NATURAL FREQUENCIES OF TUNED AND MISTUNED BLADED DISCS ON A SHAFT

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Abstract: The dynamic behaviour of a rotor consisting of two tuned and mistuned bladed discs on a solid shaft is considered. The effect of the shaft's flexibility on the dynamic characteristics of the bladed discs and the coupling effects between the shaft and bladed disc modes are investigated. Results show clearly the coupling effects in a bladed disc-shaft system. Interference diagrams are developed, from which the dynamic behaviour of a bladed disc can be predicted for varying flexibility relationships between solid shafts and bladed discs.

The global rotating mode shapes of flexible tuned and mistuned bladed disc-shaft assemblies are calculated. Rotational effects, such as centrifugal stiffening, are accounted for and all the possible couplings between the flexible parts are allowed. Calculated natural frequencies obtained from the blade, the bladed disc and the shaft with two discs are checked to discover resonance conditions and coupling effects. It is shown that blade mistuning strongly affects the interaction between flexible bladed discs and a flexible shaft. This interaction affects the flexible bladed disc modes and is not restricted to the modes with zero, one and two nodal diameters. The torsional frequency of the shaft with two bladed discs is coupled with the zero nodal diameter modes of the single bladed discs. It is shown that including the shaft in the bladed discs model is important from the designer's point of view and may alter the spectrum of frequencies considerably.

Keywords: blade, bladed disc, mistuning

Nomenclature

k – number of disc nodal diameters,
L – blade length [m],
Ω – rotation frequency.

1. Introduction

The study of the vibration characteristics of individual turbo machinery components, such as blades, discs and shafts, has been established as an important part of the design process. However, the vibration characteristics of an individual component can change considerably when these components are assembled to form a system, due to the coupling effects among them. Individual blades exhibit small structural differences referred to as blade mistuning. These variations destroy cyclic symmetry and may lead to qualitatively dynamic behaviour different than that of a tuned system.

Two independent approaches are commonly used to analyse the dynamic behaviour of rotating turbo machinery assemblies. In the first one, the rotordynamic approach is concerned with disc-shaft systems. The disc-shaft is mostly modelled using beam finite elements, with disc flexibility either ignored (Berger and Kulig [1]) or considered (Chivens and Nelson [2]). In the second approach to the tuned bladed disc one deals with flexible discs (Rządkowski [3]). There are a few papers on the free vibration of a bladed disc on a flexible shaft (Dubigeon and Michon [4], Huang and Ho [5], Jacquest-Richardet *et al.* [6], Kanki and Yammamoto [7], Khader and Masoud [8]), claiming that the inertial effects generated on the shaft by the vibration of a tuned bladed disc are important for the modes associated with zero and one nodal diameter modes.

The effects of blade mistuning on the free vibration characteristics and forced response of a bladed disc on a rigid shaft were reported by Wagner [9], Dye and Henry [10], Ewins [11], El-Bayomy and Srinivasan [12], Irretier [13], Rządkowski [14–16] and Wei and Pierre [17, 18].

The effects of mistuning on flutter boundaries also received considerable attention. Dugundji and Bundas [19], Kaza and Kielb [20], Valero and Bendiksen [21], Crawley and Hall [22], Khader and Loewy [23], Gnesin and Rządkowski [24] and Rządkowski and Kovalov [25] concluded that blade mistuning had a stabilizing effect on bladed discs supported by a rigid shaft.

Crawley [26] considered a bladed disc-shaft system as an assembly of flexible blades, cantilevered to a rigid disc. He examined the effect of rigid disc's motion on a severely mistuned bladed disc.

Khader and Masoud [8] considered the effects of blade mistuning on the free vibration characteristics of a non-rotating flexible blade-rigid disc-flexible shaft assembly.

It has been found in this paper that flexible modes of tuned bladed discs affected by shaft motion are the modes with zero and one nodal diameters. Shaft deflection is clearly visible in these modes. These results are in agreement with similar results reported in earlier papers on this subject.

