IDENTIFICATION OF POLYGONAL DOMAINS USING PIES IN INVERSE BOUNDARY PROBLEMS MODELED BY 2D LAPLACE'S EQUATION

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Abstract: The paper presents an original method to identify polygonal boundary geometry in 2D boundary problems defined by Laplace's equation using a parametric integral equation system (PIES). In the PIES, the polygonal boundary shape is defined mathematically by means of parametric linear segments, with a small number of corner points being posed. Identification of the polygonal boundary is reduced to identification of the corner points. Finally, the solution of the problem is reduced to the solution of a non-linear system of algebraic equations. Coordinates of identified corner points are obtained after solving the system of equations.

Keywords: parametric integral equation system (PIES), inverse problems, Laplace's equation

1. Introduction

Solving practical problems generally leads to solving boundary problems, which are mathematical models of real problems. Taking into account the diversity of practical problems, boundary problems, in view of searched solutions, can be divided into two groups: forward problems (analysis) and inverse problems (synthesis) [1, 2]. Numerical methods such as the Finite Element Method (FEM) [3, 4] and the Boundary Element Method (BEM) [4, 5] are used for solving such problems.

Generally, all of the problems are characterized by the fact the final solution is reduced to solving a system of algebraic equations. The main difference between the numerical methods lies in different techniques of obtaining those systems and different accuracy of their solutions. It is their common feature that all these methods require discretization: the FEM of the domain and the BEM of the boundary. These methods are also characterized by various effectiveness, which depends on the complexity of the solved problems.

From a practical point of view, inverse problems form a very significant category of problems that are more complex than simple boundary problems. Inverse problems are ill-posed [1]. Most of them are problems of identification of material parameters, boundary conditions or the boundary geometry. Various methods have been used to solve these problems, but the most frequently used method is applying an experimental choice of solutions resulting from multiple solving of modified analysis problems. Other popular methods include regularization methods [1] or those based on the sensitivity coefficients. The use of evolutionary algorithms [6] and artificial neuron networks [7] to solve inverse boundary problems is also under investigation. In view of the wide and practical application of analysis and synthesis boundary problems and the imperfections of the existing solution methods, more effective methods are certainly required.

In our paper, a method to solve analysis problems based on the PIES is proposed. The method is an alternative to the classical boundary integral equation (BIE). The PIES was obtained for Laplace's equation as a result of analytical modification of the traditional BIE. The way of analytical modification of the traditional BIE for a different boundary geometry was presented in papers [8–10]. In the PIES, a polygonal boundary geometry is mathematically defined by means of linear segments [8, 9]. The practical definition of boundary geometry in the PIES is reduced to posing a small number of corner points. An important feature of this approach is that the number of these points is independent of the domain size. The segments so created between the points do not constitute physical discretization of the boundary, as they do in the case of BEM [4, 5]. The corner points should rather be considered as a very effective method of boundary modeling. The boundary defined in that way is practically a closed broken line which creates a polygonal domain. If we move any given corner point, it will result in a modification of a considerable part of the boundary.

The purpose of this paper is to apply and analyze the effectiveness of the PIES for the identification of polygonal boundaries in inverse boundary problems. The boundary geometry is defined and modified by means of corner points. Thus, identification of an unknown part of the boundary is practically reduced to the identification of a small number of corner points. The effectiveness of the proposed method is confirmed with the examples included in the paper.

2. Definition of the problem and its solution

The problem defined here requires identification of the shape of boundary geometry in a two-dimensional potential problem. It is assumed that both empirical values at some measurement points on the analyzed part of the boundary geometry and boundary conditions are known. The solution of the problem requires reconstruction of the unknown shape of the boundary based on the known empirical values, obtained at measurement points:

a) at the boundary only, or
b) in the domain only, or
c) at the boundary and in the domain.

The PIES is used to reconstruct the unknown part of the boundary. Figure 1 shows the way in which boundary geometry is defined in the PIES.

