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A FOUR-NODE 3D ISOPARAMETRIC MEMBRANE ELEMENT

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Abstract: The paper presents a non-linear finite element procedure for analysis of membrane structures. A four-node quadrilateral finite element is formulated to represent a general curved elastic geometrically non-linear surface. The described isoparametric element is C^0-continuous, of constant thickness, and assumes a plane stress criterion. A simple numerical example is presented as an application of the described theory.

Keywords: FEM, membranes, isoparametric four-node membrane element

1. Introduction

A detailed description of the four-node, isoparametric, two-dimensional finite element in the linear case can be found in almost every book on finite element analysis, see *e.g.* [1]. However, there is no clear, straightforward description of four-node membrane isoparametric 3D finite element. This has inspired the authors of the present paper to investigate in detail the idea of finite element analysis using a four-node membrane isoparametric 3D finite element. While there numerous publications concerned with this finite element analysis (see *e.g.* [2–7]), only its general idea has been discussed so far.

2. Membrane deformation

The free bending stress state in the shell is referred to as the membrane stress state. As membrane elements are usually thin, it is possible to consider all the quantities with reference to the middle surface. The position vector, $\boldsymbol{R}$, of the middle surface (see Figure 1) refers to the initial configuration, ${}^0B$ and is specified by the following equation:

$$\boldsymbol{R} = \overrightarrow{0P} = f\left(\theta^1, \theta^2\right) = \sum_{k=1}^{3} f^k\left(\theta^1, \theta^2\right)\, \boldsymbol{i}_k = X^k\, \boldsymbol{i}_k. \tag{1}$$

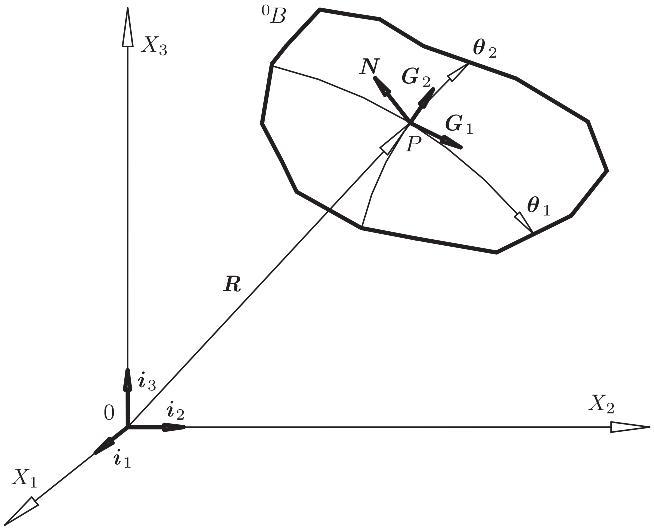

Figure 1. Visualisation of the assumed Cartesian coordinate system

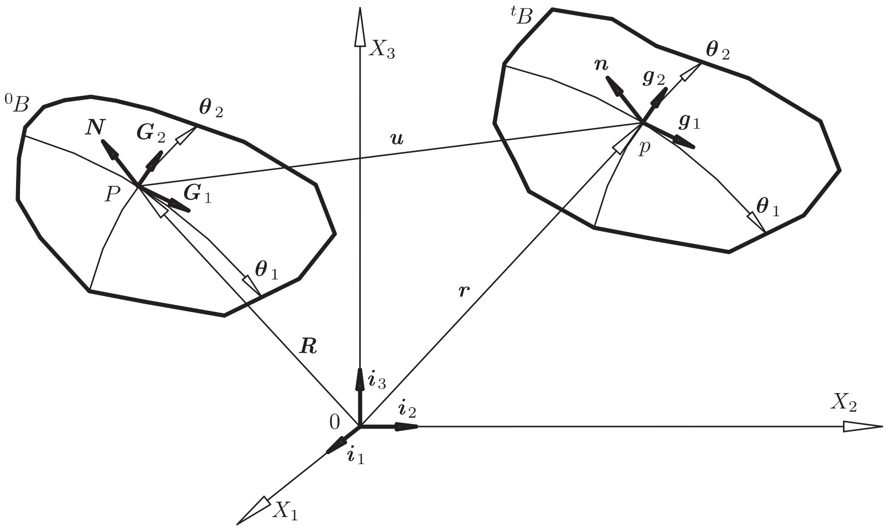

Figure 2. Deformation of the four-node membrane element

Coordinates $\theta^\beta\,(\beta=1,2)$ form a curvilinear coordinate system, the origin of which lies in the middle of the membrane's surface. Base vectors $\boldsymbol{G}_\alpha$ can be determined as:

$$\boldsymbol{G}_\alpha=\boldsymbol{R}_{,\alpha}={}^0X_{k,\alpha}\,\boldsymbol{i}_k, \tag{2}$$

whereas the middle surface normal vector, $\boldsymbol{N}$, is calculated at point P from the following equation:

$$\boldsymbol{N}=\frac{\boldsymbol{G}_1\times\boldsymbol{G}_2}{|\,\boldsymbol{G}_1\times\boldsymbol{G}_2\,|}=N^k\,\boldsymbol{i}_k. \tag{3}$$

Thus the covariant base of the surface coordinates is created. The contravariant base can be determined from the following expression:

$$\boldsymbol{G}^\alpha\cdot\boldsymbol{G}_\beta=\delta^\alpha_\beta. \tag{4}$$

It is possible to determine the covariant and contravariant components of the metric tensor (referred to as the first surface form):

$$G_{\alpha\beta} = \boldsymbol{G}_\alpha \cdot \boldsymbol{G}_\beta = X^i{}_{,\alpha} X^j{}_{,\beta} \delta_{ij}, \tag{5}$$

$$G^{\alpha\gamma} = G_{\beta\gamma} = \delta^\alpha_\beta. \tag{6}$$

The second surface form (the curvature tensor) is defined as follows (see [8] or [9]):

$$b_{\alpha\beta} = -\boldsymbol{G}_\beta \cdot \boldsymbol{N}{}_{,\alpha} = -\boldsymbol{G}_\alpha \cdot \boldsymbol{N}{}_{,\beta} = \boldsymbol{N} \cdot \boldsymbol{G}_{\beta,\alpha} = \boldsymbol{N} \cdot \boldsymbol{G}_{\alpha,\beta}. \tag{7}$$

The global coordinate system, X^1, X^2, X^3 is assumed to be orthogonal, while the local coordinate system is curvilinear, θ^α, with the axis perpendicular to the surface coordinate described by vector $\boldsymbol{N}$. The initial configuration, ${}^0B$ (undeformed state), and the actual configuration, ${}^tB$ (deformed state), are considered in the description of the strain state. The connection between the curvilinear systems is assumed to be:

$${}^t\theta^\delta = {}^0\theta^\delta. \tag{8}$$

The position vector in the actual configuration, ${}^tB$, in the assumed notation can be specified as:

$$\boldsymbol{r} = {}^tX^k\, \boldsymbol{i}_k = \boldsymbol{R}(\theta^\alpha) + \boldsymbol{u}(\theta^\alpha). \tag{9}$$