A new result found here is that zero nodal diameters of uncoupled bladed disc modes split considerably for coupled modes of a tuned bladed discs-shaft system.

It has also been found that flexible modes of tuned bladed discs affected by shaft motion are the modes with two nodal diameters. Shaft deflection is clearly visible in these modes.

Also shown are the effects of mistuning on the free vibration of a non-rotating and rotating flexible bladed disc on a flexible shaft.

Mistuned blades-disc-shaft modes are generally affected by the shaft's flexibility. A frequency split takes place in a mistuned system, unlike tuned bladed disc-shaft assemblies. Mistuning causes coupling between the torsion and bending modes, as well as coupling of individual stages.

It follows from our calculations that including a shaft in a model of bladed discs is important from the designer's point of view and may alter the spectrum of frequencies considerably.

2. The structural model

Several existing finite element codes have been considered to develop solid bladed discs-shaft dynamics procedures. The NASTRAN 2003 code takes into consideration the gyroscopic effects of a flexible shaft-discs assembly. They are calculated as separate elements of equivalent discs and fed as input into a 3D model. Difficulties arise with the inclusion of the gyroscopic effect of the bladed discs on the shaft. ANSYS has been found to be the most suitable to accomplish the task.

Both the forward whirl and the backward whirl modes have been obtained in solid rotor analysis. The forward whirls have been slightly enhanced because of the centrifugal effects, while the backward whirl modes have been significantly affected by spin softening effects.

3. Numerical results

The considered structure is composed of 24 blades on two bladed discs mounted on a simply supported shaft (see Figure 1). Its main dimensions are as follows: the disc's outer diameter is 0.7m, its inner diameter is 0.1m and its thickness is 0.02m. The blades' height is $L = 0.25$m. The blades are rectangular in cross-section, 0.0824×0.198m, with the stagger angle on the disc equal to 45°. The length of the shaft is 2.50m, its diameter – 0.1m. The first rigid bearing is placed at a distance of 0.42m from the left of the shaft. The distance from the left support to the left disc is 0.5m, that between the discs is 0.5m, with further 0.5m to the right support.

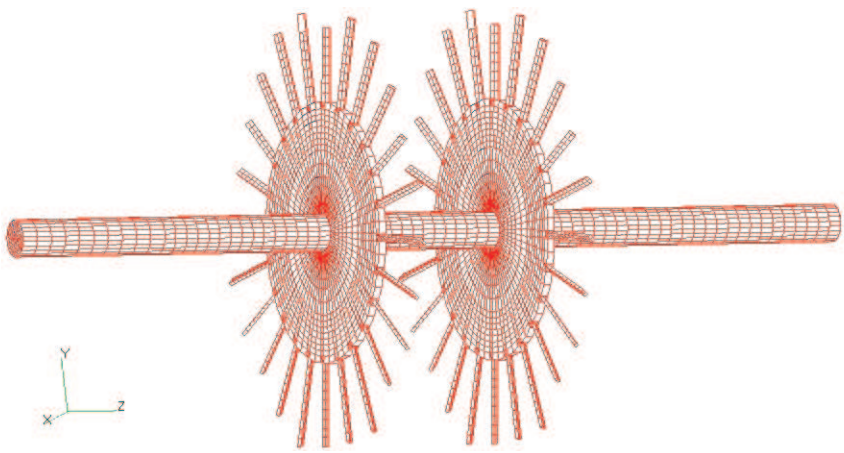

Figure 1. The bladed discs-shaft model

The mistuned blade disc is composed of 24 blades of varying length: (I) 9 blades of $L = 0.250$m, (II) 9 blades of $L = 0.225$m and (III) 6 blades of $L = 0.200$m (see Figure 2). Both discs have the same pattern and are aligned axially.

