THE USE OF COMPLEXITY HIERARCHIES IN DESCRIPTIVE SET THEORY AND AUTOMATA THEORY

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Abstract: The concept of a reduction between subsets of a given space is described, giving rise to various complexity hierarchies, studied both in descriptive set theory and in automata theory. We discuss in particular the Wadge and Lipschitz hierarchies for subsets of the Baire and Cantor spaces and the hierarchy of Borel reducibility for finitary relations on standard Borel spaces. The notions of Wadge and Lipschitz reductions are related to corresponding perfect information games.

Keywords: hierarchies, infinite games, Borel reducibility, automata theory

1. Introduction

A common feature of descriptive set theory and theoretical computer science is the prominent role of hierarchies as a tool for measuring the complexity of given objects. Here we are interested in the complexity of sets under various notions of reducibility. A very general way to compare subsets of a given non-empty set X is the following: fix $\mathcal{F} \subseteq X^X$, a family of functions from X to itself, closed under composition and containing the identity function. For $A,B \subseteq X$ define:

$$A \leq_{\mathcal{F}} B \quad \Leftrightarrow \quad \exists g \in \mathcal{F}\,(A = g^{-1}(B)).$$

In this case A is said to be $\mathcal{F}$-*reducible* to B and the function g is called a *reduction* of A to B. Note that g is then also a reduction of $\neg A$ to $\neg B$, where $\neg$ denotes complementation in the space X. So the problem of determining whether $x \in A$ reduces to establishing if $g(x) \in B$: this gives the intuition that the complexity of A does not exceed the complexity of B, assuming the function g sufficiently amenable. By the conditions on $\mathcal{F}$, the relation $\leq_{\mathcal{F}}$ is in fact a preorder on:

$$\mathscr{P}(X) \stackrel{\text{def}}{=} \{A \mid A \subseteq X\}$$

the *power-set* of X. Set also:

$$\begin{aligned} A <_{\mathcal{F}} B &\quad\Leftrightarrow\quad A \leq_{\mathcal{F}} B \wedge B \not\leq_{\mathcal{F}} A, \\ A \equiv_{\mathcal{F}} B &\quad\Leftrightarrow\quad A \leq_{\mathcal{F}} B \wedge B \leq_{\mathcal{F}} A. \end{aligned}$$

The relation $\equiv_{\mathcal{F}}$ is an equivalence relation, whose equivalence classes $[A]_{\mathcal{F}}$ are called ($\mathcal{F}$-)*degrees*. So $\leq_{\mathcal{F}}$ induces an order on $\mathcal{F}$-degrees. A set A is called ($\mathcal{F}$-)*self-dual* if $A \leq_{\mathcal{F}} \neg A$ (equivalently, $A \equiv_{\mathcal{F}} \neg A$). Being invariant under $\equiv_{\mathcal{F}}$, the definition of self duality can be extended to $\mathcal{F}$-degrees. The *dual* of a degree $[A]_{\mathcal{F}}$ is the degree $[\neg A]_{\mathcal{F}}$.

Often the condition that $\mathcal{F}$ contains all constant functions is required, to the effect that, for $\emptyset \neq A \neq X$, the inequalities $\emptyset <_{\mathcal{F}} A$, $X <_{\mathcal{F}} A$ will hold (note that the degrees $[\emptyset]_{\mathcal{F}}$, $[X]_{\mathcal{F}}$ are always singletons and they are always incomparable).

Several preorders of the form $\leq_{\mathcal{F}}$ have been extensively studied. Purpose of this note is to survey their theory in a few meaningful cases and to give some reference to how the subject is being applied in theoretical computer science. Various remarks will discuss some open questions.

The structure of $\leq_{\mathcal{F}}$ strongly depends on $\mathcal{F}$ and the set X, as it can be seen looking at two extreme cases. If $\mathcal{F} = X^X$, then the hierarchy of $\mathcal{F}$-degrees consists of two bottom degrees $[\emptyset]_{\mathcal{F}}$, $[X]_{\mathcal{F}}$ and, if X has at least two elements, a third degree containing all proper non-empty subsets of X (actually, for this picture to arise it is enough that $\mathcal{F}$ contains all two valued functions). If $\mathcal{F}$ consists only of the identity and the constant functions, then $\equiv_{\mathcal{F}}$ is equality on $\mathscr{P}(X)$; in this case there are two bottom degrees $[\emptyset]_{\mathcal{F}}$, $[X]_{\mathcal{F}}$, while all other degrees are pairwise incomparable. A degree $[A]_{\mathcal{F}}$ is a *successor* degree if it has an immediate predecessor $[B]_{\mathcal{F}}$, i.e., if $B <_{\mathcal{F}} A$ and there is no C such that $B <_{\mathcal{F}} C <_{\mathcal{F}} A$. A degree $[A]_{\mathcal{F}}$ other than $[\emptyset]_{\mathcal{F}}$ or $[X]_{\mathcal{F}}$ which is not a successor is called *limit*: in this case $[A]_{\mathcal{F}}$ is of *countable cofinality* just in case there is a sequence of degrees $[A_n]_{\mathcal{F}}$ such that $A_n <_{\mathcal{F}} A$ for all n, and there is no $B <_{\mathcal{F}} A$ such that $\forall n (A_n <_{\mathcal{F}} B)$.

2. The Wadge and Lipschitz hierarchies

The aim of this section is to introduce the Wadge and Lipschitz hierarchies and to study them in some detail for the Baire space $\mathbb{N}^{\mathbb{N}}$ and the Cantor spaces $k^{\mathbb{N}}$, with $k \geq 2$. The basic theory throughout this section will be the Zermelo–Frænkel set theory ZF augmented with the axiom of dependent choices over the reals, $\mathsf{DC}(\mathbb{R})$. This principle is strictly weaker than the full axiom of choice AC, and the reason from retreating from ZFC (the theory ZF with the axiom of choice) to $\mathsf{ZF}+\mathsf{DC}(\mathbb{R})$ is that ZFC is not suited for our investigation, as will soon become clear.

Let X be a topological space and let $\mathcal{F} = \mathcal{C}(X,X)$ be the set of continuous functions of X into itself. If X is a metric space consider also the set $\mathcal{G}$ of functions $g : X \to X$ such that

$$\forall x,y \in X \;\; d(g(x),g(y)) \leq d(x,y)$$

(these will be called *Lipschitz* functions). The preorders $\leq_{\mathcal{F}}$, $\leq_{\mathcal{G}}$ (in the sequel denoted $\leq_{\mathrm{W}}^X$, $\leq_{\mathrm{L}}^X$ or even $\leq_{\mathrm{W}}$, $\leq_{\mathrm{L}}$ when the space X is clear from the context – W standing for Wadge, L for Lipschitz) are relations arising quite naturally in topology. However it seems that the first systematical study of their structural properties was performed by W.W. Wadge in [1]. The subject has been then investigated in the Caltech-UCLA logic seminars and many results are collected in [2, 3]. These investigations exploited the relationship between games and functions on product spaces.

Endow a non-empty set Z with the discrete topology and $Z^{\mathbb{N}}$ with the product (Tychonov) topology. Then $Z^{\mathbb{N}}$ is a metric space, with the metric defined by letting:

$$d(x,y)=\begin{cases}0 & \text{if } x=y,\\ \frac{1}{2^n} & \text{if } n \text{ is least with } x(n)\neq y(n).\end{cases}$$

We shall write $Z^{<\mathbb{N}}$ for the set of finite strings (*i.e.*, finite sequences) of elements of Z, always identifying 1-tuples with elements of Z. Define also $Z^{\leq\mathbb{N}}\stackrel{\text{def}}{=}Z^{<\mathbb{N}}\cup Z^{\mathbb{N}}$, the set of finite or infinite strings from Z. If $x\in Z^{\leq\mathbb{N}}$ is such a sequence then:

$$x\restriction n\stackrel{\text{def}}{=}\langle x(0),x(1),\ldots,x(n-1)\rangle$$

is the finite string obtained from the first n values of x. (The symbol $\restriction$ usually denotes the *restriction* operator: this is consistent with our definition since in set theory a natural number is construed as the collection of its predecessors, hence $n=\{0,1,\ldots,n-1\}$.) Let $\varphi: Z^{<\mathbb{N}}\to Z^{<\mathbb{N}}$ be monotone with respect to inclusion, that is $s\subseteq s'\Rightarrow\varphi(s)\subseteq\varphi(s')$. Then φ is:

- *continuous* if $\forall x\in Z^{\mathbb{N}}\ \lim_{n\to\infty}\text{length}(\varphi(x\restriction n))=+\infty$;
- *Lipschitz* if $\forall s\in Z^{<\mathbb{N}}\ \text{length}(s)=\text{length}(\varphi(s))$.

The reason for this terminology is that, if φ is continuous or Lipschitz, then defining for $x\in Z^{\mathbb{N}}$

$$f_\varphi(x)=\bigcup_{n\in\mathbb{N}}\varphi(x\restriction n)$$

one gets a continuous, respectively Lipschitz, function $f_\varphi: Z^{\mathbb{N}}\to Z^{\mathbb{N}}$; conversely each continuous or Lipschitz function arises this way, providing thus a parametrisation $\varphi\mapsto f_\varphi$ of the set of continuous functions, or of the set of Lipschitz functions, respectively. Thus a function $f: Z^{\mathbb{N}}\to Z^{\mathbb{N}}$ is continuous just in case in order to compute $f(x)\restriction n$ it is enough to know a large enough initial segment $x\restriction m$ of x; if such m can be taken to be n, then f is Lipschitz.