It should be noted that all quantities of the actual configuration, ${}^tB$, are specified in the initial configuration, ${}^0B$ (the Lagrange approach). The displacement vector, $\boldsymbol{u}$, and the base vectors, $\boldsymbol{g}_\alpha$, can be described as:

$$\boldsymbol{u} = \boldsymbol{r} - \boldsymbol{R} = u^\alpha\, \boldsymbol{G}_\alpha + u_3\, \boldsymbol{N} = u_\alpha\, \boldsymbol{G}^\alpha + u_3\, \boldsymbol{N}, \tag{10}$$

$$\boldsymbol{g}_\alpha = {}^tX^k{}_{,\alpha}\, \boldsymbol{i}_k = \left({}^0X^k{}_{,\alpha} + u^k{}_{,\alpha}\right)\, \boldsymbol{i}_k = \boldsymbol{G}_\alpha + \boldsymbol{u}{}_{,\alpha}. \tag{11}$$

It is thus possible to write as follows:

$$\boldsymbol{G}_\beta = G_{\beta\alpha}\, \boldsymbol{G}^\alpha$$

$$\begin{aligned} u{}_{,\beta} &= (u_\alpha\, \boldsymbol{G}^\alpha + u_3\, \boldsymbol{N}){}_{,\beta} = u_\alpha|_\beta\, \boldsymbol{G}^\alpha + u_\alpha\, \boldsymbol{G}^\alpha{}_{,\beta} + u_{3,\beta} \boldsymbol{N} + u_3\, \boldsymbol{N}{}_{,\beta} = \\ &= u_\alpha|_\beta\, \boldsymbol{G}^\alpha + u_\alpha b^\beta_\alpha \boldsymbol{N} + u_{3,\beta} \boldsymbol{N} - b_{\alpha\beta} u_3\, \boldsymbol{G}^\alpha \end{aligned} \tag{12}$$

For the sake of the membrane-stress-state-only assumption, the curvature tensor, $b_{\alpha\beta}$, can be omitted, and $\boldsymbol{u}{}_{,\beta}$ takes the following simple form:

$$\boldsymbol{u}{}_{,\beta} = u_\alpha|_\beta\, \boldsymbol{G}^\alpha + u_{3,\alpha} \boldsymbol{N}. \tag{13}$$

Equation (11) can be rewritten in the form:

$$\begin{aligned} \boldsymbol{g}_\alpha &= \boldsymbol{G}_\alpha + \boldsymbol{u}{}_{,\alpha} = G_{\beta\alpha} \boldsymbol{G}^\alpha + u_\alpha|_\beta\, \boldsymbol{G}^\alpha + u_{3,\alpha} \boldsymbol{N} \\ &= \left(G_{\beta\alpha} + u_\alpha|_\beta\right) \boldsymbol{G}^\alpha + u_{3,\alpha} \boldsymbol{N} = \\ &= (G_{\alpha\beta} + \vartheta_{\alpha\beta})\, \boldsymbol{G}^\alpha + u_{3,\alpha} \boldsymbol{N} = \psi_{\lambda\alpha} \boldsymbol{G}^\lambda + \vartheta_\alpha\, \boldsymbol{N}. \end{aligned} \tag{14}$$

The following calculations give the base vector, represented in the form:

$$\boldsymbol{g}_\alpha = \psi_{\beta\alpha} \boldsymbol{G}^\beta + \vartheta_\alpha\, \boldsymbol{N} = G^{\beta\lambda} \psi_{\beta\alpha} \boldsymbol{G}_\lambda + \vartheta_\alpha \boldsymbol{N} = \psi^\lambda_{\cdot\,\alpha} \boldsymbol{G}_\lambda + \vartheta_\alpha\, \boldsymbol{N}, \tag{15}$$

where

$$\psi_{\alpha\beta} = G_{\alpha\beta} + \vartheta_{\alpha\beta}$$

$$\vartheta_{\alpha\beta} = u_\alpha|_\beta = u_{\alpha,\beta} - \Gamma^\lambda_{\alpha\beta} u_\lambda \qquad (16)$$
$$\vartheta_\alpha = u_{3,\alpha}$$

while $\Gamma^\lambda_{\alpha\beta}$ are Christoffel's symbols [10]:

$$\Gamma^r_{ij} = 0\,(i \neq j \neq r \neq i), \quad \Gamma^r_{ii} = -\frac{1}{2G_{rr}}\frac{\partial G_{ii}}{\partial \theta^r}\,(r \neq i), \quad \Gamma^i_{ij} = \Gamma^i_{ji} = \frac{1}{2G_{ii}}\frac{\partial G_{ii}}{\partial \theta^j}. \qquad (17)$$

The components of the metric tensor in the actual configuration can be calculated from the following equation:

$$g_{\alpha\beta} = \boldsymbol{g}_\alpha \cdot \boldsymbol{g}_\beta = \psi^{\lambda}_{\cdot\alpha}\psi_{\lambda\beta} + \vartheta_\alpha\vartheta_\beta = G_{\alpha\beta} + \vartheta_{\alpha\beta} + \vartheta_{\beta\alpha} + \vartheta^{\lambda}_{\cdot\alpha}\vartheta_{\alpha\beta} + \vartheta_\alpha\vartheta_\beta, \qquad (18)$$

while the strain tensor is specified by the expression:

$$\varepsilon_{\alpha\beta} = \frac{1}{2}(g_{\alpha\beta} - G_{\alpha\beta}). \qquad (19)$$

Substituting Equation (18) to Equation (19), we obtain:

$$\varepsilon_{\alpha\beta} = \frac{1}{2}\left(\psi^{\lambda}_{\cdot\alpha}\psi_{\lambda\beta} + \vartheta_\alpha\vartheta_\beta - G_{\alpha\beta}\right) = \frac{1}{2}\left(\vartheta_{\alpha\beta} + \vartheta_{\beta\alpha} + \vartheta^{\lambda}_{\cdot\alpha}\vartheta_{\alpha\beta} + \vartheta_\alpha\vartheta_\beta\right). \qquad (20)$$

Finally, the components of the strain tensor are determined from the following equation:

$$\varepsilon_{\alpha\beta} = \frac{1}{2}\left[u_\alpha|_\beta + u_\beta|_\alpha + u^\lambda\big|_\alpha\, u_\lambda|_\beta + u_{3,\alpha}u_{3,\beta}\right]. \qquad (21)$$