In order to define boundary geometry only corner points D_p $(p=0,1,\dots,6)$ are posed. Identification of the polygonal boundary's shape is reduced to identification of

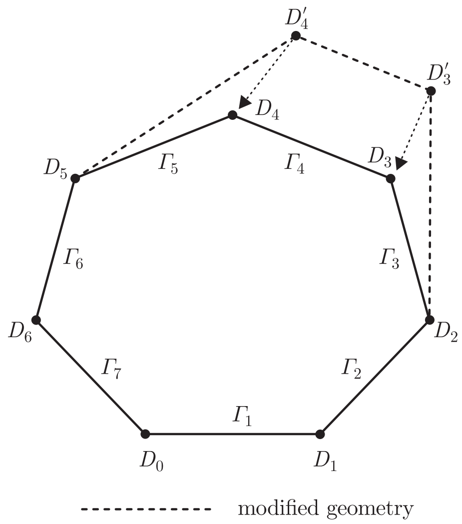

Figure 1. Polygonal boundary definition and modification by means of corner points D_p

corner points, as modification of the boundary geometry is performed with mere two corner points D_p $(p=3,4)$.

3. Mathematical foundations of the PIES and integral identity

The PIES is an effective method alternative to the traditional BIE. It was obtained as a result of analytical modification of the traditional BIE [8]. The main purpose of the modification was to separate the necessity of simultaneous approximation of the boundary geometry and the unknown boundary functions during numerical solutions in the traditional BIE. Separation of approximations makes it possible to define and modify the boundary geometry more accurately without interference in boundary functions' approximation and vice versa. Unlike the traditional BIE, in which the boundary is defined by a boundary integral, in the PIES the boundary is included in its mathematical formalism. It is presented in the following form [8, 9]:

$$\frac{1}{2}u_l(s_1)=\sum_{j=1}^{n}\int_{s_{j-1}}^{s_j}\left\{\bar{U}_{lj}^*(s_1,s)p_j(s)-\bar{P}_{lj}^*(s_1,s)u_j(s)\right\}J_j(s)\,ds, \tag{1}$$

where $l=1,2,\ldots,n$ and $s\in[s_{j-1},s_j]$.

Integrands $\bar{U}_{lj}^*(s_1,s)$ and $\bar{P}_{lj}^*(s_1,s)$ take into account the boundary geometry that can be defined by the appropriate curves:

$$\bar{U}_{lj}^*(s_1,s)=\frac{1}{2\pi}\ln\frac{1}{\eta},\quad \bar{P}_{lj}^*(s_1,s)=\frac{1}{2\pi}\frac{\eta_1 n_1^{(j)}(s)+\eta_2 n_2^{(j)}(s)}{\eta^2}, \tag{2}$$

where $\eta=\sqrt{\eta_1^2+\eta_2^2}$, $\eta_1=\Gamma_l^{(1)}(s_1)-\Gamma_j^{(1)}(s)$ and $\eta_2=\Gamma_l^{(2)}(s_1)-\Gamma_j^{(2)}(s)$.

Functions $\Gamma_p(s)$, $(p=l,j)$ in Equation (2) in this paper are parametric linear functions described as individual rectilinear segments of a polygonal domain. Only corner points are posed for their definition.

Solution in the domain, after obtaining a solution at the boundary with Equation (1), is given by the following integral identity:

$$u(x)=\sum_{j=1}^{n}\int_{s_{j-1}}^{s_j}\left\{\hat{\bar{U}}_j^*(\boldsymbol{x},s)p_j(s)-\hat{\bar{P}}_j^*(\boldsymbol{x},s)u_j(s)\right\}J_j(s)ds, \tag{3}$$

where $l=1,2,\ldots,n$, $s\in[s_{j-1},s_j]$, and

$$\hat{\bar{U}}_j^*(\boldsymbol{x},s)=\frac{1}{2\pi}\ln\frac{1}{\bar{r}},\quad \hat{\bar{P}}_j^*(\boldsymbol{x},s)=\frac{1}{2\pi}\frac{\bar{r}_1 n_1^{(j)}(s)+\bar{r}_2 n_2^{(j)}(s)}{\bar{r}^2}, \tag{4}$$

where $\bar{r}=\sqrt{\bar{r}_1^2+\bar{r}_2^2}$, $\bar{r}_1=\boldsymbol{x}-\Gamma_j^{(1)}(s)$ and $\bar{r}_2=\boldsymbol{x}-\Gamma_j^{(2)}(s)$.

Functions $\Gamma_j(s)$ found in Equation (4) are the same parametric functions as those found in Equation (2).

