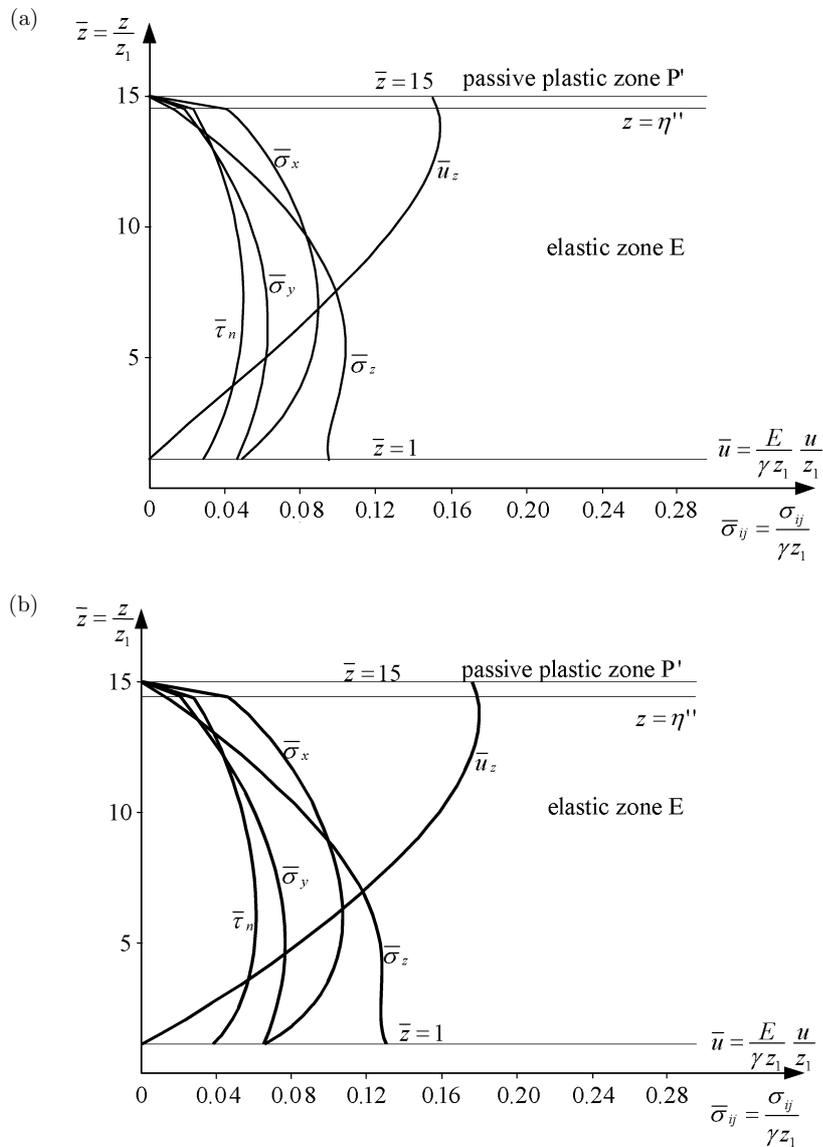
(a)

(b)

Figure 8. Stress and displacement functions in the elasto-plastic solution of the filling process (or the initial stage of the emptying process): (a) $\theta_w = 15°$, $u_z(z=1) = 0$, $\sigma_z(z=1) = 0.0971$, (b) $\theta_w = 30°$, $u_z(z=1) = 0$, $\sigma_z(z=1) = 0.133$