2.1. Lipschitz and Wadge games

Fix a non-empty set Z and a set $C\subseteq Z^{\mathbb{N}}$. The game[1] $G(C)$ is played by two players **I** and **II** who alternatively play elements $a_0,b_0,a_1,b_1,a_2,b_2,\ldots$ of Z:

$$\begin{array}{lccccccc}\mathbf{I} & a_0 & & a_1 & & a_2 & & \ldots\\ \mathbf{II} & & b_0 & & b_1 & & \ldots & \end{array}\tag{1}$$

The game is organized in *rounds* or *innings* – in the n^{th} round **I** moves first and plays a_n, and then **II** plays b_n – and since there are infinitely many rounds, the game is said to be infinite. The sequence $\langle a_0,b_0,a_1,b_1,\ldots\rangle$ is called a *play* of the game, and

1. The game $G(C)$ is often called the *Gale-Stewart game*, after [4] where some fundamental results about these games where proved.

player **I** wins if and only if the play of the game is in C. Either player is aware of the opponent's previous moves, and for this reason this is called a *perfect information* game. A *strategy* is a procedure telling a player what to do in every possible situation – this terminology might be a little misleading, since in usual games a strategy is a set of euristic principles to be used when the best move is hard to compute. What will be called a strategy here is better known as tree of analysis in real life games. There are several equivalent ways to define a strategy: in this paper a strategy for **I** is any function $\sigma: Z^{<\mathbb{N}} \to Z$ telling **I** what to play at round n when applied to the string $\langle b_0,\ldots,b_{n-1}\rangle$ of moves played by his opponent before round n. Similarly, a strategy for **II** is any function $\tau: Z^{<\mathbb{N}} \setminus \{\langle\rangle\} \to Z$, suggesting **II**'s move for every possible finite non-empty string[2] of **I**'s moves. If σ and τ are strategies for **I** and **II**, respectively, then σ and τ can be pitted against each other so that a new element

$$\sigma * \tau \stackrel{\text{def}}{=} \langle a_0, b_0, a_1, b_1, \ldots\rangle$$

of $Z^{\mathbb{N}}$ is constructed as follows:

$$\begin{aligned} a_0 &= \sigma(\langle\rangle), \\ a_{n+1} &= \sigma(\langle b_0,\ldots,b_n\rangle), \\ b_n &= \tau(\langle a_0,\ldots,a_n\rangle). \end{aligned}$$

Any element $\langle x_0, x_1, x_2, \ldots\rangle$ of $Z^{\mathbb{N}}$ can be construed as a strategy σ for **I** by letting:

$$\sigma(s) = x_{\text{length}(s)}$$

or a strategy τ for **II**:

$$\tau(s) = x_{\text{length}(s)-1}.$$

In other words, these two strategies play the given sequence irrespectively of what the opponent is playing. By identifying elements of $Z^{\mathbb{N}}$ with strategies, the expressions

$$\sigma * \langle b_0, b_1, \ldots\rangle \quad \text{and} \quad \langle a_0, a_1, \ldots\rangle * \tau$$

are well-defined: in the first case it is the play resulting when σ, a strategy for **I**, is pitted against the sequence of **II**'s moves $\langle b_0, b_1, \ldots\rangle$, in the second case it is the play resulting when τ, a strategy for **II**, is pitted against the sequence of **I**'s moves $\langle a_0, a_1, \ldots\rangle$. A strategy σ for **I** is *winning* in the game $G(C)$ if

$$\forall x \in Z^{\mathbb{N}} (\sigma * x \in C).$$

Similarly, a strategy τ for **II** is winning in $G(C)$ if

$$\forall x \in Z^{\mathbb{N}} (x * \tau \notin C).$$

Remark. Although the notions of game, strategy, rounds, etc. are extremely useful to describe certain mathematical constructions, some readers might wonder whether they are formalizable within the language of set theory. In particular: What kind of

2. The empty string, denoted by $\langle\rangle$, is really the empty-set.

set-theoretic object is the game $G(C)$? It turns out that we could dispense from using the notion of game, but not from that of strategy. For any non-empty set Z let:

$$\mathcal{S}_Z^{\mathbf{I}} = \{\sigma \mid \sigma : Z^{<\mathbb{N}} \to Z\},$$
$$\mathcal{S}_Z^{\mathbf{II}} = \{\tau \mid \tau : Z^{<\mathbb{N}} \setminus \{\langle\rangle\} \to Z\}.$$

For any $\sigma \in \mathcal{S}_Z^{\mathbf{I}}$, $\tau \in \mathcal{S}_Z^{\mathbf{II}}$, and $x \in Z^{\mathbb{N}}$ let $\sigma * \tau$, $\sigma * x$ and $x * \tau$ be defined as above. For any $C \subseteq Z^{\mathbb{N}}$ let:

$$\mathcal{WS}_Z^{\mathbf{I}}(C) = \{\sigma \in \mathcal{S}_Z^{\mathbf{I}} \mid \forall x \in Z^{\mathbb{N}} (\sigma * x \in C)\},$$
$$\mathcal{WS}_Z^{\mathbf{II}}(C) = \{\tau \in \mathcal{S}_Z^{\mathbf{II}} \mid \forall x \in Z^{\mathbb{N}} (x * \tau \notin C)\}.$$

Therefore saying that player **I** (or **II**) has a winning strategy in the game $G(C)$, simply means that $\mathcal{WS}_Z^{\mathbf{I}}(C) \neq \emptyset$ (resp. $\mathcal{WS}_Z^{\mathbf{II}}(C) \neq \emptyset$).

If τ is a strategy for **II**, $\langle a_0, a_1, \ldots\rangle \in Z^{\mathbb{N}}$ is the sequence of **I**'s moves, and $\langle b_0, b_1, \ldots\rangle$ is the sequence of **II**'s answers according to τ, then a Lipschitz map is obtained:

$$\hat{\tau} : Z^{<\mathbb{N}} \to Z^{<\mathbb{N}} \qquad \langle a_0, \ldots, a_n\rangle \mapsto \langle b_0, \ldots, b_n\rangle.$$

The Lipschitz function $f_{\hat{\tau}} : Z^{\mathbb{N}} \to Z^{\mathbb{N}}$ induced by $\hat{\tau}$ can be written as $f_{\hat{\tau}}(x) = (x * \tau)_{\mathbf{II}}$ where for any $y \in Z^{\mathbb{N}}$ the sequence $y_{\mathbf{II}} \in Z^{\mathbb{N}}$ is the sequence:

$$y_{\mathbf{II}}(n) = y(2n+1).$$

Conversely, any function $f_\varphi : Z^{\mathbb{N}} \to Z^{\mathbb{N}}$ with $\varphi : Z^{<\mathbb{N}} \to Z^{<\mathbb{N}}$ Lipschitz, gives rise to a strategy for **II** in the game on Z

$$\tau(s) = \varphi(s)(\text{length}(s) - 1),$$

such that $f_{\hat{\tau}} = f$. Similarly, any strategy σ for **I** yields a map

$$\hat{\sigma} : Z^{<\mathbb{N}} \to Z^{<\mathbb{N}} \qquad \langle b_0, \ldots, b_n\rangle \mapsto \langle a_0, \ldots, a_{n+1}\rangle$$

where $a_n = \sigma(\langle b_0, \ldots, b_n\rangle)$. Equivalently $f_{\hat{\sigma}}(x) = (\sigma * x)_{\mathbf{I}}$ where

$$y_{\mathbf{I}}(n) = y(2n)$$

for all $y \in Z^{\mathbb{N}}$. In fact the function $\hat{\sigma}$ is more than just Lipschitz, since the string $\langle b_0, \ldots, b_n\rangle$ is enough to produce the string $\langle a_0, \ldots, a_n, a_{n+1}\rangle$. This is equivalent to saying that the induced map $f : Z^{\mathbb{N}} \to Z^{\mathbb{N}}$ satisfies

$$d(f(x), f(y)) \leq \tfrac{1}{2} d(x, y).$$

An $f : Z^{\mathbb{N}} \to Z^{\mathbb{N}}$ satisfying such property is called a *contraction*, and any contraction is induced by a strategy for **I** in the game on Z. William Wadge in [1] introduced two games, now dubbed the Lipschitz and Wadge games, which are quite relevant for reducibility.

Let $A, B \subseteq Z^{\mathbb{N}}$. The *Lipschitz game* $G_{\mathrm{L}}(A, B)$ is the perfect information game on Z in which **I** and **II** take turns and play as in (1) and player **II** wins just in case

$$\langle a_0, a_1, \ldots\rangle \in A \Leftrightarrow \langle b_0, b_1, \ldots\rangle \in B.$$

The Lipschitz game is just a particular instance of the games described before: in fact $G_{\mathrm{L}}(A, B)$ is just $G(A_{\mathbf{I}} \triangle B_{\mathbf{II}})$ where

$$A_{\mathbf{I}} = \{x \in Z^{\mathbb{N}} \mid x_{\mathbf{I}} \in A\}, \quad \text{and}$$
$$B_{\mathbf{II}} = \{x \in Z^{\mathbb{N}} \mid x_{\mathbf{II}} \in B\}.$$

By definition of $G_{\mathrm{L}}(A, B)$, **II** has a winning strategy if and only if $A \leq_{\mathrm{L}} B$, and **I** has a winning strategy if and only if there is a contraction g such that $\neg B = g^{-1}(A)$ –

hence, in particular, $\neg B \leq_{\mathrm{L}} A$. Therefore Lipschitz reducibility can be characterised in terms of the existence of strategies for games on Z.