3.1. Numerical solution of the PIES

In the PIES, the approximation of the boundary geometry is separated from the approximation of boundary functions. In other words improvement in the accuracy of both approximations is independent. The accuracy of boundary geometry approximation can be increased without interference in the boundary functions' approximation and vice versa.

The boundary geometry is analytically included in the mathematical formalism of the PIES by means of linear segments. Therefore, numerical solution of the PIES is reduced to approximating boundary functions u_j, p_j on each segment. These functions are approximated according to the following formulae [8, 9]:

$$p_j(s)=\sum_{k=0}^{M}p_j^k T_j^k(s),\quad u_j(s)=\sum_{k=0}^{M}u_j^k T_j^k(s), \tag{5}$$

where $u_j^{(k)}$, $p_j^{(k)}$ are the unknown coefficients on segment j, M is the number of coefficients, whereas $T_j^{(k)}(s)$ are global base functions on the segments, *e.g.* Chebyshev polynomials.

Due to separation of the boundary geometry approximation from the boundary function approximation, analysis of solution convergence is very effective, as it is reduced to merely changing coefficient M.

4. Identification of corner points

If we use the PIES, the problem of identifying the polygonal boundary is reduced to the identification of corner points. The number of such points is much less than the number of nodes required in the traditional BEM. Identification of corner points can be achieved with experimental values $\tilde{u}_l(s_i)$, $i=1,2,\ldots,n$ obtained at n measurement points at the boundary l (or in the domain) of the problem. The least square method was used in both cases:

$$S(s,D_p)=0.5\sum_{i=1}^{m}[\tilde{u}_l(s_i,D_p)^*-u_l(s_i,D_p)]^2, \tag{6}$$

where $\tilde{u}_l(s_i,D_p)^*$ are experimental values of measurement points at a given boundary (or in the domain, where $s=x$), while $u_l(s_i,D_p)=u_l(s_l)$ are numerical values obtained from the PIES (1) or integral identity (3).

The shape of the considered domain is included in the kernels (2) of the PIES and is defined by corner points. Solutions obtained from the PIES are continuous in each segment. With their help can easily obtain values of solutions at measurement points, which depend on the shape of the identified boundary. When defining a boundary shape in the continuous way by means of linear segments, the values depend on the corner points. Therefore, minimization of formula (5) should be performed with respect to these points. Formula (5), after differentiation, can be written as follows:

$$\frac{\partial S(s,D_p)}{\partial D_p}=\sum_{i=1}^{m}[\tilde{u}_l(s_i,D_p)^*-u_l(s_i,D_p)]\frac{\partial u_l(s_i,D_p)}{\partial D_p},\quad p=1,2,....P. \tag{7}$$

In this expression, computation of the first order derivative of the boundary function $u_l(s_i)$ with respect to corner points $D_p(p=1,2,\ldots P)$ is required. This function is defined by means of the PIES (1). Therefore, the derivative can be easily computed numerically with the following formula:

$$\frac{\partial u_l(s_i,D_p)}{\partial D_p}\cong\frac{u_l(s_1,D_p+\Delta D_p)-u_l(s_1,D_p)}{\Delta D_p},\quad u_l=p_l. \tag{8}$$

In order to calculate the derivative, the PIES (1) should be solved twice for two insignificantly different (ΔD_p) corner points. Once the PIES is solved for initial geometry (defined by means of corner points D_p) and then for the geometry modified by displacement of corner points by ΔD_p.

After equating to zero Equation (7), a system of $2P$ algebraic equations with respect to the unknown coordinates of corner points is obtained:

$$\sum_{i=1}^{m}[\tilde{u}_l(s_i,D_p)^*-u_l(s_i,D_p)]\frac{\partial u_l(s_i,D_p)}{\partial D_p}=0,\quad p=1,2,\ldots,P. \tag{9}$$

Formula (9) is a non-linear system of algebraic equations with respect to the unknown corner points and Newton's iterative method is used to solve it. The system is presented in the following matrix form:

$$\left\{\nabla_p(D)\right\}_{(k)}\left\{\delta D_p\right\}=-\left\{F_p(D)\right\}_{(k)}. \tag{10}$$