passive state in the hopper. As the evolution parameter we assumed the value of η, which decreases from the initial $\eta = \eta'$ to the final value $\eta = \eta''$ when the total passive plastic state develops. From the obtained displacement distribution diagrams one can determine the bottom displacement, $u = u_1$, in terms of the parameter η. It should be noted that the motion of the interface $z = \xi$ and the pressure switch occurs for small values of u_1, hence the pressure evolution may have the characteristic of quasi-dynamic wall loading by the pressure switch. The value of overpressure due to the moving interface depends on hopper heigh and material parameters.
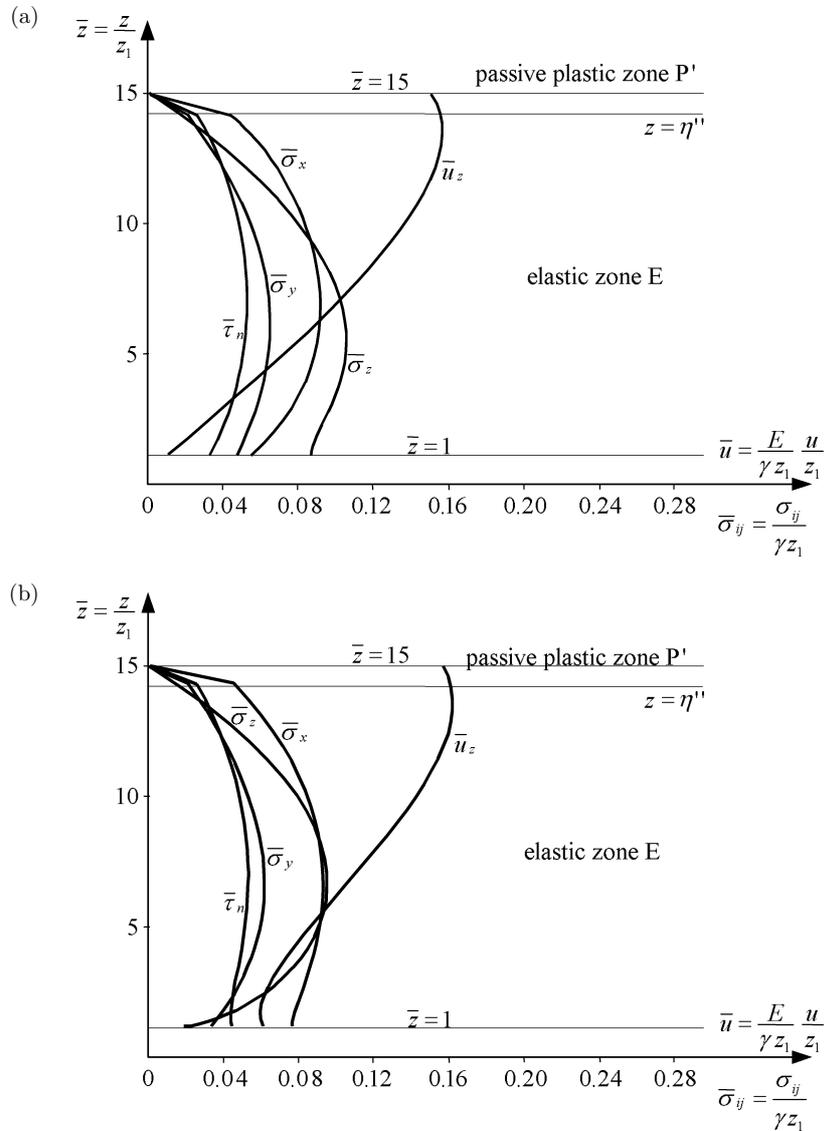
(a)

(b)

Figure 9. The elasto-plastic solution for the emptying process: the initial stage of plasticization of the bottom surface, stress and displacement functions for a wedge hopper, $\theta_w = 15°$, $\theta_w = 30°$ in the elasto-plastic solution for the initial stage of the emptying process, (a) $\theta_w = 15°$, $u_z(z=1) = \frac{1-\nu^2}{E} \cdot 0.001$, $\sigma_z(z=1) = 0.0855$, (b) $\theta_w = 15°$, $u_z(z=1) = \frac{1-\nu^2}{E} \cdot 0.005$, $\sigma_z(z=1) = 0.0395$ (*continued on the following pages*)