A similar game-theoretic characterization can be given for continuous reducibility too. The *Wadge game* $G_{\mathrm{W}}(A,B)$ where $A,B \subseteq Z^{\mathbb{N}}$ is similar to the Lipschitz game, but **II** has the option of passing at any round, with the stipulation that **II** must play infinitely often, otherwise he loses. If $\langle a_0,a_1,\ldots\rangle$ and $\langle b_0,b_1,\ldots\rangle$ are the elements of $Z^{\mathbb{N}}$ played by **I** and **II**, then **II**'s winning condition is as in the Lipschitz game. Formally $G_{\mathrm{W}}(A,B)$ is a game on $Z \cup \{\mathsf{p}\}$, where p is an element not in Z: **I**'s moves are restricted to Z while **II** cannot play a sequence that is eventually equal to p, so $G_{\mathrm{W}}(A,B)$ is just the game $G(C)$ where $C \subseteq (Z \cup \{\mathsf{p}\})^{\mathbb{N}}$ is the set of all $x \in (Z \cup \{\mathsf{p}\})^{\mathbb{N}}$ such that $x_{\mathbf{I}} \in Z^{\mathbb{N}}$ and either

(i) $\exists n \forall m \geq n\ x_{\mathbf{II}}(m) = \mathsf{p}$, or else
(ii) if $y \in Z^{\mathbb{N}}$ is the sequence obtained from $x_{\mathbf{II}}$ after erasing all p's, then $x_{\mathbf{I}} \in A \Leftrightarrow y \notin B$.

Although the Wadge game is really a game on $Z \cup \{\mathsf{p}\}$ it is more convenient to think of it as a game on Z with player **II** having the option of passing (the move p) at any stage. Thus a strategy τ for **II** need not to move at every round, but we require that $x * \tau$ be infinite, for any $x \in Z^{\mathbb{N}}$. Therefore $\hat{\tau}: Z^{<\mathbb{N}} \to Z^{<\mathbb{N}}$ the map on strings induced by τ is continuous, hence the function $f_{\hat{\tau}}: Z^{\mathbb{N}} \to Z^{\mathbb{N}}$ is continuous. Conversely, any continuous $f: Z^{\mathbb{N}} \to Z^{\mathbb{N}}$ is of the form $f_{\hat{\tau}}$, for some strategy τ for **II** in the Wadge game. Therefore $A \leq_{\mathrm{W}} B$ just in case **II** has a winning strategy in $G_{\mathrm{W}}(A,B)$. Conversely, if σ is a winning strategy for **I** in $G_{\mathrm{W}}(A,B)$, then σ is also winning in $G_{\mathrm{L}}(A,B)$, hence $\neg B$ is reducible to A via a contraction. Notice that **I** having a winning strategy in $G_{\mathrm{W}}(A,B)$ is a much stronger condition than $\neg B$ being reducible to A via a contraction, since **I** is required to play at every round, while **II** can take arbitrarily long naps.

2.2. Determinacy

Given $C \subseteq Z^{\mathbb{N}}$ it is not possible that both players have winning strategies σ and τ in $G(C)$ since otherwise $\sigma * \tau$ would be a play simultaneously inside and outside of C. The set C (or the game $G(C)$) is said to be *determined* if one of the players has a winning strategy in $G(C)$. If $\mathcal{C}$ is a non-empty family of subsets of $Z^{\mathbb{N}}$, then $\mathsf{AD}_Z(\mathcal{C})$ is the assertion:

$$\forall C \in \mathcal{C}\,(G(C) \text{ is determined}).$$

When $Z = \mathbb{N}$, the subscript is dropped and we write $\mathsf{AD}(\mathcal{C})$. The assertion $\mathsf{AD}_Z(\mathscr{P}(Z^{\mathbb{N}}))$ is called the *axiom of determinacy for games on Z*, and is usually denoted by AD_Z. If Z is a singleton, then AD_Z is trivially true, and if there is an injective map from Z to W, then $\mathsf{AD}_W \Rightarrow \mathsf{AD}_Z$. If Z has two elements then AD_Z is equivalent to AD, and it contradicts the axiom of choice.

Gale and Stewart [4] proved in ZFC that $\mathsf{AD}_Z(\mathcal{C})$ holds for any Z, when $\mathcal{C}$ is the collection of all closed subsets of $Z^{\mathbb{N}}$. The axiom of choice is needed for this result: in fact AC is equivalent (over the base theory ZF) to statement that for any non-empty Z, all closed subsets of $Z^{\mathbb{N}}$ are determined. D.A. Martin in [5] extended the result of Gale and Stewart when $\mathcal{C}$ is the collection of all Borel subsets of $Z^{\mathbb{N}}$. Martin's result is optimal, since the determinacy of all analytic[3] subsets of $2^{\mathbb{N}}$ is not

3. A set $A \subseteq Z^{\mathbb{N}}$ is analytic if it is the projection of a closed subset of $Z^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$.

provable in ZFC alone, as it implies [6] a certain large cardinal hypothesis known as $0^{\#}$. Assuming the existence of large cardinals, Woodin in unpublished work showed that the theory $\mathsf{ZF}+\mathsf{DC}(\mathbb{R})+\mathsf{AD}_{\mathbb{R}}$ is consistent, and hence the theory $\mathsf{ZF}+\mathsf{DC}(\mathbb{R})+\mathsf{AD}$ is also consistent. On the other hand AD_{ω_1} is well-known to be inconsistent with ZF.

Assuming AD_Z all games $G_{\mathrm{L}}(A,B)$ with $A,B \subseteq Z^{\mathbb{N}}$ are determined and therefore:

$$\forall A,B \in \mathscr{P}(Z^{\mathbb{N}})\,(A \leq_{\mathrm{L}} B \ \vee\ \neg B \leq_{\mathrm{L}} A).$$

The statement above is called the *semi-linear ordering principle for Lipschitz functions*, $\mathsf{SLO}^{\mathrm{L}}$ for short. Since Lipschitz functions are particular kind of continuous functions, the pre-order $\leq_{\mathrm{L}}$ could be replaced with $\leq_{\mathrm{W}}$ in the formula above, and the resulting statement is denoted by $\mathsf{SLO}^{\mathrm{W}}$. These semi-linear ordering principles assert that, up to the identification of any degree with its dual, the relations $\leq_{\mathrm{L}}$ and $\leq_{\mathrm{W}}$ are total orders. More generally, given a topological space X and a set of functions $\mathcal{F} \subseteq X^X$ containing the identity and closed under composition, it is possible to define the semi-linear ordering principle $\mathsf{SLO}^{\mathcal{F}}$:

$$\forall A,B \in \mathscr{P}(X)\,(A \leq_{\mathcal{F}} B \ \vee\ \neg B \leq_{\mathcal{F}} A).$$

We now focus on the case when the space is $\mathbb{N}^{\mathbb{N}}$ or, more generally, $Z^{\mathbb{N}}$ with Z countable. Then the following implications hold:

$$\begin{tikzcd}
 & \mathsf{AD}^{\mathrm{L}} \arrow[r, Rightarrow] & \mathsf{SLO}^{\mathrm{L}} \arrow[d, Rightarrow] \\
\mathsf{AD} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \\
 & \mathsf{AD}^{\mathrm{W}} \arrow[r, Rightarrow] & \mathsf{SLO}^{\mathrm{W}}
\end{tikzcd}$$

It has been conjectured that $\mathsf{SLO}^{\mathrm{W}} \Rightarrow \mathsf{AD}$, at least if $\mathrm{V}=\mathrm{L}(\mathbb{R})$. Moreover, in [7, 8] it is shown that if all subsets of $\mathbb{N}^{\mathbb{N}}$ have the property of Baire, then:

$$\mathsf{AD}^{\mathrm{W}} \Leftrightarrow \mathsf{AD}^{\mathrm{L}} \Leftrightarrow \mathsf{SLO}^{\mathrm{L}} \Leftrightarrow \mathsf{SLO}^{\mathrm{W}}.$$

The assumption of these principles yield a very regular picture of Lipschitz and Wadge hierarchies. However all these principles, just like AD, are inconsistent with the full axiom of choice, justifying the restriction to a weaker basic theory, like $\mathsf{ZF}+\mathsf{DC}(\mathbb{R})$. To see this, recall that a subset of a topological space is perfect if it is closed and it has no isolated points. The axiom of choice implies that in an uncountable Polish (i.e., separable and completely metrizable) space there are *Bernstein sets*, i.e. sets which do not contain, nor are disjoint from any non-empty perfect subsets (see [9], Example 8.24). On the other hand the following holds.

Proposition 1. [*Wadge*] *Assume* $\mathsf{SLO}^{\mathrm{W}}$. *Then all uncountable subsets of* $\mathbb{N}^{\mathbb{N}}$ *contain a non-empty perfect set.*

2.3. The structure of Lipschitz degrees on the Baire space

We will investigate how Lipschitz degrees on the Baire and Cantor spaces look like, under suitable determinacy assumptions, starting here with the Baire space. By the discussion above, we will be allowed to show existence of Lipschitz or continuous reductions either directly or by use of game theoretic arguments.

Lemma 1. *Let* $A,B \subseteq \mathbb{N}^{\mathbb{N}}$. *If* $A \leq_{\mathrm{L}} B$ *and* $s,t \in \mathbb{N}^{<\mathbb{N}}$ *are sequences of the same length, then* $s^\frown A \leq_{\mathrm{L}} t^\frown B$.