Matrix $\nabla_p(D)$ found on the left-hand side of system (10) is presented in the following form:

$$\nabla_p(D)=\begin{bmatrix}\frac{\partial F_p(s,D)}{\partial D_p} & \frac{\partial F_p(s)}{\partial D_{p+1}} & \cdots & \frac{\partial F_p(s)}{\partial D_{p+P}}\\ \frac{\partial F_{p+1}(s)}{\partial D_p} & \frac{\partial F_{p+1}(s)}{\partial D_{p+1}} & \cdots & \frac{\partial F_{p+1}(s)}{\partial D_{p+P}}\\ \cdots & \cdots & \cdots & \cdots\\ \frac{\partial F_{p+P}(s)}{\partial D_p} & \frac{\partial F_{p+P}(s)}{\partial D_{p+1}} & \cdots & \frac{\partial F_{p+P}(s)}{\partial D_{p+P}}\end{bmatrix}_{(k)}, \tag{11}$$

where $F_p(s,D)=\sum_{l=1}^{m}\left[\tilde{u}_l(s_i,D_p)^*-u_l(s_i,D_p)\right]\frac{\partial u_l(s_i,D_p)}{\partial D_p}$.

Coefficients of matrix (11) are obtained after analytical differentiation of function $F_p(s_i)$ $(p=1,2,\dots,P)$ with respect to all corner points. The following formulae are obtained as the result:

$$\begin{aligned}\frac{\partial F_p(s,D)}{\partial D_p} &= \sum_{i=1}^{m}\Bigg\{-\frac{\partial u_l(s_i,D_p)}{\partial D_p}\frac{\partial u_l(s_i,D_p)}{\partial D_p}+\\ &\qquad\big[\tilde{u}_l(s_i,D_p)^*-u_l(s_i,D_p)\big]\frac{\partial^2 u_l(s_i,D_p)}{\partial D_p^2}\Bigg\},\\ \frac{\partial F_p(s,D)}{\partial D_{p+1}} &= \sum_{i=1}^{m}\Bigg\{-\frac{\partial u_l(s_i,D_p)}{\partial D_{p+1}}\frac{\partial u_l(s_i,D_p)}{\partial D_{p+1}}+\\ &\qquad\big[\tilde{u}_l(s_i,D_p)^*-u_l(s_i,D_p)\big]\frac{\partial^2 u_l(s_i,D_p)}{\partial D_p\partial D_{p+1}}\Bigg\}.\end{aligned} \tag{12}$$

To compute the elements of matrix (11) with formulae (12) it is necessary to compute the second order derivative of the boundary functions with respect to corner points. An approximate way of derivative computation with the following formula can be used for this purpose:

$$\frac{\partial^2 u_l(s,D_p)}{\partial D_p^2}\cong\frac{u_l(s,D_p-\Delta D_p)-2u_l(s,D_p)+u_l(s,D_p+\Delta D_p)}{[\Delta D_p]^2}. \tag{13}$$

Columned matrix $F_p(D)$ found on the right-hand side of system (11) is presented in the following form:

$$\{F_p(D)\}=\begin{Bmatrix}F_p(D)\\ F_{p+1}(D)\\ \dots\\ F_{p+P}(D)\end{Bmatrix}_{(k)}, \tag{14}$$

where $F_p(s,D)=\sum_{l=1}^{m}[u_l(s_i,D_p)^*-u_l(s_i,D_p)]\dfrac{\partial u_l(s_i,D_p)}{\partial D_p}$.

The first order derivative of the boundary function with respect to corner points is computed numerically by means of formula (12).

New values of corner points $D_p^{(k+1)}$ in the following steps of the iteration process are as follows:

$$D_p^{(k+1)}=D_p^{(k)}+\delta D_p. \tag{15}$$

We can identify corner points as a result of the iteration process assuming any initial corner points $D_p^{(0)}$ $(p=1,2,\dots,P)$ in Equation (13). The iteration process is completed when the last two values of corner points are the same or when the difference between empirical and numerical values at measurement points becomes minimal.

5. Testing examples

In order to define a polygonal boundary geometry in the PIES, only corner points are set. The arrangement and number of the points depends on the actual shape of the considered boundary geometry. Their number depends on the geometry's complexity. Therefore, identification of the boundary geometry is ultimately reducible to identification of corner points' coordinates.