5. Concluding remarks

A simplified elasto-plastic analysis of transient stress and wall pressure evolution during the emptying process of granular materials in wedge hoppers has been presented in this paper. The analysis indicates that the transient evolution is characterised by a moving interface between the passive plastic state and the elastic state from the bottom surface upwards toward the upper free surface. Such tran-
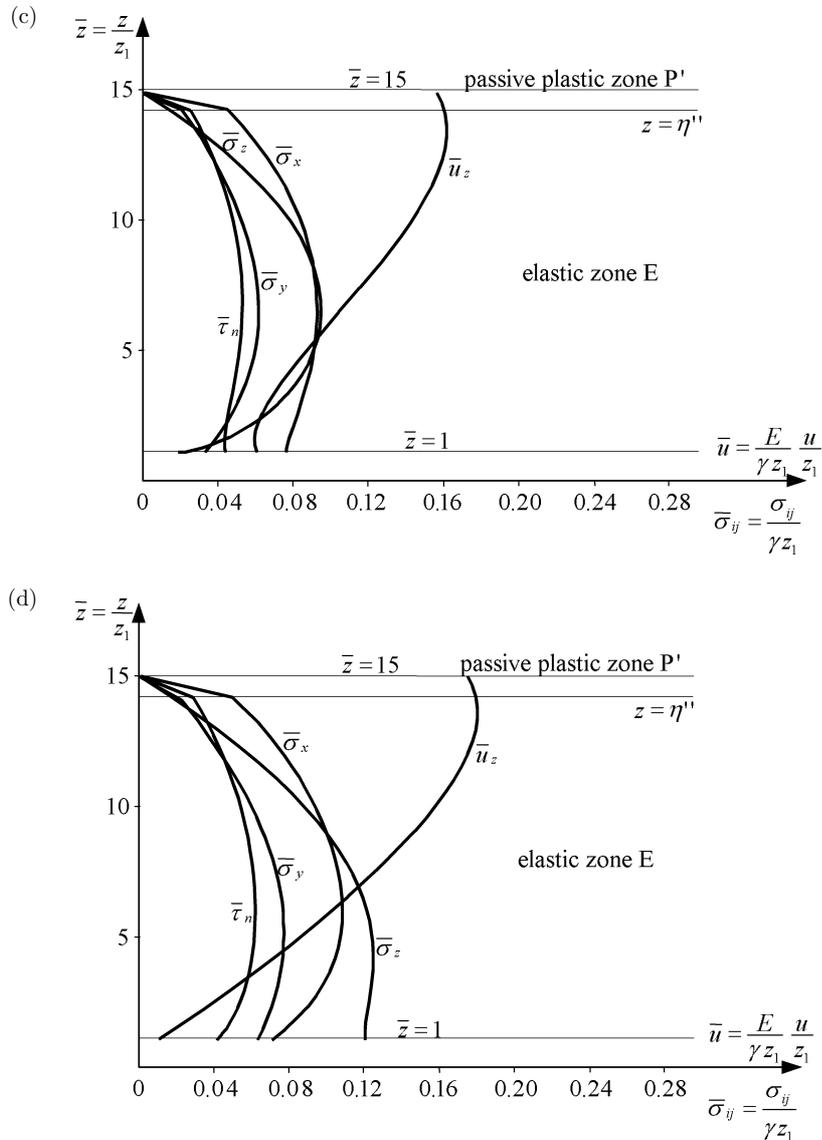
(c)

(d)

Figure 9 – continued. (c) $\theta_w = 15°$, $u_z(z=1) = \frac{1-\nu^2}{E} \cdot 0.007$, $\sigma_z(z=1) = 0.0164$,
(d) $\theta_w = 30°$, $u_z(z=1) = \frac{1-\nu^2}{E} \cdot 0.001$, $\sigma_z(z=1) = 0.1222$

sient switch between the passive and elastic zones generates excessive wall pressure with its maximum at the interface $z = \xi$. As significant interface motion occurs due to small discharge displacements at the bottom surface, a quasi-dynamic character of the pressure evolution manifests itself. This pressure evolution may induce hopper vibration.
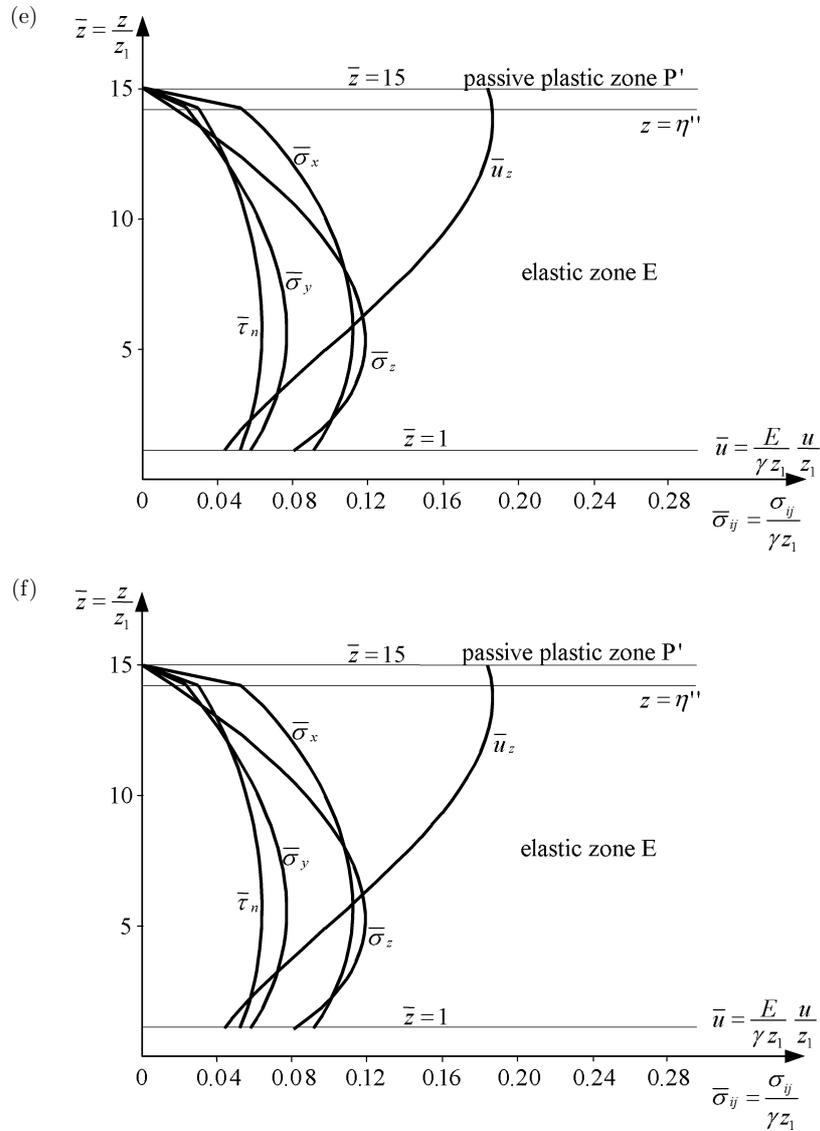
Acknowledgements