Proof. By hypothesis, **II** has a winning strategy in $G_{\mathrm{L}}(A,B)$. Then player **II** wins $G_{\mathrm{L}}(s^\frown A,t^\frown B)$ as follows. As long as **I** enumerates s, player **II** enumerates t; if at some round $n < \mathrm{length}(s)$ player **I** deviates from s, then **II**'s reply deviates from t. If the two players have produced s,t respectively with their first length(s) moves, **II** makes use from this move on of the winning strategy he has in $G_{\mathrm{L}}(A,B)$. ∎

For Z a non-empty set, $A \subseteq Z^{\mathbb{N}}$ and $s \in Z^{<\mathbb{N}}$, define

$$A_{\lfloor s\rfloor} = \{x \in Z^{\mathbb{N}} \mid s^\frown x \in A\}.$$

Lemma 2. *For all* $A,B \subseteq \mathbb{N}^{\mathbb{N}}$ *and* $s,t \in \mathbb{N}^{<\mathbb{N}}$*:*

a) $A \leq_{\mathrm{L}} s^\frown A$*;*
b) $\mathrm{length}(s) \leq \mathrm{length}(t) \wedge A \leq_{\mathrm{L}} B \Rightarrow s^\frown A \leq_{\mathrm{L}} t^\frown B$
c) $A_{\lfloor s\rfloor} \leq_{\mathrm{L}} A$*;*
d) *if* A *is self dual, then* $s^\frown A$ *is self dual.*

Proof.

a) To win $G_{\mathrm{L}}(A,s^\frown A)$, player **II** produces s, then copies all moves **I** is being playing since the beginning.

b) To win $G_{\mathrm{L}}(s^\frown A,t^\frown B)$, player **II** first builds the sequence t (unless **I** deviates from s, in which case **II** plays anything else), then makes use of his winning strategy in $G_{\mathrm{L}}(A,B)$.

c) To win $G_{\mathrm{L}}(A_{\lfloor s\rfloor},A)$, player **II** first produces s, then copies all moves **I** is being playing since the beginning.

d) To win $G_{\mathrm{L}}(s^\frown A,\neg(s^\frown A))$, player **II** plays s as long as **I** plays s (deviating from s as soon as **I** does) and then making use of his winning strategy in $G_{\mathrm{L}}(A,\neg A)$. ∎

Lemma 3. *Suppose* $A \leq_{\mathrm{L}} \neg A$ *and let* $n \in \mathbb{N}$. *Then:*

a) $A <_{\mathrm{L}} n^\frown A$*;*
b) *Assume* AD^{L}. *Then* $[n^\frown A]_{,\mathrm{L}}$ *is the immediate successor of* $[A]_{\mathrm{L}}$.

Proof.

a) Let τ be a winning strategy for **II** in $G_{\mathrm{L}}(A,\neg A)$. Then the strategy σ dfined by $\sigma(\langle\rangle) = n$ and $\sigma(s) = \tau(s)$, if length$(s) \neq 0$, is a winning strategy for **I** in $G_{\mathrm{L}}(n^\frown A,A)$. Therefore **II** does not have a winning strategy in $G_{\mathrm{L}}(n^\frown A,A)$, hence $n^\frown A \not\leq_{\mathrm{L}} A$; by part (a) of Lemma 2, $A <_{\mathrm{L}} n^\frown A$.

b) Suppose $B <_{\mathrm{L}} n^\frown A$. Then $n^\frown A \not\leq_{\mathrm{L}} B$ and since $n^\frown A$ is self-dual, it follows $n^\frown A \not\leq_{\mathrm{L}} \neg B$. Thus **II** does not have a winning strategy in $G_{\mathrm{L}}(n^\frown A,\neg B)$ hence by AD^{L} **I** has a winning strategy σ for such game. Thus $\sigma\restriction(\mathbb{N}^{\mathbb{N}} \setminus \{\langle\rangle\})$ is a winning strategy for **II** in $G_{\mathrm{L}}(B,A)$. The uniqueness of $[n^\frown A]_{\mathrm{L}}$ as immediate successor of $[A]_{\mathrm{L}}$ follows because $[n^\frown A]_{,\mathrm{L}}$ is self-dual and by the semi-linear ordering principle. ∎

Proposition 2.

a) *If* $A \leq_{\mathrm{L}} \neg A$*, then* $\forall n\ (A_{\lfloor n\rfloor} <_{\mathrm{L}} A)$.
b) $(\mathsf{AD}^{\mathrm{L}})$ *If* $\forall n\ (A_{\lfloor n\rfloor} <_{\mathrm{L}} A)$*, then* $A \leq_{\mathrm{L}} \neg A$.

Proof.

a) By part (c) of Lemma 2, $A_{\lfloor n\rfloor} \leq_{\mathrm{L}} A$. If $A_{\lfloor n\rfloor} \equiv_{\mathrm{L}} A$, then $A_{\lfloor n\rfloor}$ is Lipschitz self-dual and so $A_{\lfloor n\rfloor} <_{\mathrm{L}} n^\frown A_{\lfloor n\rfloor} \leq_{\mathrm{L}} A$, the second inequality holding as $A \neq \mathbb{N}^{\mathbb{N}}$: a contradiction.

b) Since $A \not\leq_{\mathrm{L}} A_{\lfloor n\rfloor}$, let σ_n be a winning strategy for **I** in $G_{\mathrm{L}}(A, A_{\lfloor n\rfloor})$. Then $\tau(n^\frown s) = \sigma_n(s)$ defines a strategy for **II** We claim that it is winning for the game $G_{\mathrm{L}}(A, \neg A)$ since

$$n^\frown a \in A \Leftrightarrow a \in A_{\lfloor n\rfloor} \Leftrightarrow ((n^\frown a) * \tau)_{\mathbf{II}} = (\sigma_n * a)_{\mathbf{I}} \in \neg A.$$

■

For $A, B, A_n \subseteq \mathbb{N}^{\mathbb{N}}$, define

$$A \oplus B = \bigcup_{n\in\mathbb{N}} ((2n^\frown A) \cup ((2n+1)^\frown B)),$$

$$\bigoplus\nolimits_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} n^\frown A_n.$$

Lemma 4. *Let $A, A', B, B', A_n, A'_n \subseteq \mathbb{N}^{\mathbb{N}}$.*

a) *$A \leq_{\mathrm{L}} A \oplus B$, $B \leq_{\mathrm{L}} A \oplus B$, and $A_i \leq_{\mathrm{L}} \bigoplus_{n\in\mathbb{N}} A_n$ for all i.*
b) *$A \leq_{\mathrm{L}} A' \wedge B \leq_{\mathrm{L}} B' \;\Rightarrow\; A \oplus B \leq_{\mathrm{L}} A' \oplus B'$.*
c) *$\forall n \in \mathbb{N}\ (A_n \leq_{\mathrm{L}} A'_n) \;\Rightarrow\; \bigoplus_{n\in\mathbb{N}} A_n \leq_{\mathrm{L}} \bigoplus_{n\in\mathbb{N}} A'_n$.*

Proof.

a) To win $G_{\mathrm{L}}(A, A \oplus B)$, player **II** begins with 0 and then copies **I**'s moves; similarly **II** wins $G_{\mathrm{L}}(B, A \oplus B)$ (respectively, $G_{\mathrm{L}}(A_i, \bigoplus_{n\in\mathbb{N}} A_n)$) by playing 1 (respectively, i) and then copying **I**'s moves.

b) If τ and τ' are winning strategies for **II** in $G_{\mathrm{L}}(A, A')$ and $G_{\mathrm{L}}(B, B')$, respectively, then **II** wins $G_{\mathrm{L}}(A \oplus B, A' \oplus B')$ by copying **I**'s first move a_0 and then employing τ or τ' according to whether a_0 is even or odd.

c) Similarly as above, if τ_n is a winning strategy for **II** in $G_{\mathrm{L}}(A_n, A'_n)$, a winning strategy τ for **II** in $G_{\mathrm{L}}(\bigoplus_{n\in\mathbb{N}} A_n, \bigoplus_{n\in\mathbb{N}} A'_n)$ is defined by letting $\tau(n) = n$ and $\tau(n^\frown s) = \tau_n(s)$. ■

By Lemma 4, the operations $\oplus, \bigoplus$ extend to Lipschitz degrees:

$$[A]_{\mathrm{L}} \oplus [B]_{\mathrm{L}} = [A \oplus B]_{\mathrm{L}},$$

$$\bigoplus\nolimits_{n\in\mathbb{N}} [A_n]_{\mathrm{L}} = [\bigoplus\nolimits_{n\in\mathbb{N}} A_n]_{\mathrm{L}}.$$

Note also that $\neg(A \oplus B) = \neg A \oplus \neg B, \neg \bigoplus_{n\in\mathbb{N}} A_n = \bigoplus_{n\in\mathbb{N}} \neg A_n$.