5.1. Example 1

The investigated problem is concerned with the identification of an unknown part of the boundary geometry $B_2 = \Gamma_2 + \Gamma_3$ shown in Figure 2 for stationary heat flow with a known, constant part of the boundary $B_1 = \Gamma_1$ and the following measurement values at the known boundary B_1 and boundary conditions at the boundary to be identified:

a) temperature $\tilde{u}_i$ was measured at the known boundary B_1 at five ($i = 5$) measurement points: $\tilde{u}_1(s = \frac{1}{5}) = 21.33$, $\tilde{u}_2(s = \frac{1}{3}) = 28.02$, $\tilde{u}_3(s = \frac{1}{2}) = 30.83$, $\tilde{u}_4(s = \frac{2}{3}) = 28.01$, $\tilde{u}_5(s = \frac{4}{5}) = 21.33$,

b) a constant temperature $u = 0$ was set at the B_2 boundary to be identified.

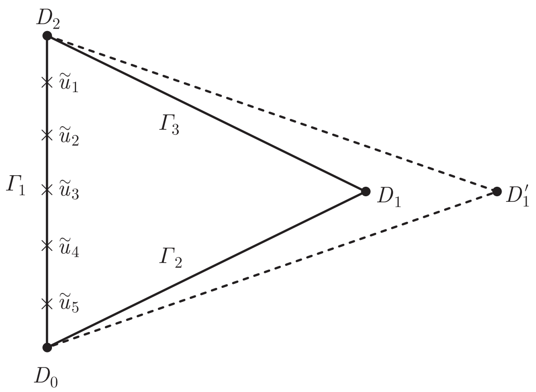

Figure 2. Definition of the problem considered in example 1

The considered boundary geometry is modelled by means of three linear segments (three corner points). Identification of an unknown part of the boundary is reduced to identification of one corner point, D_1.

In order to solve the problem various coordinates of the initial corner points were considered. As a result of the performed identification, coordinates of the corner points searched for were obtained. In order to verify the reliability of the boundary identification procedure a forward problem was solved and numerical values at measurement points were compared with experimental ones. The difference between these values was calculate with the following formula:

$$\xi = \frac{\sum_{i=1}^{n} \left| \frac{\tilde{u}_i - u_i}{\tilde{u}_i} \right|}{n} \cdot 100\%, \tag{16}$$

where n is the number of measurement points.

Results of the performed identifications are presented in Table 1.

As can be seen from the above table, the iterative identification process always converges to the same corner point coordinates. The geometry was found with high accuracy, after a small number of iterations.

Table 1. Input data and results from chosen identification processes

Initial corner point		Identified corner point		ξ[%]	Number of iterations
x	y	x	y		
2	0	3.99764	0.00055148	0.010712	6
5	0	3.99766	0.000550532	0.023658	5
3	1.5	3.99757	0.000629397	0.092963	7
4	–1	3.99762	0.000560582	0.027736	5
2	1	3.99763	0.00056758	0.038855	7
5	–0.5	3.99766	0.000545098	0.088707	6

5.2. Example 2

Solving this problem involves identification of the unknown part of boundary geometry $B_2=\Gamma_3+\Gamma_4$ for the boundary geometry presented in Figure 3. The input data for this problem are the following measurement values in the domain and set boundary conditions at the unknown and constant boundary $B_1=\Gamma_1+\Gamma_2+\Gamma_5$:

a) a constant temperature of u=25 was set at the known boundary B_1,
b) a constant temperature of u=0 was set at the B_2, boundary to be identified,
c) temperature $\tilde{u}_i$ was measured in the domain at five ($i=5$) measurement points: $\tilde{u}_1(1,1.75)=12.86$, $\tilde{u}_2(1,1.5)=16.26$, $\tilde{u}_3(1,1.25)=18.95$, $\tilde{u}_4(1,1)=20.94$, $\tilde{u}_5(1,0.5)=23.44$.