Figure 9 – continued. (e) $\theta_w = 30°$, $u_z(z=1) = \frac{1-\nu^2}{E} \cdot 0.005$, $\sigma_z(z=1) = 0.0797$,
(f) $\theta_w = 30°$, $u_z(z=1) = \frac{1-\nu^2}{E} \cdot 0.007$, $\sigma_z(z=1) = 0.0583$

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(a)

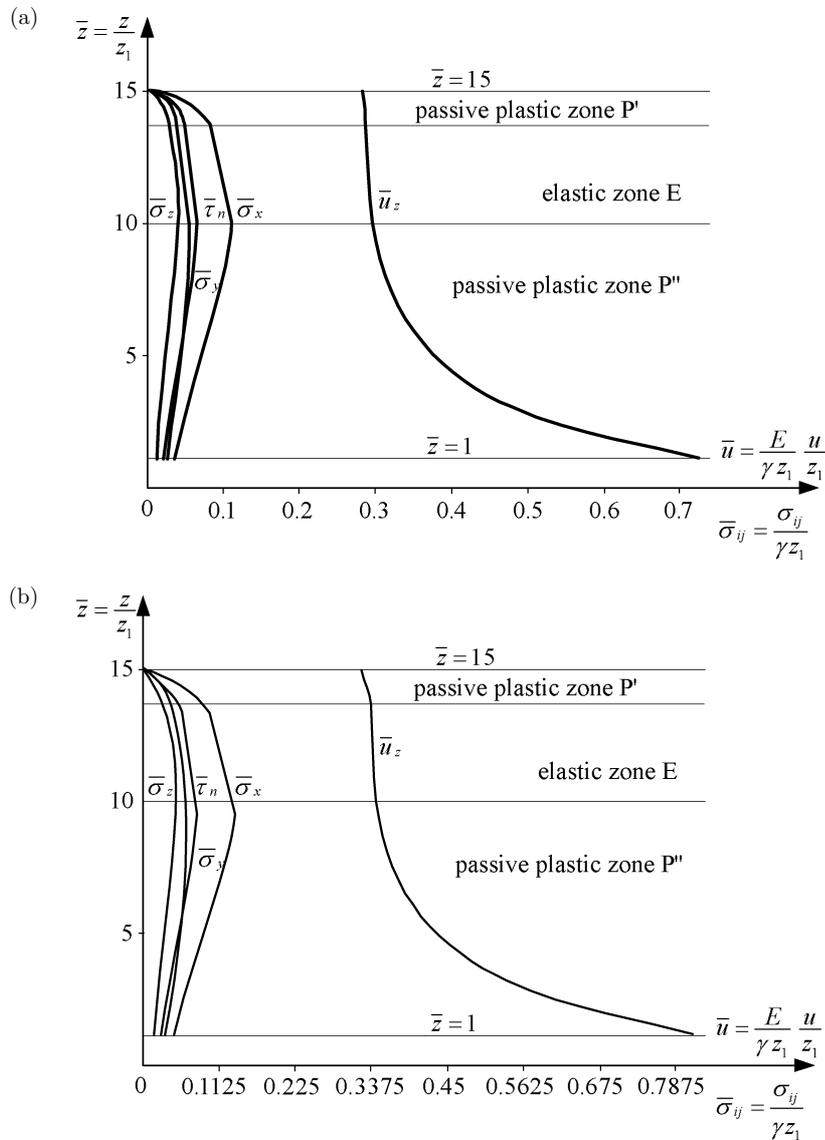
(b)