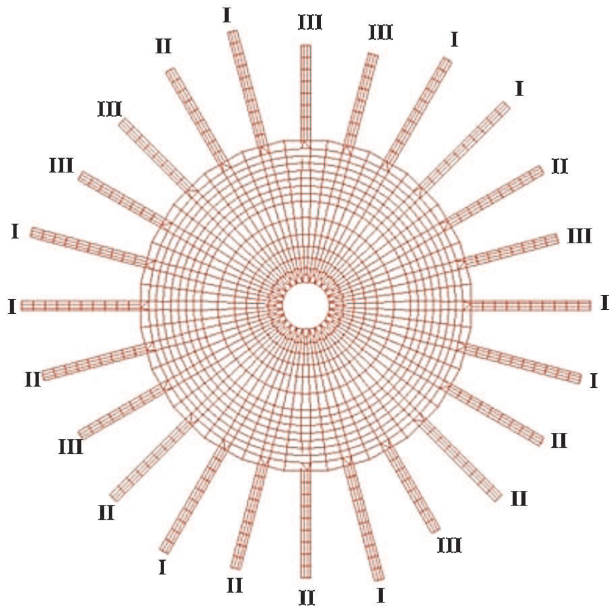

Figure 2. Mistuned bladed disc

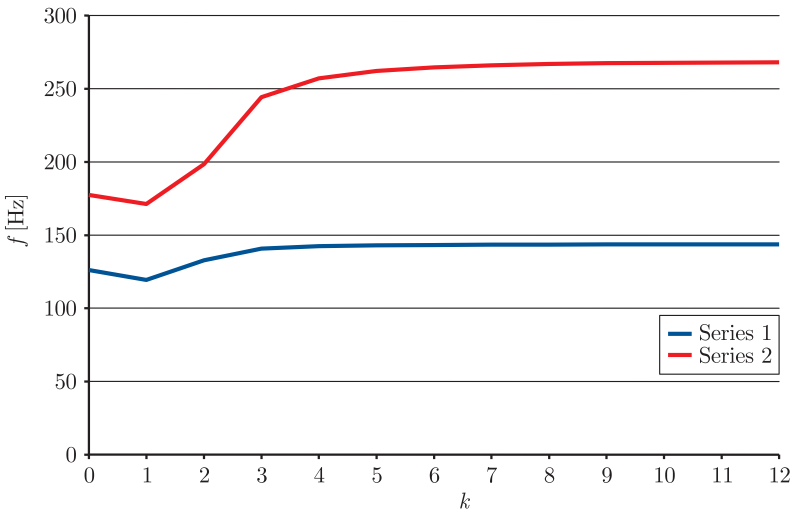

Figure 3. Interference diagram of a tuned bladed disc ($L = 0.250$m)

Isoparametric brick elements with 8 nodes and 3 degrees of freedom per node are used.

The natural frequencies of the non-rotating bladed disc with 24 blades of $L = 0.250$m are shown in Figure 3 versus the number of nodal diameters. The bladed disc modes are classified by analogy with axisymmetric modes, which are mainly characterized by nodal lines lying along the structure's diameters with constant angular spacing. There are zero ($k = 0$), one ($k = 1$), two ($k = 2$) or more ($k > 2$) nodal diameter bending or torsion modes. Series 1 is associated with the first natural frequency of the single cantilever blade. Series 2 is associated with the second natural frequency of the single cantilever blade; and so on, with k as the number of nodal diameters (see Figure 3).

In order to discuss the natural frequencies of the mistuned bladed disc on the shaft, the natural frequencies of the two tuned bladed discs on the shaft were calculated for various blade lengths: (I) $L = 0.250$m, (II) $L = 0.225$m, (III) $L = 0.200$m (Figure 2). The frequencies of the tuned bladed disc are discussed below, followed by a comparison with the frequencies of a mistuned system.

The natural frequencies of tuned bladed discs on the shaft are divided into bladed disc-dominated modes and shaft with disc-dominated modes.

The natural frequencies of the cantilever blade, the single bladed disc and the shaft with two discs have been checked to discover resonance conditions and coupling effects in bladed discs on a flexible shaft.