The problem will be solved by the finite element method. This method requires proper discretization of the structure. The orthogonal coordinate system is assumed in every finite element; this assumption enables a simplification, *viz.*:

$$G_{\alpha\beta} = \delta_{\alpha\beta}, \quad G^{\alpha\beta} = \delta^{\alpha\beta}, \qquad (22)$$

so that the covariant derivatives are turned into ordinary derivatives,

$$\boldsymbol{u}_\alpha|_\beta = \boldsymbol{u}_{\alpha,\beta}, \quad \boldsymbol{u}^\lambda\big|_\alpha = \boldsymbol{u}^\lambda_{,\beta}. \qquad (23)$$

According to Eqations (22) and (21), the strain tensor is as follows:

$$\varepsilon_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha,\beta} + u_{\beta,\alpha} + u^{\lambda}{}_{,\alpha}u_{\lambda,\beta} + u_{3,\alpha}u_{3,\beta}\right). \qquad (24)$$

The $u^{\lambda}{}_{,\alpha}$ component can be expressed as:

$$u^{\lambda}{}_{,\alpha} = G^{\lambda\rho}u_{\rho,\alpha} = \delta^{\lambda\rho}u_{\rho,\alpha}. \qquad (25)$$

Finally, the components of the strain tensor are calculated from the following equation:

$$\varepsilon_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha,\beta} + u_{\beta,\alpha} + \delta^{\lambda\rho}u_{\rho,\alpha}u_{\lambda,\beta} + u_{3,\alpha}u_{3,\beta}\right), \qquad (26)$$

It should be noted that the plane stress state is assumed in the membrane elements (the strain component perpendicular to the surface is neglected according to the Kirchhoff assumption). However, the notion of plane stress state has to be considered in the space analysis (the three components of displacement, u_1, u_2, u_3). For detailed studies of the shell mechanics the reader is referred to [11], where over 2500 references are given.

3. The FEM approach

In general, it is possible to specify two families of elements: *strain elements* and *stress elements*. The former group uses the assumption of strain (displacement) interpolation function analysis. In the latter group of elements, the stress interpolation function (an Airy-type stress function) is assumed. Our considerations are restricted to isoparametric strain-type membrane elements. An element can be called isoparametric when the displacement interpolation functions are assumed to be the same as the shape interpolation functions. The family of isoparametric elements has been introduced by Zienkiewicz *et al.* (see *e.g.* [12]).

The geometry of the described C^0 class of quadrilateral membrane elements is specified by four nodes, each attributed with three translational degrees of freedom. Constant thickness, t, is assumed. It is also assumed that mechanical behaviour is adequately represented by the enforced plane-stress criterion. Displacements and geometry are interpolated from nodal quantities, with the use of identical shape functions, according to the isoparametric approach.

In the first step, the finite element discretization of the curvilinear domain is necessary. The global coordinates can be specified, but the local quantities should be also determined before calculating the global matrices of the FEM (*e.g.* the stiffness matrix). Therefore, the global coordinates and displacements,

$$\begin{aligned}
\boldsymbol{X}_1 &= \{{}_iX_1 \;\; {}_jX_1 \;\; {}_kX_1 \;\; {}_mX_1\}^T, \quad \boldsymbol{Q}_1 = \{{}_iU_1 \;\; {}_jU_1 \;\; {}_kU_1 \;\; {}_mU_1\}^T \\
\boldsymbol{X}_2 &= \{{}_iX_2 \;\; {}_jX_2 \;\; {}_kX_2 \;\; {}_mX_2\}^T, \quad \boldsymbol{Q}_2 = \{{}_iU_2 \;\; {}_jU_2 \;\; {}_kU_2 \;\; {}_mU_2\}^T \\
\boldsymbol{X}_3 &= \{{}_iX_3 \;\; {}_jX_3 \;\; {}_kX_3 \;\; {}_mX_3\}^T, \quad \boldsymbol{Q}_3 = \{{}_iU_3 \;\; {}_jU_3 \;\; {}_kU_3 \;\; {}_mU_3\}^T
\end{aligned} \tag{27}$$

have to be transformed to the assumed local orthogonal coordinate system:

$$\begin{aligned}
\boldsymbol{x}_1 &= \{{}_ix_1 \;\; {}_jx_1 \;\; {}_kx_1 \;\; {}_mx_1\}^T, \quad \boldsymbol{q}_1 = \{{}_iu_1 \;\; {}_ju_1 \;\; {}_ku_1 \;\; {}_mu_1\}^T \\
\boldsymbol{x}_2 &= \{{}_ix_2 \;\; {}_jx_2 \;\; {}_kx_2 \;\; {}_mx_2\}^T, \quad \boldsymbol{q}_2 = \{{}_iu_1 \;\; {}_ju_1 \;\; {}_ku_1 \;\; {}_mu_1\}^T \\
\boldsymbol{x}_3 &= \{{}_ix_3 \;\; {}_jx_3 \;\; {}_kx_3 \;\; {}_mx_3\}^T, \quad \boldsymbol{q}_3 = \{{}_iu_1 \;\; {}_ju_1 \;\; {}_ku_1 \;\; {}_mu_1\}^T.
\end{aligned} \tag{28}$$

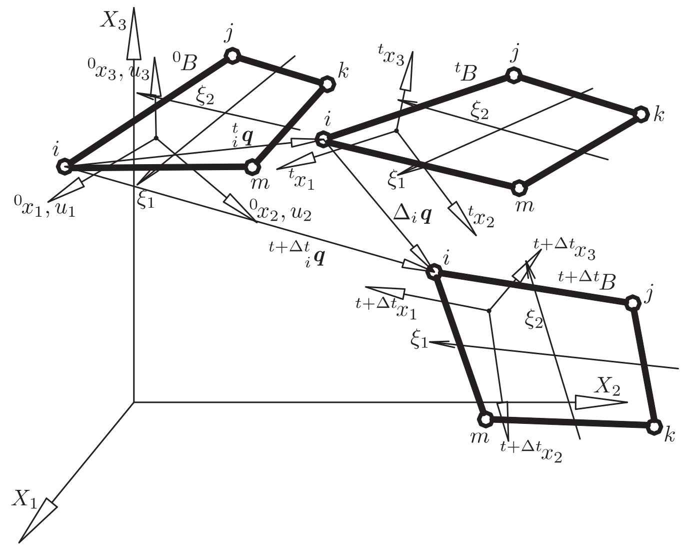

Figure 3. Deformation of the four node membrane element

The shape of a membrane element can be expressed in terms of the interpolation function and its nodal coordinates,