Figure 10. The elasto-plastic solution for the emptying process, an advanced stage of the process: (a) $\theta_w = 15°$, $u_z(z=1) = \frac{1-\nu^2}{E} \cdot 0.0961$, $\sigma_z(z=1) = 0.010109$, (b) $\theta_w = 30°$, $u_z(z=1) = \frac{1-\nu^2}{E} \cdot 0.1044$, $\sigma_z(z=1) = 0.009795$

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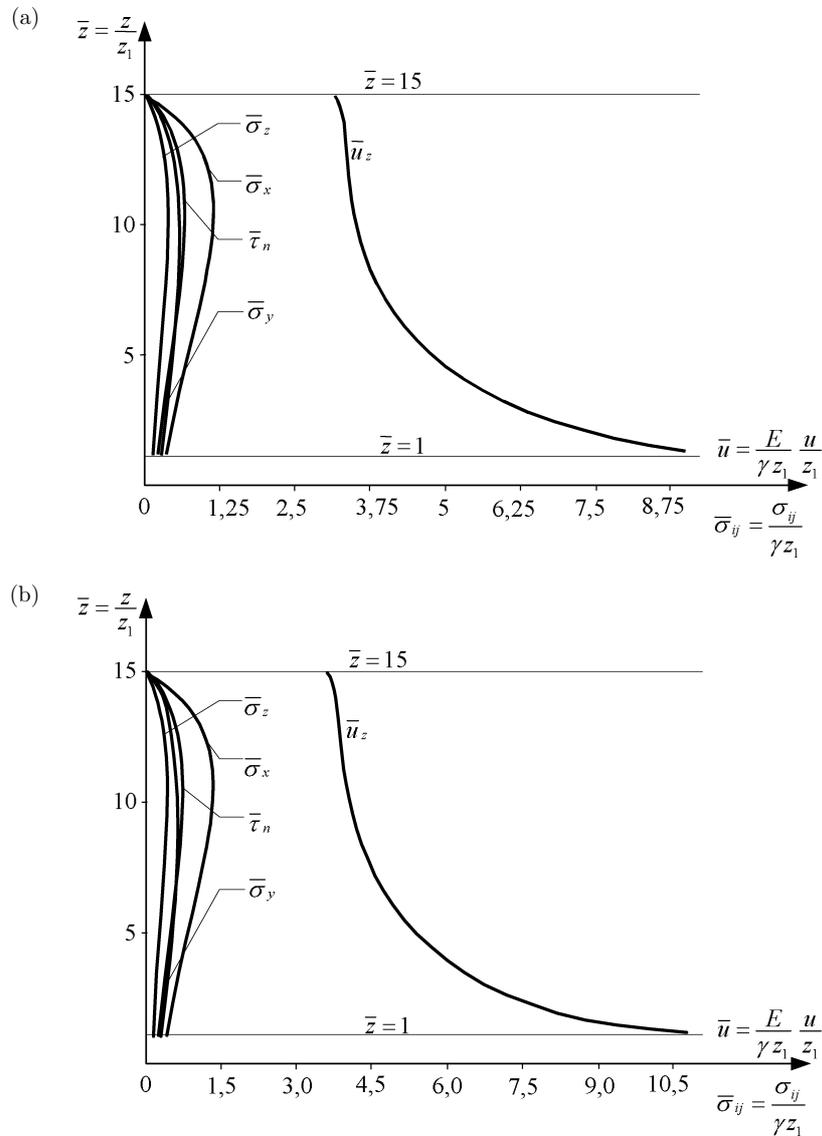[18] Zhang Q, Bergen J I and Britton M G 1997 *Can. Agric. Engng.* **39** 3

Figure 11. The emptying process: full plasticization of the material, (a) $\theta_w = 15°$, (b) $\theta_w = 30°$

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