4. Tuned structures at rest

The leftmost axis of Figure 6 indicates the uncoupled natural frequencies of the cantilever blade. The next axis to the right shows the natural frequencies of the tuned bladed disc. The coupled natural frequencies of the two tuned bladed discs on the shaft are given on the third axis from the left. Next, the natural frequencies of the shaft with two discs are shown. The numbers in brackets show the number of nodal diameters of the tuned bladed discs modes. Colour blue marks the bending mode of the shaft, green – its torsion mode, and red – the bending mode of the bladed disc, where the shaft's motion is not visible.

The first two coupled bending natural frequencies of the system, equal to 46.59Hz, are the shaft with two disc-dominated modes, where the blades' vibration is very small (Figure 4).

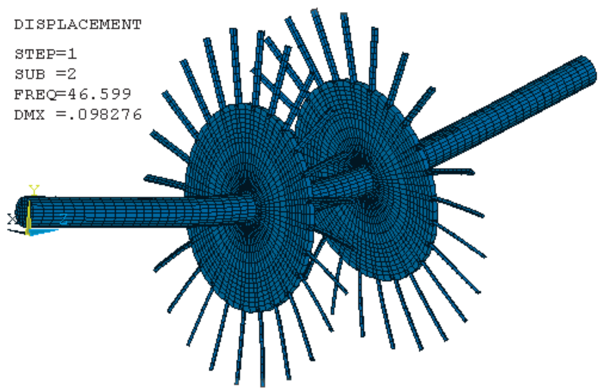

Figure 4. The mode's shape at 45.59Hz

As expected, the strongest coupling between the modes takes place when the uncoupled shaft-with-two-discs frequencies are close to the uncoupled flexible-bladed-disc frequencies.

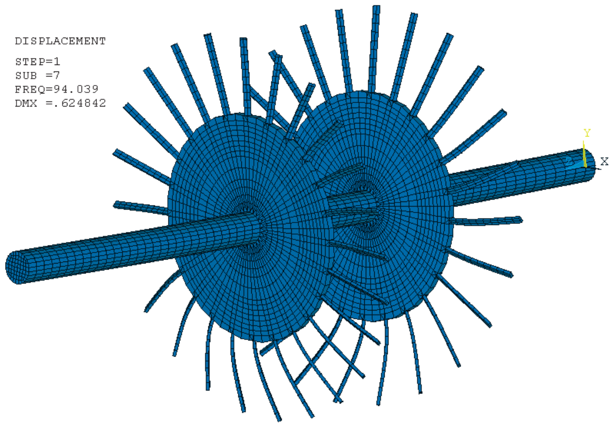

Figure 5. The mode shape at 94.04 Hz

The uncoupled tuned bladed disc's natural frequency of the zero nodal diameter mode ($k=0$) is 101.65 Hz (colour red), while the uncoupled torsional natural frequency of the shaft with two discs is 103.05 Hz (colour green). The coupled natural frequencies of bladed discs on the shaft with zero nodal diameters split into the values of 78.74 Hz (with a predominant first bladed disc vibration) and 98.0 Hz (with a predominant second bladed disc vibration). When the two discs placed on the shaft are in their single torsion modes with zero nodal diameters and staggered blades are attached to the disc, the zero nodal diameter modes split because of the uncoupled torsion shaft with two disc modes are close to the uncoupled bladed discs with zero nodal diameters.

Two uncoupled bladed discs bending modes of one nodal diameter ($k=1$) are 97.0 Hz, while the coupled natural frequencies of the shaft with two bladed discs are 93.10 Hz for the first predominant bladed disc vibration mode (colour blue) and 94.04 Hz for the second (Figure 5).

In this case, one nodal diameter doubled modes of the bladed disc influence the modes with one nodal diameter of the bladed disc on the shaft, which splitting in two.

These results are in agreement with similar results reported in earlier papers on this subject. It has been found that flexible modes of tuned bladed discs affected by shaft motion are the modes with zero and one nodal diameters. This strong coupling in the zero nodal diameter modes is due to them being not self-equilibriated and thus producing a net force on the shaft. One-nodal diameter modes produce a net moment on the shaft.