$$\begin{Bmatrix} x_1(\boldsymbol{\xi}) \\ x_2(\boldsymbol{\xi}) \\ x_3(\boldsymbol{\xi}) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} {}_ix_1 \\ {}_ix_2 \\ {}_ix_3 \\ {}_jx_1 \\ {}_jx_2 \\ {}_jx_3 \\ {}_kx_1 \\ {}_kx_2 \\ {}_kx_3 \\ {}_mx_1 \\ {}_mx_2 \\ {}_mx_3 \end{Bmatrix} = \tag{29}$$

$$= [N_1\boldsymbol{I}_3 \quad N_2\boldsymbol{I}_3 \quad N_3\boldsymbol{I}_3 \quad N_4\boldsymbol{I}_3]\,\boldsymbol{x} = \boldsymbol{N}\,\boldsymbol{x}$$

where

$$\boldsymbol{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{30}$$

In the above equations, $N_i(\boldsymbol{\xi}) = N_i$ are interpolation functions in the curvilinear coordinates, as given below:

$$\begin{aligned} N_1(\boldsymbol{\xi}) &= \tfrac{(1+\xi_1)(1+\xi_2)}{4}, & N_2(\boldsymbol{\xi}) &= \tfrac{(1-\xi_1)(1+\xi_2)}{4}, \\ N_3(\boldsymbol{\xi}) &= \tfrac{(1-\xi_1)(1-\xi_2)}{4}, & N_4(\boldsymbol{\xi}) &= \tfrac{(1+\xi_1)(1-\xi_2)}{4}. \end{aligned} \tag{31}$$

The displacement functions, referring to the local coordinate system, are given by:

$$\begin{Bmatrix} u_1(\boldsymbol{\xi}) \\ u_2(\boldsymbol{\xi}) \\ u_3(\boldsymbol{\xi}) \end{Bmatrix} = [N_1\boldsymbol{I}_3 \;\; N_2\boldsymbol{I}_3 \;\; N_3\boldsymbol{I}_3 \;\; N_4\boldsymbol{I}_3]\,\boldsymbol{q} = \boldsymbol{N}\,\boldsymbol{q}, \tag{32}$$

where $\boldsymbol{q} = \{{}_iu_1 \;\; {}_iu_2 \;\; {}_iu_3 \;\; {}_ju_1 \;\; {}_ju_2 \;\; {}_ju_3 \;\; {}_ku_1 \;\; {}_ku_2 \;\; {}_ku_3 \;\; {}_mu_1 \;\; {}_mu_2 \;\; {}_mu_3\}^T$.

According to Equation (26), the total strain field, $\boldsymbol{\varepsilon}$, can be divided into linear and non-linear parts,

$$\boldsymbol{\varepsilon} = \boldsymbol{e} + \boldsymbol{n} = \begin{Bmatrix} u_{1,1} \\ u_{2,2} \\ u_{1,2} + u_{2,1} \end{Bmatrix} + \frac{1}{2} \begin{Bmatrix} (u_{1,1})^2 + (u_{2,1})^2 + (u_{3,1})^2 \\ (u_{1,2})^2 + (u_{2,2})^2 + (u_{3,2})^2 \\ 2(u_{1,1}u_{1,2} + u_{2,1}u_{2,2} + u_{3,1}u_{3,2}) \end{Bmatrix}. \tag{33}$$

The linear part of the strain vector is specified as:

$$\boldsymbol{e} = \begin{Bmatrix} u_{1,1} \\ u_{2,2} \\ u_{1,2} + u_{2,1} \end{Bmatrix} = \begin{bmatrix} {}_ib & 0 & 0 & {}_jb & 0 & 0 & {}_kb & 0 & 0 & {}_mb & 0 & 0 \\ 0 & {}_ic & 0 & 0 & {}_jc & 0 & 0 & {}_kc & 0 & 0 & {}_mc & 0 \\ {}_ic & {}_ib & 0 & {}_jc & {}_jb & 0 & {}_kc & {}_kb & 0 & {}_mc & {}_mb & 0 \end{bmatrix} \cdot \begin{Bmatrix} {}_iu_1 \\ {}_iu_2 \\ {}_iu_3 \\ {}_ju_1 \\ {}_ju_2 \\ {}_ju_3 \\ {}_ku_1 \\ {}_ku_2 \\ {}_ku_3 \\ {}_mu_1 \\ {}_mu_2 \\ {}_mu_3 \end{Bmatrix} = B_e\,\boldsymbol{q} \tag{34}$$

where

$$
\begin{aligned}
{}_jb &= \bar{J}_{11}\cdot N_{2,\xi_1}+\bar{J}_{12}\cdot N_{2,\xi_2}\,, \quad {}_ib=\bar{J}_{11}\cdot N_{1,\xi_1}+\bar{J}_{12}\cdot N_{1,\xi_2}\\
{}_kb &= \bar{J}_{11}\cdot N_{3,\xi_1}+\bar{J}_{12}\cdot N_{3,\xi_2}\,, \quad {}_mb=\bar{J}_{11}\cdot N_{4,\xi_1}+\bar{J}_{12}\cdot N_{4,\xi_2}
\end{aligned}
\tag{35}
$$

$$
\begin{aligned}
{}_jc &= \bar{J}_{21}\cdot N_{2,\xi_1}+\bar{J}_{22}\cdot N_{2,\xi_2}\,, \quad {}_ic=\bar{J}_{21}\cdot N_{1,\xi_1}+\bar{J}_{22}\cdot N_{1,\xi_2}\\
{}_kc &= \bar{J}_{21}\cdot N_{3,\xi_1}+\bar{J}_{22}\cdot N_{3,\xi_2}\,, \quad {}_mc=\bar{J}_{21}\cdot N_{4,\xi_1}+\bar{J}_{22}\cdot N_{4,\xi_2}
\end{aligned}
\tag{36}
$$

The components of the inverse Jacobi matrix, $\bar{J}_{\alpha\beta}$, can be determined as:

$$
\begin{aligned}
\bar{J}_{11} &= \tfrac{J_{22}}{\det(\boldsymbol{J})}, \quad \bar{J}_{12}=-\tfrac{J_{12}}{\det(\boldsymbol{J})},\\
\bar{J}_{21} &= -\tfrac{J_{21}}{\det(\boldsymbol{J})}, \quad \bar{J}_{22}=\tfrac{J_{11}}{\det(\boldsymbol{J})},
\end{aligned}
\tag{37}
$$

where the components of the Jacobian $\boldsymbol{J}$ are specified as follows:

$$
\boldsymbol{J}=\begin{bmatrix} N_{1,\xi_1} & N_{2,\xi_1} & N_{3,\xi_1} & N_{4,\xi_1}\\ N_{1,\xi_2} & N_{2,\xi_2} & N_{3,\xi_2} & N_{4,\xi_2}\end{bmatrix}\begin{bmatrix} {}_ix_1 & {}_ix_2\\ {}_jx_1 & {}_jx_2\\ {}_kx_1 & {}_kx_2\\ {}_mx_1 & {}_mx_2\end{bmatrix}.
\tag{38}
$$