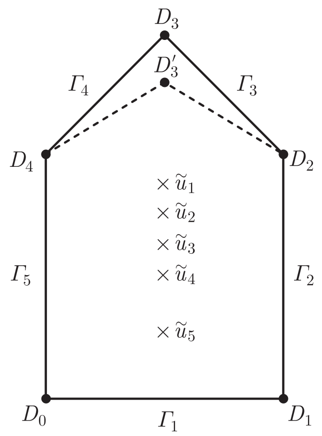

Figure 3. Definition of the problem considered in Example 2

Various initial corner points were considered in order to solve the problem. The identification problem was reduced to the identification of only one corner point, D_3'. A forward problem was solved for obtained geometries and only those geometries were considered in which the difference between numerical and empirical values at measurement points was less than 0.05%. Selected obtained geometries are presented in Figure 4.

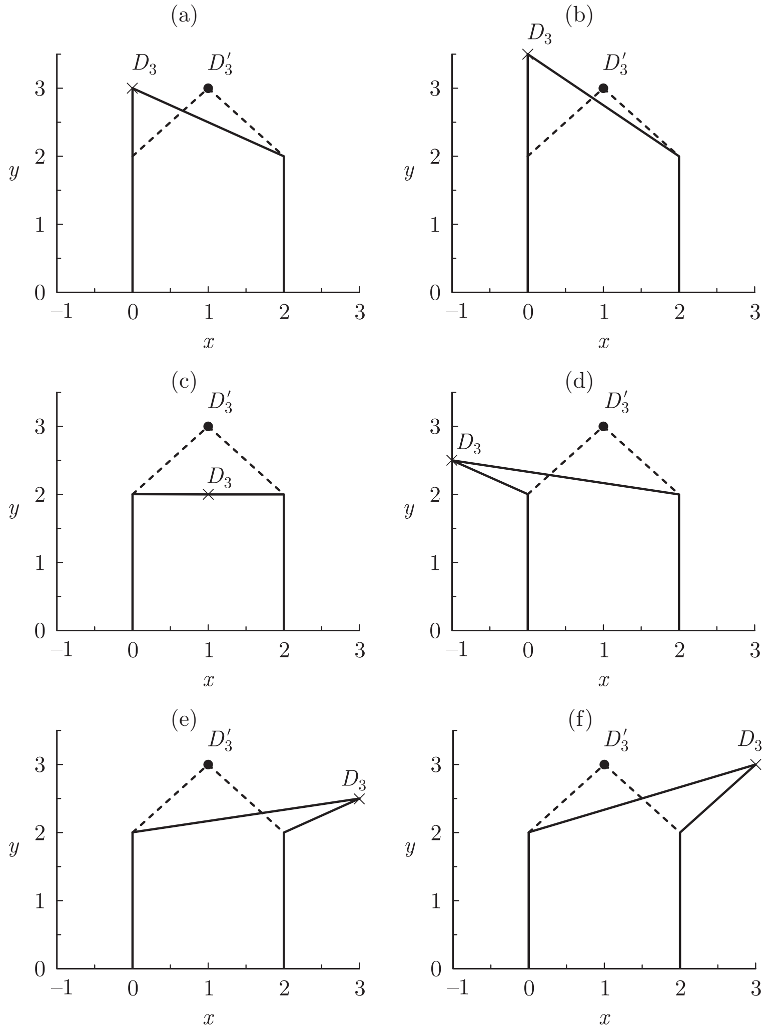

Figure 4. Selected initial and identified geometries (solid and dashed lines, respectively); • – identified corner point × – initial corner point

An additional stop condition was applied in the identification process, stopping the iteration process when corner point coordinates were the same in two subsequent iterations.

As can be seen in Figure 4, the identification process started from various initial corner points, but the identified geometry was the same, which means that the identification process was unambiguous and effective.

5.3. Example 3

In our final example both the known boundary $B_1 = \Gamma_1 + \Gamma_4$ and the searched boundary $B_2 = \Gamma_2 + \Gamma_3 + \Gamma_5 + \Gamma_6$ were modelled by means of six corner points $P_i\,(i = 0,1,...,6)$. The considered domain is presented in Figure 5.