A new result found here has been that zero nodal diameters of coupled bladed discs on a shaft split because of the influence of the single torsion of two discs on the shaft modes and the zero nodal diameter of the bladed disc.

The uncoupled bladed disc bending mode of two nodal diameters is 104.06 Hz, while the coupled natural frequencies of the shaft with two bladed discs are 103.85 Hz with predominant vibration of the first bladed disc and 103.85 Hz with predominant vibration of the second bladed disc. In this case, the coupled modes are bladed disc-dominated modes, the shaft causing a slight decrease of the frequencies from 104.06 Hz to 103.85 Hz, splitting not observed.

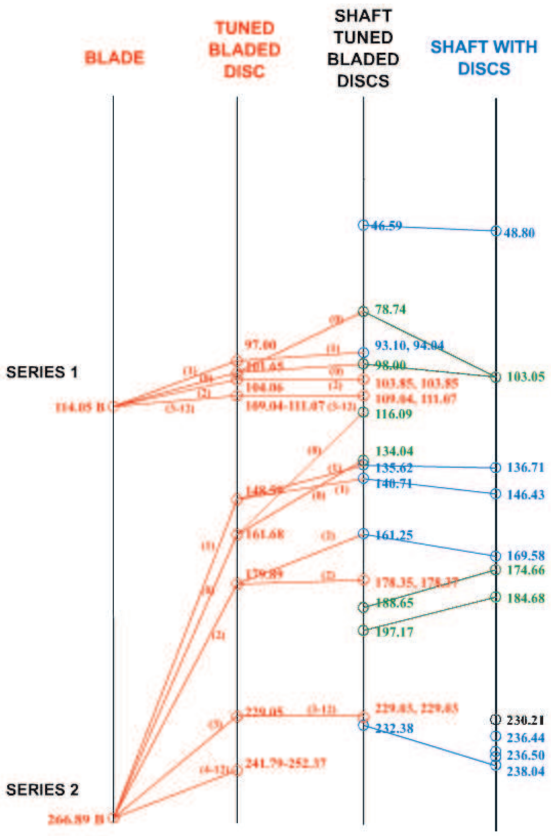

Figure 6. Natural frequencies of bladed discs on a shaft for blade length $L = 0.250$m

Other coupled modes of Series 1, from three to twelve nodal diameters of bladed discs, do not interact with the shaft's motion (colour red). The uncoupled bladed disc's natural frequencies from three to twelve nodal diameter modes are in the 109.04–111.07Hz range (colour red) and the coupled natural frequencies of the shaft with two bladed discs are in the 109.04–111.07Hz range (colour red) for the first or the second predominant disc vibration modes.

The uncoupled tuned bladed disc's natural frequency of the zero nodal diameter mode of the second series is 161.68Hz. The coupled torsion natural frequencies of bladed discs on the shaft with zero nodal diameters split into values of 116.09Hz torsion vibration mode when the first bladed disc is vibrating predominantly (colour green) and 134.04Hz for the second bladed disc. These coupled modes are also influenced by the uncoupled torsion mode of two discs on the shaft (103.03Hz). The torsion mode of two discs on the shaft (103.05Hz) are coupled with the zero nodal diameter modes of the bladed disc from the first (101.65Hz) and the second series (161.68Hz).

The uncoupled shaft with two discs' bending mode (colour blue, shaft motion visible) of one nodal diameter is 136.71Hz, while the coupled natural frequencies of the shaft with two bladed discs are 135.62Hz (colour blue) for the first predominant bladed disc's vibration mode and 140.71Hz for the second. Mode of 140.71Hz is coupled with