The non-linear part of the strain tensor is determined from the following relation:

$$
\boldsymbol{n}\frac{1}{2}\begin{bmatrix} u_{1,1} & 0 & u_{2,1} & 0 & u_{3,1} & 0\\ 0 & u_{1,2} & 0 & u_{2,2} & 0 & u_{3,2}\\ u_{1,2} & u_{1,1} & u_{2,2} & u_{2,1} & u_{3,2} & u_{3,1}\end{bmatrix}\begin{Bmatrix} u_{1,1}\\ u_{1,2}\\ u_{2,1}\\ u_{2,2}\\ u_{3,1}\\ u_{3,2}\end{Bmatrix}=\frac{1}{2}\boldsymbol{A}\,\boldsymbol{G}\,\boldsymbol{q}=\boldsymbol{B}_N\,\boldsymbol{q}
\tag{39}
$$

where matrices $\boldsymbol{A}$ and $\boldsymbol{G}$ have the form:

$$
\boldsymbol{G}=\begin{bmatrix}
{}_ib & 0 & 0 & {}_jb & 0 & 0 & {}_kb & 0 & 0 & {}_mb & 0 & 0\\
{}_ic & 0 & 0 & {}_jc & 0 & 0 & {}_kc & 0 & 0 & {}_mc & 0 & 0\\
0 & {}_ib & 0 & 0 & {}_jb & 0 & 0 & {}_kb & 0 & 0 & {}_mb & 0\\
0 & {}_ic & 0 & 0 & {}_jc & 0 & 0 & {}_kc & 0 & 0 & {}_mc & 0\\
0 & 0 & {}_ib & 0 & 0 & {}_jb & 0 & 0 & {}_kb & 0 & 0 & {}_mb\\
0 & 0 & {}_ic & 0 & 0 & {}_jc & 0 & 0 & {}_kc & 0 & 0 & {}_mc
\end{bmatrix},
\tag{40}
$$

$$
\boldsymbol{A}=\begin{bmatrix} B_u & 0 & B_v & 0 & B_w & 0\\ 0 & C_u & 0 & C_v & 0 & C_w\\ C_u & B_u & C_v & B_v & C_w & B_w\end{bmatrix},
\tag{41}
$$

while the components of matrix $\boldsymbol{A}$ are:

$$
\begin{aligned}
B_u &= [{}_ib_iu_1+{}_jb_ju_1+{}_kb_ku_1]\\
B_v &= [{}_ib_iu_2+{}_jb_ju_2+{}_kb_ku_2]\\
B_w &= [{}_ib_iu_3+{}_jb_ju_3+{}_kb_ku_3]\\
C_u &= [{}_ic_iu_1+{}_jc_ju_1+{}_kc_ku_1]\\
C_v &= [{}_ic_iu_2+{}_jc_ju_2+{}_kc_ku_2]\\
C_w &= [{}_ic_iu_3+{}_jc_ju_3+{}_kc_ku_3]
\end{aligned}
\tag{42}
$$

Consequently, the local elastic stiffness matrix of the element, $\boldsymbol{K}_L^e$, and the geometric stiffness matrix $\boldsymbol{K}_\sigma^e$, can be calculated from the following equations:

$$
\boldsymbol{K}_L^e=\frac{1}{|\boldsymbol{J}|}\int_{-1}^{+1}\int_{-1}^{+1}(\boldsymbol{B_e}+\boldsymbol{B_e}(\boldsymbol{q}))^T\,\boldsymbol{D}\,(\boldsymbol{B_e}+\boldsymbol{B_e}(\boldsymbol{q}))^T\,t\,d\xi_1 d\xi_2,
\tag{43}
$$

$$\boldsymbol{K}_\sigma^e = \frac{1}{|\boldsymbol{J}|}\int_{-1}^{+1}\int_{-1}^{+1} \boldsymbol{G}^T \boldsymbol{H}\,\boldsymbol{G}\, t\, d\xi_1 d\xi_2, \tag{44}$$

where

$$\boldsymbol{H} = \begin{bmatrix} \sigma_{11}\boldsymbol{I} & \tau_{12}\boldsymbol{I} \\ \tau_{12}\boldsymbol{I} & \sigma_{22}\boldsymbol{I} \end{bmatrix}. \tag{45}$$

The sum of matrices given by Equations (44) and (45) is equal to the total stiffness matrix of the element.

4. Example of the strain state analysis

The values of the nodal coordinates and displacements in the global orthogonal coordinate system are assumed as:

$$\begin{Bmatrix} {}_1X_1 \\ {}_1X_2 \\ {}_1X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} {}_2X_1 \\ {}_2X_2 \\ {}_2X_3 \end{Bmatrix} = \begin{Bmatrix} 6 \\ -8 \\ 5 \end{Bmatrix}, \quad \begin{Bmatrix} {}_3X_1 \\ {}_3X_2 \\ {}_3X_3 \end{Bmatrix} = \begin{Bmatrix} 6 \\ 2 \\ 3 \end{Bmatrix}, \quad \begin{Bmatrix} {}_4X_1 \\ {}_4X_2 \\ {}_4X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 5 \\ 2 \end{Bmatrix}, \tag{46}$$

$$\begin{Bmatrix} {}_1U_1 \\ {}_1U_2 \\ {}_1U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} {}_2U_1 \\ {}_2U_2 \\ {}_2U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} {}_3U_1 \\ {}_3U_2 \\ {}_3U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}, \quad \begin{Bmatrix} {}_4U_1 \\ {}_4U_2 \\ {}_4U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{47}$$

The distribution of the strain vector in the isoparametric finite element is investigated. The values at the four integration points are calculated (see Figure 4).