Measurement points were points on the known boundary B_1. The investigations were performed on the following basis:

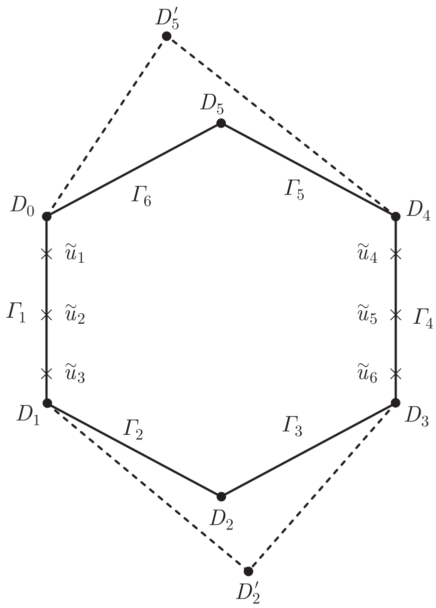

Figure 5. Definition of the boundary geometry identification problem

a) a constant temperature of $u=0$ was set at the B_2 boundary to be identified,
b) temperature $\tilde{u}_i$ was measured at the known B_1 boundary at six ($i=6$) measurement points: $\tilde{u}_1(s=\frac{1}{3})=12.6$, $\tilde{u}_2(s=\frac{1}{2})=13.5$, $\tilde{u}_3(s=\frac{2}{3})=12.6$, $\tilde{u}_4(s=\frac{1}{3})=12.6$, $\tilde{u}_5(s=\frac{1}{2})=13.5$ and $\tilde{u}_6(s=\frac{2}{3})=12.6$.

To solve the problem it is enough to identify the coordinates of two corner points, D_2' and D_5'. Results of the performed identifications are presented in Figure 6.

The identification process in all cases (Figures 6a–6f) converges to the same ultimate boundary geometry. The coordinates of the identified corner points are $D_2'(1.99,0.04)$ and $D_5'(2.00,4.95)$. The identification process turned out to be very fast (it took 5–10 iterations) and accurate, as the difference between empirical and numerical values at measurement points does not exceed 0.8%.

6. Conclusions

The method presented in the paper is a combination of the PIES, the least square approach and Newton's method. It is an original and highly effective tool for identification of polygonal boundary geometries in boundary problems. The PIES, when applied to solving analytical problems, makes it possible to easily modify the boundary geometry by means of a small number of corner points. Proposed boundary modification method is more effective than methods which require repetitions of the traditional discretization of the domain (like in the FEM) and the boundary (like in the BEM). Identifying the boundary's polygonal shape is practically reduced to identifying a small number of corner points.

Our investigations have shown that the method is not only highly accurate, but very economical as well.

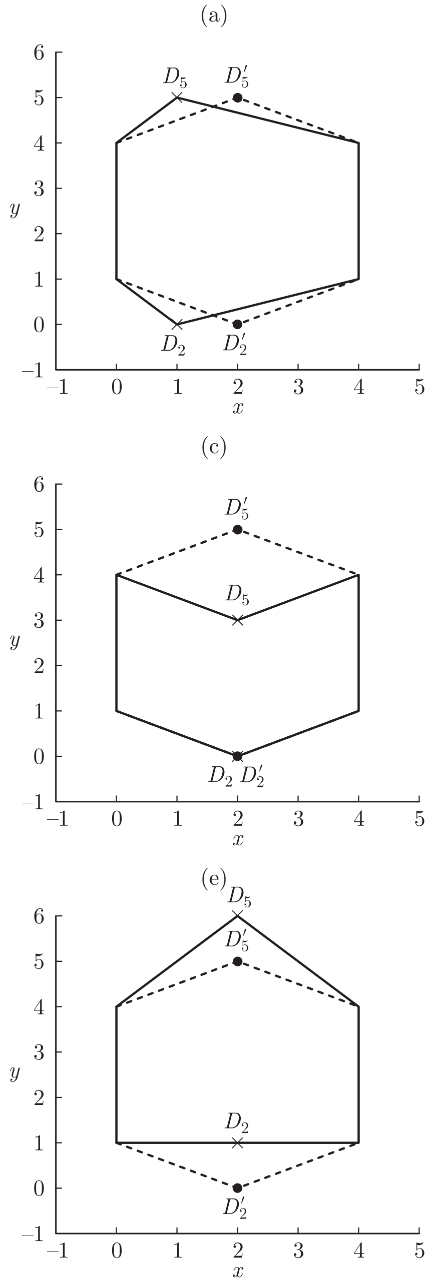

Figure 6. Selected initial and identified geometries (solid and dashed lines, respectively); • – identified corner point, × – initial corner point

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