Lemma 5.

a) *$A \oplus \neg A$ is Lipschitz self-dual.*
b) (AD^{L}) *There is no B such that $A <_{\mathrm{L}} B <_{\mathrm{L}} A \oplus \neg A$.*
c) (AD^{L}) *If A is non-self-dual, then $[A \oplus \neg A]_{\mathrm{L}} = \sup([A]_{\mathrm{L}}, [\neg A]_{\mathrm{L}})$.*

Proof.

a) The Lipschitz function $g:\mathbb{N}^{\mathbb{N}}\to\mathbb{N}^{\mathbb{N}}$

$$g(2n^\frown x)=1^\frown x$$
$$g((2n+1)^\frown x)=0^\frown x$$

witnesses $A\oplus\neg A\leq_{\mathrm{L}}\neg A\oplus A=\neg(A\oplus\neg A)$.

b) Suppose $B<_{\mathrm{L}}A\oplus\neg A$, in order to show $B\leq_{\mathrm{L}}A\vee B\leq_{\mathrm{L}}\neg A$, which is enough to conclude both in the case A is self-dual and when it is not. Since there is no winning strategy for **II** in $G_{\mathrm{L}}(A\oplus\neg A,B)$, let σ be a winning strategy for **I** in this game. Then $\sigma\restriction\mathbb{N}^{<\mathbb{N}}\setminus\{\langle\rangle\}$ is a winning strategy for **II** in $G_{\mathrm{L}}(B,\neg A)$ or in $G_{\mathrm{L}}(B,A)$ according to whether $\sigma(\langle\rangle)$ is even or odd.

c) Since A is not self-dual, by Lemma 4(a) and part (a) above we have that $A<_{\mathrm{L}}A\oplus\neg A$ and $\neg A<_{\mathrm{L}}A\oplus\neg A$. ■

Lemma 6. *Let $A_n\subseteq\mathbb{N}^{\mathbb{N}}$ be such that $\forall n\ \exists m>n\ (A_n<_{\mathrm{L}}A_m)$.*

a) ($\mathsf{SLO}^{\mathrm{L}}$) $\bigoplus_{n\in\mathbb{N}}A_n$ *is Lipschitz self-dual.*
b) (AD^{L}) $[\bigoplus_{n\in\mathbb{N}}A_n]_{\mathrm{L}}=\sup_{i\in\mathbb{N}}[A_i]_{\mathrm{L}}$.
c) (AD^{L}) *If B_n are any subsets of $\mathbb{N}^{\mathbb{N}}$ such that $\forall n\ \exists m\ (A_n\leq_{\mathrm{L}}B_m)$, then* $\bigoplus_{n\in\mathbb{N}}A_n\leq_{\mathrm{L}}\bigoplus_{n\in\mathbb{N}}B_n$.

Proof. Let $j(n)$ be the least $m>n$ such that $A_n<_{\mathrm{L}}A_m$.

a) By $\mathsf{SLO}^{\mathrm{L}}$ the inequalities $A_n<_{\mathrm{L}}\neg A_{j(n)}$ also hold: let τ_n be a winning strategy for **II** in $G_{\mathrm{L}}(A_n,\neg A_{j(n)})$. Then $\tau(n)=j(n)$, $\tau(n^\frown s)=\tau_n(s)$ defines a winning strategy for **II** in $G_{\mathrm{L}}(\bigoplus_{n\in\mathbb{N}}A_n,\neg\bigoplus_{n\in\mathbb{N}}A_n)$.

b) Let $B\subseteq\mathbb{N}^{\mathbb{N}}$ be such that $\forall n\ (A_n\leq_{\mathrm{L}}B)$. If $\bigoplus_{n\in\mathbb{N}}A_n\not\leq_{\mathrm{L}}B$, then **I** has a winning strategy σ in $G_{\mathrm{L}}(\bigoplus_{n\in\mathbb{N}}A_n,B)$. Let $n=\sigma(\langle\rangle)$. Then $B\leq_{\mathrm{L}}\neg A_n<_{\mathrm{L}}A_{j(n)}$, a contradiction.

c) By Lemma 4(a) and part (b) above. ■

Proposition 3. (AD^{L}) *A limit Lipschitz degree is self-dual if and only if it is of countable cofinality.*

Proof. If the sets A_n witness that $[A]_{\mathrm{L}}$ is of countable cofinality, then Lemma 6 implies that $[\bigoplus_{n\in\mathbb{N}}A_n]_{\mathrm{L}}=[A]_{\mathrm{L}}$ is self-dual.

Conversely, suppose $[A]_{\mathrm{L}}$ is self-dual. By Proposition 2, $\forall n\ (A_{\lfloor n\rfloor}<_{\mathrm{L}}A)$. Let $B_0=A_{\lfloor 0\rfloor}$

$$B_{n+1}=\begin{cases}A_{\lfloor n+1\rfloor} & \text{if } B_n<_{\mathrm{L}}A_{\lfloor n+1\rfloor},\\ 0^\frown(B_n\oplus\neg B_n) & \text{otherwise.}\end{cases}$$

By induction it follows that $A_{\lfloor n\rfloor}\leq_{\mathrm{L}}B_n<_{\mathrm{L}}B_{n+1}<_{\mathrm{L}}A$, so $\bigoplus_{n\in\mathbb{N}}B_n$ is Lipschitz self-dual and

$$A=\bigoplus_{n\in\mathbb{N}}A_{\lfloor n\rfloor}\leq_{\mathrm{L}}\bigoplus_{n\in\mathbb{N}}B_n\leq_{\mathrm{L}}A$$

by Lemma 6 and part (c) of Lemma 4. Thus the degrees $[B_n]_{\mathrm{L}}$ witness that A is of countable cofinality. ■

Under AD^{L} one gets a first bit of the picture of Lipschitz degrees on the Baire space. Suppose $[A]_{\mathrm{L}}$ is self-dual. By Lemma 3, $[0^\frown A]_{\mathrm{L}}$ is the immediate successor of

$[A]_{\mathrm{L}}$ and it is self-dual. Therefore it is possible to define a successor operator $\mathcal{L}$ on self-dual Lipschitz degrees: $\mathcal{L}([A]_{\mathrm{L}}) = [0^\frown A]_{\mathrm{L}}$. Using Lemma 6, the operation can be iterated through the countable ordinals by letting:

$$\begin{aligned} \mathcal{L}^0([A]_{\mathrm{L}}) &= [A]_{\mathrm{L}}, \\ \mathcal{L}^{\alpha+1}([A]_{\mathrm{L}}) &= \mathcal{L}(\mathcal{L}^{\alpha}([A]_{\mathrm{L}})), \\ \mathcal{L}^{\lambda}([A]_{\mathrm{L}}) &= \big[\bigoplus_{n\in\mathbb{N}} A_n\big]_{\mathrm{L}}, \end{aligned}$$

where $A_n \in \mathcal{L}^{\alpha_n}([A]_{\mathrm{L}})$ for a sequence α_n increasing and cofinal in the limit countable ordinal λ. This operation is well defined at limit steps by part (c) of Lemma 6. Thus following any self-dual Lipschitz degree there is an ω_1 sequence of self-dual Lipschitz degrees.

Assuming AD, one can recover a complete picture of the Lipschitz hierarchy using the following theorem of Martin (building on partial results of L. Monk). For a proof, see [2] (Theorem 2.2).

Theorem 1. (AD) *The relation* $<_{\mathrm{L}}$ *is well founded on* $\mathscr{P}(\mathbb{N}^{\mathbb{N}})$.

In view of this result, one can define the *Lipschitz rank* $\|A\|_{\mathrm{L}}$ of a set $A \subseteq \mathbb{N}^{\mathbb{N}}$ as the rank of A in the well founded relation $<_{\mathrm{L}}$; for technical reasons non-zero ordinals are used, so starting with $\|\emptyset\|_{\mathrm{L}} = \|\mathbb{N}^{\mathbb{N}}\|_{\mathrm{L}} = 1$.

Summarising the results of this section, under AD the complete picture of the Lipschitz hierarchy for the Baire space can be described as follows. The preorder $\leq_{\mathrm{L}}$ on Lipschitz degrees of $\mathbb{N}^{\mathbb{N}}$ is well founded and its anti-chains have size at most 2. The hierarchy begins with a pair of non-self-dual degrees $[\emptyset]_{\mathrm{L}}$, $[\mathbb{N}^{\mathbb{N}}]_{\mathrm{L}}$ and each pair of non-self-dual degrees is followed by an ω_1-sequence of self-dual degrees. At limit levels of countable cofinality there is a self-dual degree, while at limit levels of uncountable cofinality there is a pair of non-self-dual degrees.

2.4. Lipschitz hierarchy on Cantor spaces

The analysis of the previous section can be adapted for the study of Lipschitz degrees of the Cantor space $k^{\mathbb{N}}$. The notion of infinite sum $\bigoplus_{n\in\mathbb{N}} A_n$ does not make sense any longer, but the definition of sum $A \oplus B$ can be adapted to this case, by setting $A \oplus B = (0^\frown A) \cup (1^\frown B)$. By the analogous of part (c) of Lemma 5 and part (a) of Lemma 3, above any pair of non-self-dual degrees there is an ω-sequence of self-dual degrees; however no self-dual degree $[A]_{\mathrm{L}}$ is limit, since $[A]_{\mathrm{L}} = [\bigcup_{n<k}(n^\frown A_{\lfloor n\rfloor})]_{\mathrm{L}} = \sup_{n<k}[A_{\lfloor n\rfloor}]_{\mathrm{L}}$. Since the Martin-Monk Theorem holds also for $k^{\mathbb{N}}$, the structure of Lipschitz degrees in Cantor spaces, under AD, is determined as follows. The preorder $\leq_{\mathrm{L}}$ on Lipschitz degrees of $k^{\mathbb{N}}$ is well founded and its anti-chains have size at most 2. The hierarchy begins with a pair of non-self-dual degrees $[\emptyset]_{\mathrm{L}}$, $[k^{\mathbb{N}}]_{\mathrm{L}}$ and each pair of non-self-dual degrees is followed by an ω-sequence of self-dual degrees. At limit levels there are pairs of non-self-dual degrees.

2.5. The Wadge hierarchy

Our aim is now to study the hierarchy of Wadge degrees on the Baire space and the Cantor space. The first observation is that every Lipschitz function is continuous, in other words $A \leq_{\mathrm{L}} B \Rightarrow A \leq_{\mathrm{W}} B$; so Lipschitz hierarchy is a refinement of Wadge hierarchy and a Lipschitz self-dual set is also Wadge self-dual. In fact, the following holds.