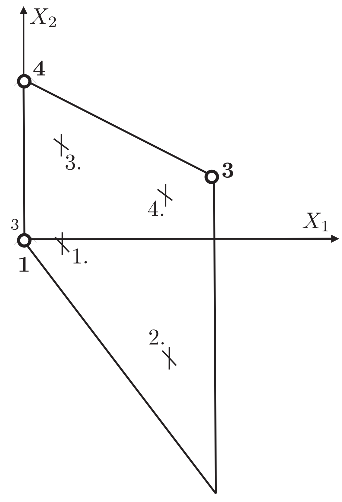

Figure 4. Plane visualization of the isoparametric membrane element

The values of $\xi_1 = +0.57735$, $\xi_2 = +0.57735$ are taken for the first integration point. Consequently, the shape functions and its derivatives are specified as:

$$\boldsymbol{N} = [N_1 \quad N_2 \quad N_3 \quad N_4] = [0.62201 \quad 0.16667 \quad 0.04466 \quad 0.16667], \tag{48}$$

$$\begin{bmatrix} N_{1,\xi_1} & N_{2,\xi_1} & N_{3,\xi_1} & N_{4,\xi_1} \\ N_{1,\xi_2} & N_{2,\xi_2} & N_{3,\xi_2} & N_{4,\xi_2} \end{bmatrix} = \begin{bmatrix} 0.39434 & -0.39434 & -0.10566 & 0.10566 \\ 0.39434 & 0.10566 & -0.10566 & -0.39434 \end{bmatrix}. \tag{49}$$

The position vector, $\boldsymbol{R}(\boldsymbol{\xi})$, at the first integration point can be described as:

$$\begin{aligned}\boldsymbol{R}(\xi) &= (\boldsymbol{N}\cdot\boldsymbol{X}_1)\,\boldsymbol{i}_1 + (\boldsymbol{N}\cdot\boldsymbol{X}_2)\,\boldsymbol{i}_2 + (\boldsymbol{N}\cdot\boldsymbol{X}_3)\,\boldsymbol{i}_3 = \\ &= 1.26795\,\boldsymbol{i}_1 - 0.41068\,\boldsymbol{i}_2 + 1.3006\,\boldsymbol{i}_3.\end{aligned} \tag{50}$$

It is necessary to specify the derivatives of the position vector, $\boldsymbol{R}(\boldsymbol{\xi})$, with respect to ξ_1 and ξ_2 as follows:

$$\begin{aligned}\boldsymbol{R}_{,\xi_1} &= (N_{,\xi_1}\cdot\boldsymbol{X}_1)\,\boldsymbol{i}_1 + (N_{,\xi_1}\cdot\boldsymbol{X}_2)\,\boldsymbol{i}_2 + (N_{,\xi_1}\cdot\boldsymbol{X}_3)\,\boldsymbol{i}_3 = \\ &= -3\,\boldsymbol{i}_1 + 3.47169\,\boldsymbol{i}_2 - 2.07735\,\boldsymbol{i}_3 \\ \boldsymbol{R}_{,\xi_2} &= (N_{,\xi_2}\cdot\boldsymbol{X}_1)\,\boldsymbol{i}_1 + (N_{,\xi_2}\cdot\boldsymbol{X}_2)\,\boldsymbol{i}_2 + (N_{,\xi_2}\cdot\boldsymbol{X}_3)\,\boldsymbol{i}_3 = \\ &= 0\,\boldsymbol{i}_1 - 3.028313\,\boldsymbol{i}_2 - 0.57735\,\boldsymbol{i}_3\end{aligned} \tag{51}$$

It is useful to build vectors $\boldsymbol{t}_\alpha$, $\boldsymbol{s}_\alpha$ and $\boldsymbol{d}_\alpha$, instrumental in creating the local Cartesian coordinate system,

$$\boldsymbol{t}_1 = \frac{\boldsymbol{R}_{,\xi_1}}{|\boldsymbol{R}_{,\xi_1}|},\ \boldsymbol{t}_2 = \frac{\boldsymbol{R}_{,\xi_2}}{|\boldsymbol{R}_{,\xi_2}|},\ \ \boldsymbol{s}_1 = \boldsymbol{t}_1 + \boldsymbol{t}_2,\ \boldsymbol{s}_2 = \boldsymbol{t}_1 - \boldsymbol{t}_2,\ \ \boldsymbol{d}_1 = \frac{\boldsymbol{s}_1}{\sqrt{2}|\boldsymbol{s}_1|},\ \boldsymbol{d}_2 = \frac{\boldsymbol{s}_2}{\sqrt{2}|\boldsymbol{s}_2|} \tag{52}$$

Finally, the local orthogonal unit vector can be obtained as follows:

$$\begin{aligned}\boldsymbol{j}_1 &= \boldsymbol{d}_1 + \boldsymbol{d}_2 = -0.70625\,\boldsymbol{i}_1 + 0.42917\,\boldsymbol{i}_2 - 0.56304\,\boldsymbol{i}_3 \\ \boldsymbol{j}_2 &= \boldsymbol{d}_1 - \boldsymbol{d}_2 = -0.23534\,\boldsymbol{i}_1 - 0.892397\,\boldsymbol{i}_2 - 0.38502\,\boldsymbol{i}_3 \\ \boldsymbol{j}_3 &= \boldsymbol{j}_1\times\boldsymbol{j}_2 = -0.6677\,\boldsymbol{i}_1 - 0.139416\,\boldsymbol{i}_2 + 0.73126\,\boldsymbol{i}_3\end{aligned} \tag{53}$$

It is easy to prove that

$$|\boldsymbol{j}_1| = 1,\ |\boldsymbol{j}_2| = 1,\ |\boldsymbol{j}_3| = 1 \ \text{ and }\ \boldsymbol{j}_3\times\boldsymbol{j}_1 = \boldsymbol{j}_2 = -0.23534\boldsymbol{i}_1 - 0.892397\,\boldsymbol{i}_2 - 0.38502\,\boldsymbol{i}_3.$$

In the next step, it is necessary to transform the system of global coordinates of nodes, and define displacements in this system, to the local coordinate system. The transformation matrix, $\boldsymbol{L}$, is used in this operation,

$$\boldsymbol{L} = \begin{bmatrix}\{\boldsymbol{j}_1\}^T \\ \{\boldsymbol{j}_2\}^T \\ \{\boldsymbol{j}_3\}^T\end{bmatrix} = \begin{bmatrix}-0.70625 & 0.4291746 & -0.56304 \\ -0.23534 & -0.892397 & -0.38502 \\ -0.6677 & -0.139416 & 0.73126\end{bmatrix}. \tag{54}$$

Thus, the new local coordinates of nodes are as follows:

$$\begin{aligned}&\begin{Bmatrix}{}_1x_1 \\ {}_1x_2 \\ {}_1x_3\end{Bmatrix} = \begin{Bmatrix}1.8041 \\ 0.4327 \\ -0.1617\end{Bmatrix}, \quad \begin{Bmatrix}{}_2x_1 \\ {}_2x_2 \\ {}_2x_3\end{Bmatrix} = \begin{Bmatrix}-8.6821 \\ 4.2346 \\ 0.60369\end{Bmatrix}, \\ &\begin{Bmatrix}{}_3x_1 \\ {}_3x_2 \\ {}_3x_3\end{Bmatrix} = \begin{Bmatrix}-3.2642 \\ -3.9192 \\ -2.2530\end{Bmatrix}, \quad \begin{Bmatrix}{}_4x_1 \\ {}_4x_2 \\ {}_4x_3\end{Bmatrix} = \begin{Bmatrix}2.8238 \\ -4.7993 \\ 0.60369\end{Bmatrix},\end{aligned} \tag{55}$$