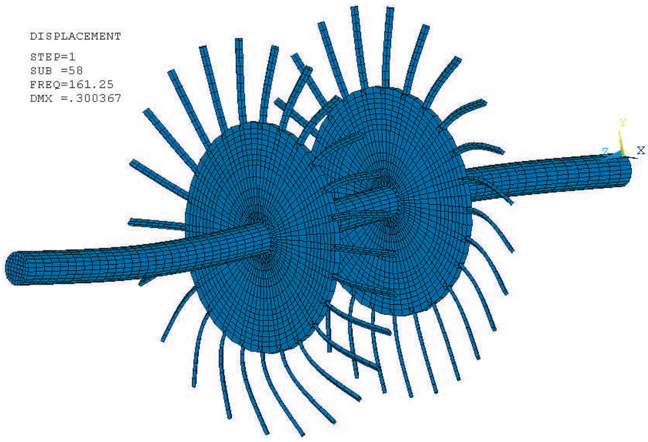

Figure 7. Mode shape at 161.25Hz

the shaft with two discs' bending mode equal to 146.43Hz. Two bending modes of the shaft with two discs (136.71Hz, 146.43Hz) are coupled with the bladed disc with one nodal diameter modes (135.62Hz, 140.71Hz) causing a frequency split greater than in Series 1. Shaft deflection is clearly visible in these modes. The split is influenced by close frequencies of two discs shaft modes of 136.71Hz and 146.43Hz, and the 148.59Hz bladed disc mode with one nodal diameter.

The uncoupled shaft with two discs' bending mode of two nodal diameters is 169.58Hz and the coupled natural frequency of the shaft with two bladed discs is 161.25Hz (colour blue). It has been found that the considered flexible modes of tuned bladed discs affected by shaft motion are modes with two nodal diameters. In these modes, shaft deflection is clearly visible (Figure 7).

The coupled modes of the bladed discs-shaft system are 178.35Hz for the first predominant bladed disc's vibration mode and 178.37Hz for the second when shaft vibration is not visible (colour red). A small splitting can be observed in this case. The corresponding uncoupled bladed discs' modes of two nodal diameters are 179.89Hz.

In a coupled multistage model, adjacent stages will not have the same number of blades. Thus, coupling of individual stages introduces a level of disorder to each stage (Rządkowski *et al.* [27]). In our case, the adjacent stages will have the same number of blades.

In order to better understand the influence of mistuning on the bladed discs-shaft system, natural frequencies were calculated for tuned bladed discs on a shaft with blade lengths of $L = 0.225$m and 0.2m (see Figures 8 and 9). When the blade length was decreased, the natural frequencies of the bladed disc-shaft system increased, the general conclusions being similar to the case of bladed discs on a shaft with blade length $L = 0.250$m.

5. Mistuned structure at rest

For mistuned results, a single random mistuning pattern was used for each stage. Three blades with lengths of 0.250m, 0.225m and 0.2m were arranged around

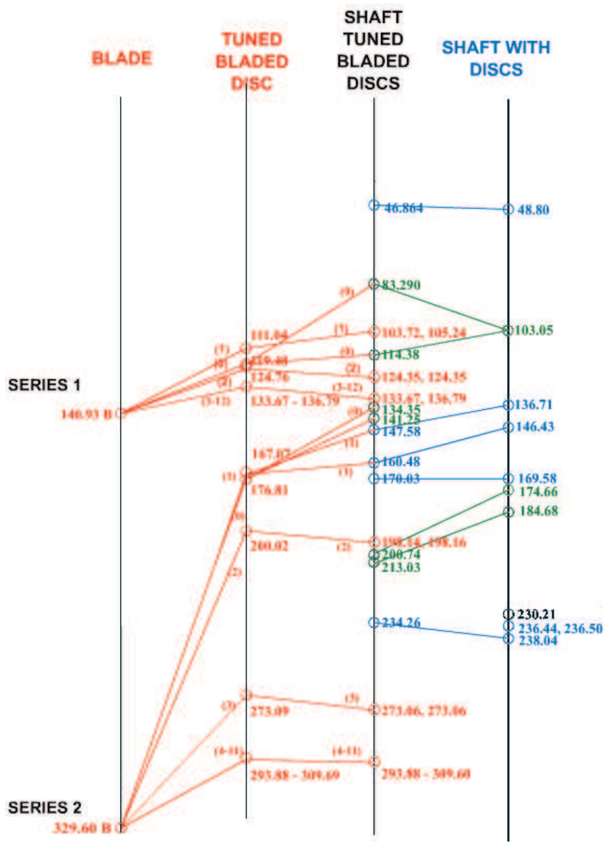

Figure 8. Natural frequencies of bladed discs on a shaft for blade length $L = 0.225$m

the disc. Deviation in blade lengths of +11% and −9% was introduced. The natural frequencies of mistuned bladed discs on the shaft were divided into bladed disc-dominated modes and shaft with disc-dominated modes.