Lemma 7. *Let* $A, A_n \subseteq \mathbb{N}^{\mathbb{N}}$.

a) $A \neq \mathbb{N}^{\mathbb{N}} \Rightarrow \forall n\ (n^\frown A \equiv_{\mathrm{W}} A)$.
b) $\forall n\ (A_n \leq_{\mathrm{W}} A) \Rightarrow \bigoplus_{n\in\mathbb{N}} A_n \leq_{\mathrm{W}} A$.

Proof.

a) The inequality $A \leq_{\mathrm{W}} n^\frown A$ follows from $A \leq_{\mathrm{L}} n^\frown A$.
Let $y \in \neg A$. Define $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ by letting:

$$f(m^\frown x) = \begin{cases} x & \text{if } m = n, \\ y & \text{if } m \neq n. \end{cases}$$

Then $f^{-1}(A) = n^\frown A$.

b) Let $f_n : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be such that $f_n^{-1}(A) = A_n$. If $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}, f(n^\frown x) = f_n(x)$, then $f^{-1}(A) = \bigoplus_{n\in\mathbb{N}} A_n$. ■

Therefore, by induction on $\alpha < \omega_1$, it follows that $\mathcal{L}^{\alpha}([A]_{\mathrm{L}}) \subseteq [A]_{\mathrm{W}}$. So, under AD^{L}, every ω_1-sequence of consecutive self-dual Lipschitz degrees is contained in a single self-dual Wadge degree. The following theorem of Steel and Van Wesep [2] (Theorem 3.1) implies that any self-dual Wadge degree is the union of the Lipschitz degrees contained in it.

Theorem 2. *Assume* AD. *Then* $A \leq_{\mathrm{W}} \neg A \Rightarrow A \leq_{\mathrm{L}} \neg A$.

Corollary 1. (AD) *Let* $A \subseteq \mathbb{N}^{\mathbb{N}}$.

a) *A is Lipschitz non-self-dual if and only if it is Wadge non-self-dual. In this case* $[A]_{\mathrm{W}} = [A]_{\mathrm{L}}$ *and*

$$\{B \in \mathscr{P}(\mathbb{N}^{\mathbb{N}}) \mid B \leq_{\mathrm{W}} A\} = \{B \in \mathscr{P}(\mathbb{N}^{\mathbb{N}}) \mid B \leq_{\mathrm{L}} A\}.$$

b) *Every self-dual Wadge degree* $[A]_{\mathrm{W}}$ *is the union of an* ω_1*-sequence of consecutive Lipschitz self-dual degrees and*

$$\{B \in \mathscr{P}(\mathbb{N}^{\mathbb{N}}) \mid B \leq_{\mathrm{L}} A\} \subset \{B \in \mathscr{P}(\mathbb{N}^{\mathbb{N}}) \mid B \leq_{\mathrm{W}} A\}.$$

c) *Every self-dual Wadge degree has a non-self-dual pair of immediate successor degrees.*
d) *Every non-self-dual pair of Wadge degrees has a self-dual degree as immediate successor.*
e) *A limit Wadge degree is self-dual if and only if it has countable cofinality.*

Assuming AD, Theorem 1 (together with the observation that well foundedness of $<_{\mathrm{L}}$ implies the well foundedness of $<_{\mathrm{W}}$) and Corollary 1 give a complete picture of the Wadge hierarchy on $\mathbb{N}^{\mathbb{N}}$.

Analogous results – again assuming AD – hold for Cantor space as well: every self-dual Wadge degree is the union of an ω-sequence of consecutive self-dual Lipschitz degrees, non-self-dual Wadge degrees coincide with the corresponding non-self-dual Lipschitz degrees, and all limit Wadge degrees are non-self-dual.

Remark. The use of the axiom of determinacy in the above results is local, in the following sense. If $\mathcal{A}$ is a class of sets that is an initial segment with respect to $\leq_W$ and determinacy is assumed for sets in $\mathcal{A}$, then the structure of Lipschitz and Wadge degrees restricted to $\mathcal{A}$ is as described above. In particular, since Borel sets are detemined by Martin's theorem, the results on $\leq_L$, $\leq_W$ on Borel sets hold without additional assumptions.

2.6. Lipschitz and Wadge hierarchies in other spaces

Some of the results of the previous sections generalise to spaces of the form $[T]$ where T is a pruned tree on $\mathbb{N}$. For example, assuming AD, the Lipschitz and Wadge hierarchies on $[T]$ are well founded and antichains have size at most 2. However, the structure of these hierarchies is wide open in general spaces. The following remark points out that, under AC, the failure of continuum hypothesis yields arbitrarily big antichains.

Theorem 3. *Assume* AC. *For every cardinal* κ, *with* $\aleph_0 \leq \kappa \leq 2^{\aleph_0}$, *there is* $A_\kappa \subseteq [0,1]$ *such that whenever* $\kappa \neq \kappa'$ *the sets* A_κ, $A_{\kappa'}$ *are Wadge incomparable.*

Proof. Let $A_{2^{\aleph_0}}$ be any Bernstein set in $[0,1]$ and for $\aleph_0 \leq \kappa < 2^{\aleph_0}$ let A_κ be a set of cardinality κ such that $\operatorname{card}(A_\kappa \cap [a,b]) = \kappa$ for any subinterval $[a,b] \subseteq [0,1]$. (One way to get such A_κ is to fix an enumeration I_n $(n \in \mathbb{N})$ of open subintervals of $[0,1]$ with rational end points, choose sets $B_n \subseteq I_n$ of size κ, and set $A_\kappa = \bigcup_n B_n$.) Suppose $g:[0,1] \to [0,1]$ be a continuous function witnessing $A_\kappa = g^{-1}(A_{\kappa'})$. Since g cannot be constant, let $[a,b] \subseteq [0,1]$ be its range. As $\operatorname{card}(A_{\kappa'} \cap [a,b]) = \kappa'$, it follows $\kappa' \leq \kappa$. If $\kappa' < \kappa$, there are $y \in A_{\kappa'} \cap [a,b]$ and a cardinal $\lambda > \kappa'$ such that $\operatorname{card}(g^{-1}(\{y\})) = \lambda$. Being $g^{-1}(\{y\})$ closed, it follows that $\lambda = 2^{\aleph_0}$, so $g^{-1}(\{y\}) \subseteq A_\kappa$ contradicts either $\kappa < 2^{\aleph_0}$ or the fact that $A_{2^{\aleph_0}}$ does not contain perfect non-empty subsets of $[0,1]$, since the closed set $g^{-1}(\{y\})$ does contain some. ∎

3. Borel reducibility for finitary relations

So far we have looked at Lipschitz and continuous reducibilities, but a similar analysis can be carried out for the the notion of *Borel reducibility.* In order to define the notion in the proper setting, recall that a *standard Borel space* is a set X endowed with a σ-algebra which is the family of Borel subsets of X for some carefully chosen Polish topology; for this reason, the sets in such σ-algebra are called Borel sets. If X is standard Borel $A,B \subseteq X$, set

$$A \leq_{\mathbf{B}} B \Leftrightarrow \exists f \in \mathcal{B}\big(A = f^{-1}(B)\big)$$

where $\mathcal{B}$ is the collection of all Borel functions from X to X. Any two uncountable standard Borel spaces are Borel-isomorphic [9] (Theorem 15.6), so the notion of Borel reducibility $\leq_{\mathbf{B}}$ does not depend on the underlying space. The pre-order $\leq_{\mathbf{B}}$ and the associated equivalence classes $[A]_{\mathbf{B}}$, called the *Borel-Wadge degrees*, have been investigated in [10]. In that paper it is shown that, assuming AD, the structure of the Borel-Wadge degrees is analogous to the structure of the Wadge hierarchy on $\mathbb{N}^{\mathbb{N}}$: the relation $<_{\mathbf{B}}$ is well-founded, self-dual and non-self-dual pairs of degrees alternate, with self-dual degrees at limit levels of countable cofinality and non-self-dual pairs at limit levels of uncountable cofinality.

We are now going to focus on an important development: the theory of Borel reducibility for n-ary relations. This can be thought as the n-dimensional generalization of the reducibility relation exposed in the preceding paragraph. Let X, X' be standard Borel spaces and let R, R' be n-ary relations on X and X', respectively. Then R is *Borel reducible* to R', in symbols

$$R \leq^n_{\mathbf{B}} R',$$

if and only if there is a Borel function $g: X \to X'$ such that, for all $x_1, \ldots, x_n \in X$,

$$R(x_1, \ldots, x_n) \Leftrightarrow R'(g(x_1), \ldots, g(x_n)).$$

This notion is really of interest only when X and X' are uncountable, and since two uncountable such spaces are Borel isomorphic, all relations can be thought as living on the same space X. Thus the relation $\leq^n_{\mathbf{B}}$ is a preorder on the class of n-ary relations on X, and if no confusion arises, we will drop the superscipt and simply write $\leq_{\mathbf{B}}$. To see that it can be studied within the framework of the relations $\leq_{\mathcal{F}}$, consider the family of all functions $f: X^n \to X^n$ that are of the form $f = g \times \ldots \times g$ for some Borel $g: X \to X$. Then $R \leq_{\mathbf{B}} R'$ if and only if $R \leq_{\mathcal{F}} R'$ as subsets of X^n. In the last decade much work has been done on the study of $\leq_{\mathbf{B}}$, especially for some classes of binary relations. This section will briefly survey some results in this field, to see another application of the notion of reduction. We will concentrate on two kinds of relations: essentially countable Borel equivalence relations and analytic preorders (a subset of a standard Borel space is *analytic* if it is the Borel image of a Borel set, while it is *coanalytic* if its complement is analytic). It will turn out that the structure of $\leq_{\mathbf{B}}$ on binary relations is much more complicated than the almost well order of $\leq_{\mathrm{L}}$ or $\leq_{\mathrm{W}}$ on subsets (unary relations) of $\mathbb{N}^{\mathbb{N}}$.