while the new local displacements can be calculated as:

$$\begin{aligned}&\begin{Bmatrix}{}_1u_1 \\ {}_1u_2 \\ {}_1u_3\end{Bmatrix} = \begin{Bmatrix}0 \\ 0 \\ 0\end{Bmatrix}, \quad \begin{Bmatrix}{}_2u_1 \\ {}_2u_2 \\ {}_2u_3\end{Bmatrix} = \begin{Bmatrix}0 \\ 0 \\ 0\end{Bmatrix}, \\ &\begin{Bmatrix}{}_3u_1 \\ {}_3u_2 \\ {}_3u_3\end{Bmatrix} = \begin{Bmatrix}-0.56304 \\ -0.385021 \\ 0.731261\end{Bmatrix}, \quad \begin{Bmatrix}{}_4u_1 \\ {}_4u_2 \\ {}_4u_3\end{Bmatrix} = \begin{Bmatrix}0 \\ 0 \\ 0\end{Bmatrix}.\end{aligned} \tag{56}$$

Then, the following B_e, $\boldsymbol{A}$, and $\boldsymbol{G}$ matrices can be obtained:

$$\boldsymbol{B}_e = \begin{bmatrix} 0.1434 & 0 & 0 & -0.0793 & 0 & 0 & -0.0384 & 0 & 0 & -0.0256 & 0 & 0 \\ 0 & 0.1826 & 0 & 0 & 0.0097 & 0 & 0 & -0.0489 & 0 & 0 & -0.1434 & 0 \\ 0.1826 & 0.1434 & 0 & 0.0097 & -0.0793 & 0 & -0.0489 & -0.0384 & 0 & -0.1434 & -0.0256 & 0 \end{bmatrix}, \tag{57}$$

$$\boldsymbol{A} = \begin{bmatrix} 0.02163 & 0 & 0.01479 & 0 & -0.02809 & 0 \\ 0 & 0.02755 & 0 & 0.01884 & 0 & -0.03578 \\ 0.02755 & 0.02163 & 0.01884 & 0.01479 & -0.03578 & -0.02809 \end{bmatrix}, \tag{58}$$

$$\boldsymbol{G} = \begin{bmatrix} 0.14337 & 0 & 0 & -0.0793 & 0 & 0 & -0.0384 & 0 & 0 & -0.02567 & 0 & 0 \\ 0.18260 & 0 & 0 & 0.0097 & 0 & 0 & -0.0489 & 0 & 0 & -0.1434 & 0 & 0 \\ 0 & 0.14337 & 0 & 0 & -0.0793 & 0 & 0 & -0.0384 & 0 & 0 & -0.02567 & 0 \\ 0 & 0.18260 & 0 & 0 & 0.0097 & 0 & 0 & -0.0489 & 0 & 0 & -0.1434 & 0 \\ 0 & 0 & 0.14337 & 0 & 0 & -0.0793 & 0 & 0 & -0.0384 & 0 & 0 & -0.02567 \\ 0 & 0 & 0.18260 & 0 & 0 & 0.0097 & 0 & 0 & -0.0489 & 0 & 0 & -0.1434 \end{bmatrix}. \tag{59}$$

As the matrices of the finite element method have been created, the components of the strain vector, ${}_1\boldsymbol{\varepsilon}$, at the first integration point can be determined,

$${}_1\boldsymbol{\varepsilon} = \boldsymbol{e} + \boldsymbol{n} = \boldsymbol{B}_e\,\boldsymbol{q} + \frac{1}{2}\boldsymbol{A}\,\boldsymbol{G}\,\boldsymbol{q} = \begin{Bmatrix} 0.021630 \\ 0.018838 \\ 0.042340 \end{Bmatrix} + \begin{Bmatrix} 0.000738 \\ 0.001197 \\ 0.001880 \end{Bmatrix} = \begin{Bmatrix} 0.022368 \\ 0.020035 \\ 0.044220 \end{Bmatrix}. \tag{60}$$

If the same calculation procedure is repeated for the other integration points, the following values of the strain component vector can be obtained for the second, third and fourth integration point:

$${}_2\boldsymbol{\varepsilon} = \boldsymbol{e} + \boldsymbol{n} = \begin{Bmatrix} 0.038749 \\ 0.009405 \\ 0.068848 \end{Bmatrix} + \begin{Bmatrix} 0.003231 \\ 0.008559 \\ 0.010517 \end{Bmatrix} = \begin{Bmatrix} 0.0419799 \\ 0.0179642 \\ 0.0793649 \end{Bmatrix} \tag{61}$$

$${}_3\boldsymbol{\varepsilon} = \boldsymbol{e} + \boldsymbol{n} = \begin{Bmatrix} 0.037077 \\ -0.00826 \\ 0.024506 \end{Bmatrix} + \begin{Bmatrix} 0.01191 \\ 0.009789 \\ 0.021596 \end{Bmatrix} = \begin{Bmatrix} 0.0489875 \\ 0.0015318 \\ 0.0461014 \end{Bmatrix} \tag{62}$$

$${}_4\boldsymbol{\varepsilon} = \boldsymbol{e} + \boldsymbol{n} = \begin{Bmatrix} 0.041528 \\ 0.019049 \\ 0.057720 \end{Bmatrix} + \begin{Bmatrix} 0.008039 \\ 0.002337 \\ 0.008669 \end{Bmatrix} = \begin{Bmatrix} 0.0495664 \\ 0.0213861 \\ 0.0663889 \end{Bmatrix} \tag{63}$$

For the sake of the presented procedure, a verification of the geometrical non-linear analysis was carried out in the MSC.Marc commercial system. The results of the strain state distribution obtained from the commercial program (see Figures 5–7) and from the described procedure are exactly the same. It should be noted that the strain values presented in the Figures 5–7 correspond to the integration points (the TRANSLATE option of the MSC.Marc system was applied).

5. Remarks and conclusions

A four-node, isoparametric, membrane element analysis has been performed. It should be noted that this element has no bending stiffness and is very unstable – membrane analysis is extremely difficult due to rigid body modes. As the membrane element is used with geometric non-linear analysis, the tensile initial stress stiffness increases the element's rigidity. Due to bilinear interpolation, the surface forms a hyperbolic paraboloid, which is allowed to degenerate into a plane.