It is difficult to identify the number of nodal diameters in a mistuned bladed disc. Generally, mistuned blades-disc-shaft modes are affected by the shaft's flexibility. A frequency split occurs in mistuned systems, which is not observed for tuned bladed disc-shaft assemblies. The mistuning leads to a coupling between the torsion and the bending modes.

Only the first series (24 frequencies) will be analysed here. The left axis of Figure 10 indicates the uncoupled natural frequencies of mistuned cantilever blades. The next axis shows the natural frequencies of mistuned bladed disc of Series 1. The coupled natural frequencies of two mistuned bladed discs on a shaft are given on the third axis. Next, the natural frequencies of a shaft with two discs are shown.

The first coupled natural frequencies of the mistuned system are 46.830Hz and 46.839Hz. This is the shaft with two disc-dominated modes, where blade vibration is very limited.

The mistuning has caused a splitting of the doubled bending frequencies of the tuned system (46.864Hz, see Figure 8). The splitting is very small because of the negligible coupling of the shaft with the bladed-disc modes.

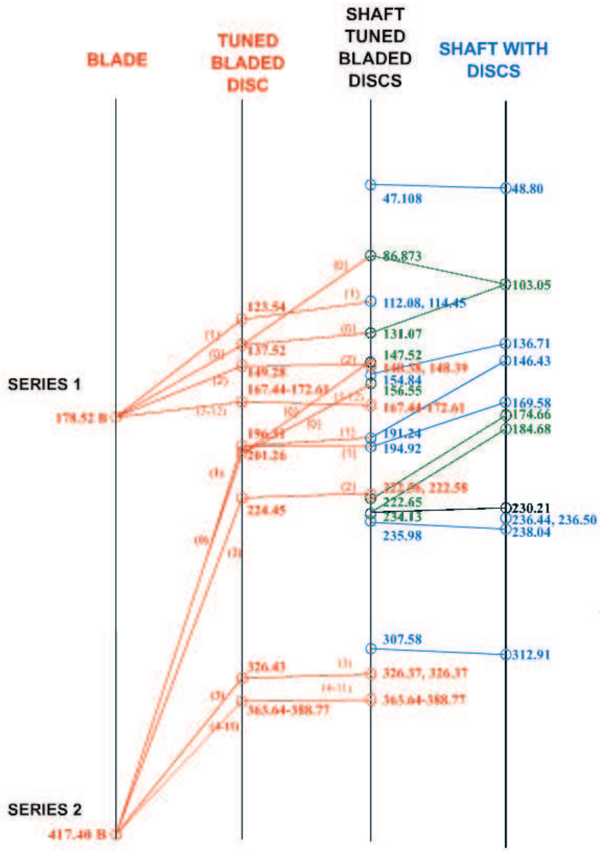

Figure 9. Natural frequencies of bladed discs on a shaft for blade length $L = 0.200\,\mathrm{m}$

The uncoupled mistuned bladed disc's natural frequency of the zero nodal diameter mode is 106.01 Hz (colour red) and the uncoupled torsion natural frequency of the shaft with two discs is 103.05 Hz (colour green). The coupled natural frequencies of the mistuned bladed discs on the shaft with zero nodal diameters split into values of 82.117 Hz and 103.51 Hz. In the 82.117 Hz mode, the torsion-bending motion of the bladed discs is observed and only the torsion of the shaft is noticeable. In the 103.51 Hz mode, the torsion-bending motion of the bladed discs and the shaft are observed (colours blue and green). The torsion-bending motion of the shaft in this mode is due to mistuning. In the case of a tuned system, the corresponding modes have frequencies of 78.74 Hz and 98.0 Hz (see Figure 6). In these modes, the shaft performed the torsion motion only.