3.1. Smooth equivalence relations

An equivalence relation E on a standard Borel space X is called *smooth* if $E \leq_{\mathbf{B}} =_X$, $E \leq_{\mathbf{B}} =_X$, where $=_X$ is equality on X. This means that there is a Borel function $g: X \to X$ such that:

$$\forall x, x' \in X \ (xEx' \Leftrightarrow g(x) = g(x')).$$

Note that such an E is Borel as a subset of X^2; moreover, if E is smooth equivalence relation on X and F is any binary relation on X with $F \leq_{\mathbf{B}} E$, then F is a smooth equivalence relation as well. The function witnessing smoothness assigns to each equivalence class of a smooth equivalence relation an element of a standard Borel space (equivalently: a real number). So a smooth equivalence relation E is also called *concretely classifiable*, since the elements are classified by a concrete object which distinguishes the equivalence classes, and moreover this is done in a Borel way. The cardinality of the quotient space X/E can be finite, countably infinite or the continuum. A fundamental result on Borel reducibility of equivalence relations is the following result, known as the Silver dichotomy:

Theorem 4. [*Silver*] *If E is a coanalytic equivalence relation on an uncountable standard Borel space, then either*

- *E has countably many classes or else*
- *$=_{\mathbb{R}} \leq_{\mathbf{B}} E$.*

Therefore two smooth equivalence relations E and F are always comparable with respect to $\leq_{\mathbf{B}}$ and they are ranked according to the cardinality of their quotient spaces:

$$E \leq_{\mathbf{B}} F \;\Leftrightarrow\; \mathrm{card}(X/E) \leq \mathrm{card}(X/F).$$

So, up to $\equiv_{\mathbf{B}}$, there are countably many smooth equivalence relations, which can be listed as follows:

$$1 <_{\mathbf{B}} 2 <_{\mathbf{B}} \ldots <_{\mathbf{B}} n <_{\mathbf{B}} \ldots <_{\mathbf{B}} \mathbb{N} <_{\mathbf{B}} \mathbb{R}. \tag{2}$$

Examples.

1. For n a positive natural number, let $\mathcal{M}(n\times n,\mathbb{C})$ be the space of all $n\times n$ complex matrices. The equivalence relation of similarity on $\mathcal{M}(n\times n,\mathbb{C})$ is a smooth equivalence relation: such matrices are classified by their Jordan normal form.

2. Isomorphism for Bernoulli measure preserving automorphisms on $[0,1]$ is smooth by [11], since these are classified by their entropy, a real number.

3. If U is the Urysohn space, the isometry relation on $\mathbf{K}(U)$ (the space of compact subsets of U) is smooth (see [12]). The situation is very different for isometry on all of $\mathbf{F}(U)$ (the space of closed subsets of U). The position of the latter equivalence relation in the $\leq_{\mathbf{B}}$ hierarchy has been computed precisely in [13].

Smoothness is perhaps the most desirable situation when dealing with an equivalence relation, in the sense that in this case one usually has a very good understanding of the equivalence. However this turns out to be a rather uncommon situation in mathematics.

3.2. Countable Borel equivalence relations

A *countable Borel* equivalence relation E on a standard Borel space X is an equivalence relation that is Borel as a subset of X^2 and such that all equivalence classes are countable. If F is a Borel equivalence relation, then F is *essentially countable* if, for some countable Borel equivalence relation E, the inequality $F \leq_{\mathbf{B}} E$ holds. So essentially countable Borel equivalence relations form an initial segment with respect to $\leq_{\mathbf{B}}$. Smooth equivalence relations provide the easiest examples of essentially countable Borel equivalence relations. Other examples of countable Borel equivalence relations are orbit equivalences induced by Borel actions of countable groups on standard Borel spaces. In fact, this is the only actual case by the following theorem of [14].

Theorem 5. *If E is a countable Borel equivalence relation on a standard Borel space X, then there are a countable group G and a Borel action $G\times X \to X$ inducing E.*

A very interesting fact concerning Borel equivalence relations is the Glimm-Effros dichotomy which, in the following general form, is due to [15].

Theorem 6. *There is a non-smooth countable Borel equivalence relation E_0 such that, for any Borel equivalence relation E, either E is smooth or $E_0 \leq_{\mathbf{B}} E$.*

So, for Borel equivalence relations, the chain an be continued as:

$$1 <_{\mathbf{B}} 2 <_{\mathbf{B}} \ldots <_{\mathbf{B}} n <_{\mathbf{B}} \ldots <_{\mathbf{B}} \mathbb{N} <_{\mathbf{B}} \mathbb{R} <_{\mathbf{B}} E_0 \tag{3}$$

and this chain contains all Borel equivalence relations (up to $\equiv_{\mathbf{B}}$) that are Borel reducible to E_0.

A *universal* countable Borel equivalence relation is a countable Borel equivalence relation E_∞ such that, for all countable Borel equivalence relations E, the inequality $E \leq_{\mathbf{B}} E_\infty$ holds.

Theorem 7. [16] *There is a universal countable Borel equivalence relation.*

Examples.

1. [16] The universal equivalence relation usually denoted by E_∞ is defined as the orbit equivalence relation induced by the free group F_2 on two generators on its power set $\mathscr{P}(2^{F_2})$ by translation: $(g,A) \mapsto gA \stackrel{\text{def}}{=} \{ga \mid a \in A\}$.

2. [17] For $n \geq 2$ consider the equivalence relation on the Cantor space $n^{\mathbb{N}}$ defined by letting $x \cong_r^n y$ just in case there is a recursive permutation φ such that $x \circ \varphi = y$. So this equivalence relation is induced by a right action on $n^{\mathbb{N}}$ by the group of recursive permutations. For $n \geq 5$, $\cong_r^n$ is a universal countable Borel equivalence relation. The case $2 \leq n \leq 4$ is still open.

3. [17] If G is a countable group containing a copy of F_2, then conjugacy equivalence relation on subgroups of G is a universal countable Borel equivalence relation.

4. [18] The equivalence relations of recursive isomorphism on countable trees, groups, Boolean algebras, fields, total orderings are all universal countable Borel equivalence relations.

The structure of $\leq_{\mathbf{B}}$ is linear on countable Borel equivalence relations that are Borel reducible to E_0, as depicted in relation (3). However the general structure is very different as shown by the following theorem of [19].

Theorem 8. *There is a map $A \mapsto E_A$ assigning to each Borel subset $A \subseteq \mathbb{R}$ a countable Borel equivalence relation E_A such that, for all Borel $A, B \subseteq \mathbb{R}$,*

$$A \subseteq B \;\Leftrightarrow\; E_A \leq_{\mathbf{B}} E_B.$$

3.3. Analytic equivalence relations and preorders

Analytic equivalence relations and analytic preorders also form initial segments under $\leq_{\mathbf{B}}$. The general study of $\leq_{\mathbf{B}}$ on them has revealed a very rich structure. In particular, by results of [20], there are $\leq_{\mathbf{B}}$-universal elements both for analytic equivalence relations and for analytic preorders; moreover, if P is a universal analytic preorder, then the associated equivalence relation is a universal analytic equivalence relation.

Various examples of universal analytic preorders are given in [20–23]. A common procedure to show a given analytic preorder is universal is to Borel reduce to it another analytic preorder already known to be universal. To illustrate this technique in a simple case, we give an example arising in the form of a preorder $\leq_{\mathcal{F}}$. Recall that, by [20], the relation of embeddability for combinatorial trees on $\mathbb{N}$ is a universal analytic preorder.

Theorem 9. *There is a closed monoid $\mathcal{F} \subseteq \mathbb{N}^{\mathbb{N}}$ containing all constant functions such that $\leq_{\mathcal{F}}$ is a universal analytic preorder.*

Proof. The result will be shown for the set $\mathbb{N}^2$. So, let $\mathcal{F} \subseteq (\mathbb{N}^2)^{\mathbb{N}^2}$ be the space of all functions that are constants or are the Cartesian product $g \times g$ for some injection $g : \mathbb{N} \to \mathbb{N}$. As constant functions form a closed set and the same is true for Cartesian products of an injection times itself and since identity on $\mathbb{N}^2$ is in $\mathcal{F}$, which moreover is closed under composition, it turns out that $\mathcal{F}$ is a closed monoid of functions, so $\leq_{\mathcal{F}}$ is indeed an analytic preorder. Since combinatorial trees on $\mathbb{N}$ are a family of subsets of $\mathbb{N}^2$ it will be enough to observe that on them embeddability and $\leq_{\mathcal{F}}$ coincide. Let G, H be combinatorial trees on $\mathbb{N}$. Suppose there is an embedding $g : \mathbb{N} \to \mathbb{N}$ from G into H. Then $g \times g \in \mathcal{F}$ witnesses $G \leq_{\mathcal{F}} H$. Conversely, let $\varphi \in \mathcal{F}$ be such that $\varphi^{-1}(H) = G$. Since $\emptyset \neq G \neq \mathbb{N}^2$, the function φ is not a constant and must be of the form $g \times g$ for some injection $g : \mathbb{N} \to \mathbb{N}$, which is an embedding of G into H. So the identity function witnesses that embeddability for combinatorial trees on $\mathbb{N}$ Borel reduces to $\leq_{\mathcal{F}}$. ■

4. Interplay with automata theory

4.1. Automata on infinite strings

In theoretical computer science there are various notions of automata reading infinite words. Each of these automata accepts a set of infinite strings from a given alphabet. The problem arises of which are the sets of strings that are accepted by some automaton of a given kind. This question and its relations with what has been discussed above will be illustrated by sketching the case of Büchi automata, studied by K. Wagner in [24] and of deterministic pushdown automata, investigated by J. Duparc in [25]. It is worth mentioning here that further work is being done on hierarchies of sets accepted by automata. See, for instance, [26, 27]. A more detailed survey on these and related topics is [28].