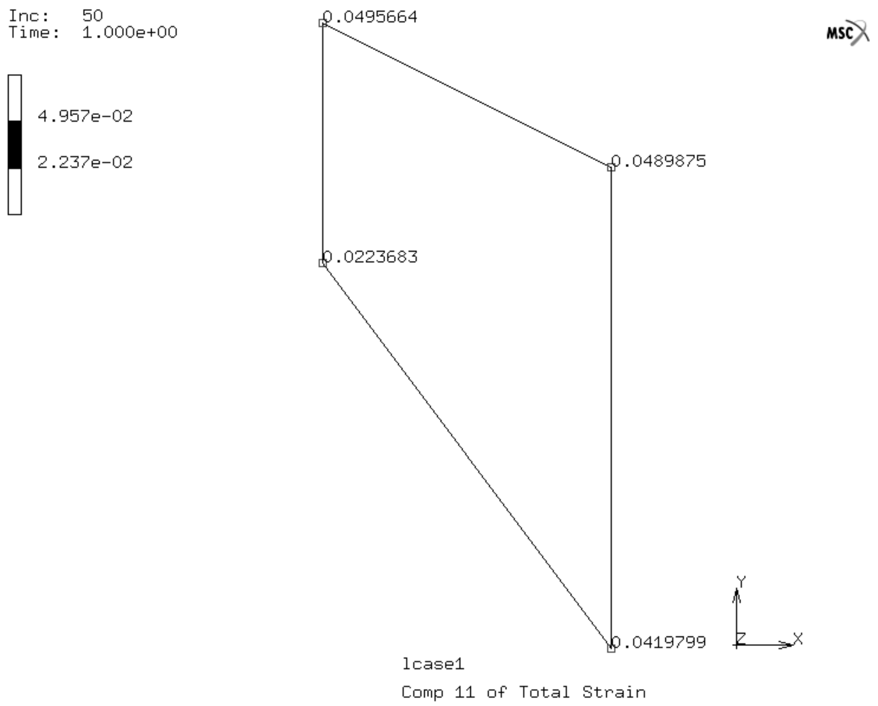

Figure 5. Integration point values of the strain component ε_{11}

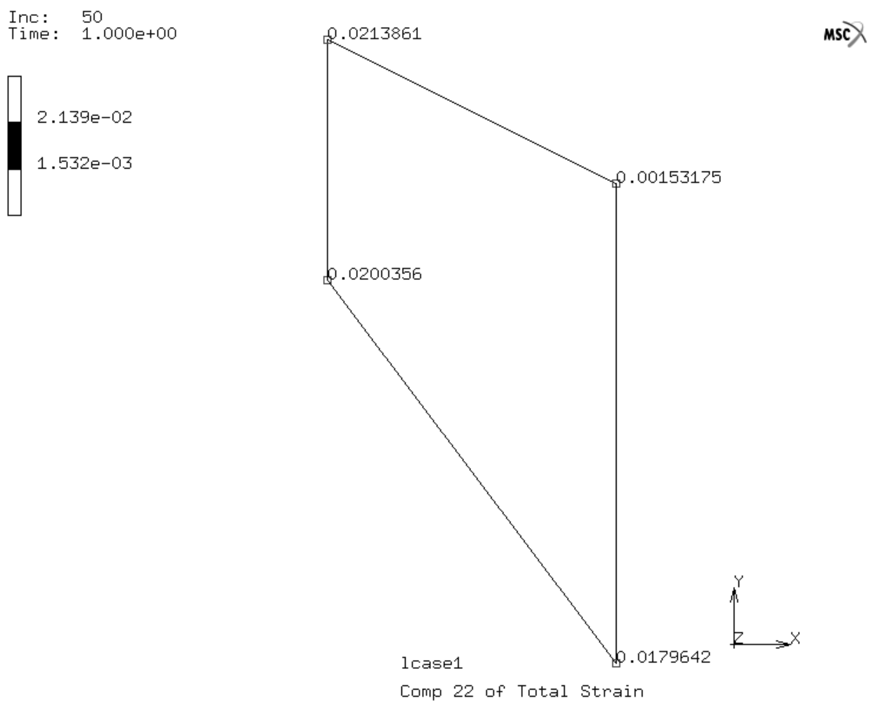

Figure 6. Integration point values of the strain component ε_{22}

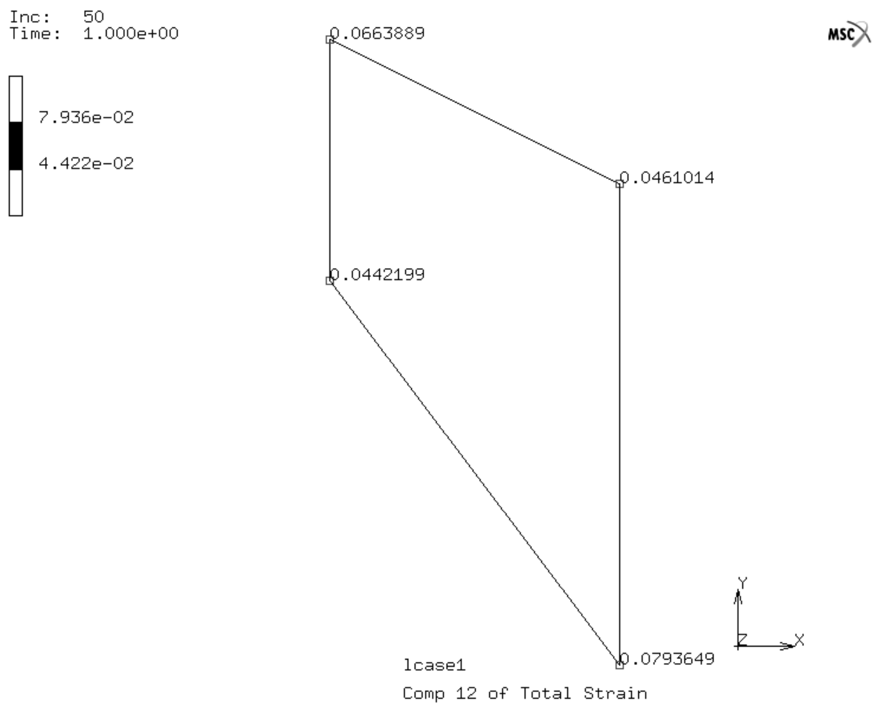

Figure 7. Integration point values of the strain component ε_{12}

Generally, membrane elements are used for analysis of membrane structures. Usually, membrane sheet constructions are made from the glass fibres or carbon fibres covered by a plastics coating (*e.g.* PTFE, PVC), referred to as technical woven fabric. A number of theoretical models have been developed to describe the behaviour of technical woven fabrics (see *e.g.* [13, 14]).The choice of the constitutive model assumed to describe the fabric's behaviour is always a disputable problem ([15, 16]). The authors have successfully used this type of finite elements in their analysis of membrane hanging roofs, with various types of constitutive modelling used to describe the behaviour of the fabric membrane ([17–19]) in which the viscoelastic characteristic [20], the viscoplastic characteristic ([21, 22]) or the non-linear elastic characteristic ([23, 24]) can be applied.

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