Let Z be a finite non-empty set, called the *alphabet*. A *Büchi automaton* over Z is a quadruple:

$$B = (Q, q_0, \Delta, F),$$

where:

- Q is a finite non-empty set, whose elements are called *states*;
- $q_0 \in Q$ is the *initial state*;
- $\Delta \subseteq Q \times B \times Q$ is the *transition relation*;
- $F \subseteq Q$ is the set of *final states*.

Let $x \in Z^{\mathbb{N}}$. Then x is *accepted* by B if there exists $s \in Q^{\mathbb{N}}$ such that:

- $s(0) = q_0$,
- $\forall i \in \mathbb{N}\ (s(i), x(i), s(i+1)) \in \Delta$,
- $\exists^{\infty} i \in \mathbb{N}\ s(i) \in F$.

The set of elements of $Z^{\mathbb{N}}$ that are accepted by the automaton B is the *set accepted* by B and is still denoted by B, identifying it with the accepting automaton. By [29], sets accepted by Büchi automata are exactly the ω-regular sets (for a definition, see [30]).

A *deterministic pushdown automaton* $D = (\Gamma, Q, \delta, \perp, q^0, \mathcal{F})$ in the alphabet Z consists of:

- a finite non-empty set Γ, called the *pushdown alphabet*;
- a finite *set of states* Q;

- a *transition function*

$$\delta:\Omega\to Q\times\Gamma^{<\mathbb{N}}$$

where $\Omega\subseteq Q\times(Z\cup\{\varepsilon\})\times\Gamma$, and ε is a new symbol (a *dummy character*);
- a *bottom symbol* $\perp\in\Gamma$;
- a set $\mathcal{F}\subseteq\mathscr{P}(Q)$ of *accepting conditions*.

The transition function must satisfy the following conditions:

(1) for all $q\in Q$ and $g\in\Gamma$, either

$$(q,\varepsilon,g)\notin\Omega \text{ and } \forall a\in Z\ (q,a,g)\in\Omega$$

or

$$(q,\varepsilon,g)\in\Omega \text{ and } \forall a\in Z\ (q,a,g)\notin\Omega;$$

(2) for all $q\in Q$ and $a\in Z\cup\{\varepsilon\}$, if $(q,a,\perp)\in\Omega$ then

$$\exists q'\in Q\ \exists\gamma\in\Gamma^{<\mathbb{N}}\ (\delta(q,a,\perp)=(q',\perp^\frown\gamma)).$$

If $\delta(q,a,g)=(q',\gamma')$, write $a:(q,\gamma^\frown g)\vdash_D(q',\gamma^\frown\gamma')$.

Let $x\in Z^{\mathbb{N}}$. Reading x character by character, the automaton D will change configuration using auxiliary strings in the alphabet Γ. At each step, D checks its current state q and the last symbol g of the auxiliary string. If $\delta(q,\varepsilon,g)=(q',\gamma)$ is defined, then D goes to state q', replace the last letter g in the auxiliary string with the string γ and does not procede in the reading of x. If $\delta(q,\varepsilon,g)$ is not defined, then D reads the first unread entry a of x; since, by condition (1), $\delta(q,a,g)=(q',\gamma)$ is defined, the automaton passes to state q' and replaces the last character of the auxiliary string with γ. Note that by condition (2) the symbol $\perp$ in the auxiliary string can never be erased.

To describe this more formally, x determines unique $x'\in(Z\cup\{\varepsilon\})^{\mathbb{N}}$, $p\in Q^{\mathbb{N}}$, $\rho\in(\Gamma^{<\mathbb{N}})^{\mathbb{N}}$ such that:

- $x'(0)=x(0)$, $p(0)=q_0$, $\rho(0)=\perp$,
- $x'(i):(p(i),\rho(i))\vdash_D(p(i+1),\rho(i+1))$,
- if $\tilde{x}$ is obtained by erasing all ε values of x', then $\tilde{x}\subseteq x$.

The sequence x is *accepted* by D if $\tilde{x}=x$ (this means that, though it can make ε-moves infinitely often, the automaton actually reads x entirely) and $\{q\in Q\mid\exists^\infty i\in\mathbb{N}\ p(i)=q\}\in\mathcal{F}$.

Wagner's and Duparc's works allow to locate along the Wadge hierarchy of $Z^{\mathbb{N}}$ the levels of sets recognized by these automata. Since Z is finite, each self-dual Wadge degree is the successor of a pair of non-self dual degrees, thus we can restrict to non-self-dual levels. Denote by $\mathcal{W}$ the Wadge hierachy on $Z^{\mathbb{N}}$ restricted to non-self-dual degrees. Let $d_{\mathcal{W}}(A)$ be the rank of A in $\mathcal{W}$, again starting with $d_{\mathcal{W}}(\emptyset)=d_{\mathcal{W}}(Z^{\mathbb{N}})=1$. It is then possible to define a family of operations on sets in $\mathcal{W}$ (see [28]):

- an addition operation $(A,A') \mapsto A+A'$ such that:
$$d_{\mathcal{W}}(A+A') = d_{\mathcal{W}}(A) + d_{\mathcal{W}}(A');$$
- for each $1 \leq \alpha \leq \omega_1$ a multiplication operation $A \mapsto A\alpha$ such that:
$$d_{\mathcal{W}}(A\alpha) = d_{\mathcal{W}}(A) \cdot \alpha.$$

Let $\mathrm{BA} \subseteq \mathcal{W}$ be the class of (sets in $\mathcal{W}$ accepted by some) Büchi automata. Similarly, let $\mathrm{DPDA} \subseteq \mathcal{W}$ be the class of (sets in $\mathcal{W}$ accepted by) deterministic pushdown automata. Denoting by [BA] the downward closure in $\mathcal{W}$ of BA under $\leq_{\mathrm{W}}$, the relations
$$\mathrm{BA} \subseteq \mathrm{DPDA} \subseteq [\mathrm{BA}] \subseteq \boldsymbol{\Delta}^0_3(Z^{\mathbb{N}})$$
hold.

Theorem 10. [*Wagner, Duparc*]

a) *Up to complementation and Wadge equivalence, the class* BA *is the closure of* $\{\emptyset\}$ *under addition and multiplication by* ω_1*; its lentgh is* ω^ω.
b) *Up to complementation and Wadge equivalence, the class* DPDA *is the closure of* $\{\emptyset\}$ *under sum, multiplication by* ω *and by* ω_1*; its length is* ω^{ω^2}.
c) *The class* [BA] *is the initial segment of* $\mathcal{W}$ *of length* ω_1^ω.

4.2. The conciliatory hierarchy

The Lipschitz and Wadge hierarchies were defined in terms of existence of functions and, for product spaces, they admitted equivalent game theoretic characterisations. We end this paper by definining the *conciliatory hierarchy*, whose definition is in terms of games. Introduction and more details are in [28]. The Wadge game is not symmetric, since player **II** is allowed to pass, while player **I** is not. The conciliatory game is defined so to permit both players to skip moves.

Let Z be a countable set and let $A,B \subseteq Z^{\leq\mathbb{N}}$. In the *conciliatory game* $G_{\mathrm{c}}(A,B)$ each player is allowed, at each of his moves, to either play an element of Z or to pass (say, playing an element $\mathsf{p} \notin Z$). Note that the sequences x and y of elements of Z played by **I** and **II**, respectively, may now be finite, since a player is allowed to pass as long as he likes. Player **I** wins if $x \in A \Leftrightarrow y \notin A$, otherwise **II** wins. The concept of winning strategies for the two players in this game are defined in the natural way.

Define $A \leq_{\mathrm{c}} B$ if and only if there is a winning strategy for **II** in $G_{\mathrm{c}}(A,B)$. Then $\leq_{\mathrm{c}}$ is a preorder on $\mathscr{P}(Z^{\leq\mathbb{N}})$, with $\equiv_{\mathrm{c}}$ the associated equivalence relation. Denote by $\mathcal{C}$ this hierarchy of sets.

Note that if $A \subseteq Z^{\leq\mathbb{N}}$, a winning strategy for **I** in $G_{\mathrm{c}}(A,\neg A)$ is established by passing at the start of the game and then copying **II**'s moves. Consequently $A \not\leq_{\mathrm{c}} \neg A$. So $\mathcal{C}$ does not have self-dual sets.

The following result (see [28]) relates the conciliatory hierarchy with our earlier discussion.

Theorem 11.

a) *The hierarchy* $\mathcal{W}$ *restricted to* [BA] *and the hierarchy* $\mathcal{C}$ *restricted to* [BA] *are isomorphic.*
b) *The hierarchy* $\mathcal{W}$ *restricted to Borel sets and the hierarchy* $\mathcal{C}$ *restricted to Borel sets are isomorphic.*

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