The first uncoupled mistuned bladed discs' bending mode of "one nodal diameter" is 101.38 Hz (colour red) and the coupled natural frequencies of the shaft with two mistuned bladed discs are 98.023 Hz and 98.840 Hz (both in blue). In this case, two bladed discs vibrate and shaft motion is noticeable (colour blue). This is the difference between the mistuned and tuned system, where for the mode corresponding to one nodal diameter (*cf.* Figure 6) with a frequency of 93.10 Hz it was the first tuned bladed disc that predominantly vibrated and for 94.04 Hz the second bladed disc on the

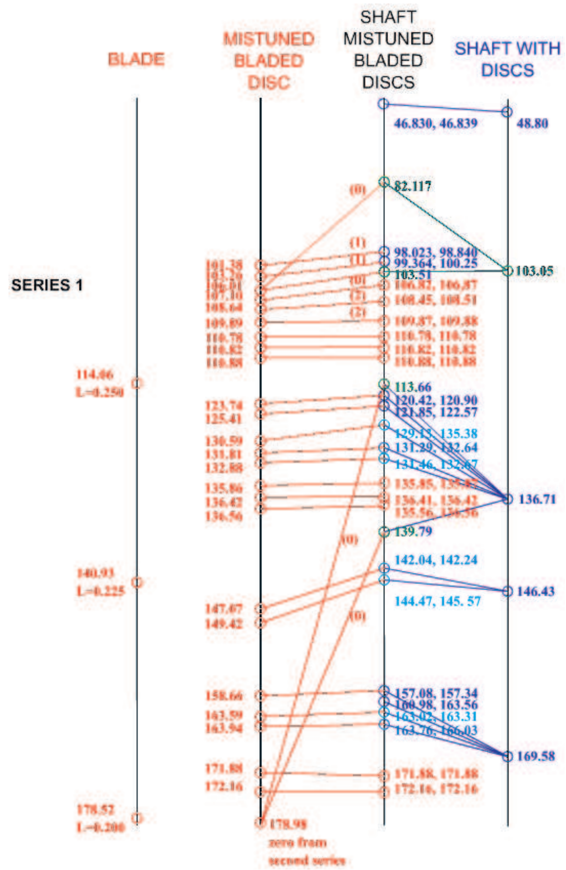

Figure 10. Natural frequencies of non-rotating mistuned bladed discs on a shaft

shaft was vibrated predominantly, so that the mistuning led to a stronger interaction between the bladed discs.

The second uncoupled mistuned bladed discs' bending mode of "one nodal diameter" is 103.26 Hz and the coupled natural frequencies of the shaft with two mistuned bladed discs are 99.364 Hz and 100.25 Hz (both in blue). In this case, two bladed discs vibrate and shaft motion is noticeable. Mistuning leads to the coupling of individual stages.

The first uncoupled mistuned bladed discs' bending mode of "two nodal diameters" is 107.10 Hz (Figure 10) and the coupled natural frequencies of the shaft with two mistuned bladed discs are 106.82 Hz and 106.87 Hz (both in red). In these modes, two mistuned bladed discs vibrate and shaft motion is not noticeable (colour red). This is the difference between the mistuned and tuned system, where for the corresponding frequency equal to 103.85 Hz (see Figure 6) only the first tuned bladed disc vibrated predominantly.

The uncoupled second mistuned bladed discs' bending mode of "two nodal diameters" is 108.64 Hz and the coupled natural frequencies of the shaft with two mistuned bladed discs are 108.45 Hz and 108.51 Hz (both in red). Here, two mistuned bladed discs vibrate and the shaft does not interact with the bladed discs' motion (colour red). In the tuned system for the corresponding mode of two nodal diameters with frequency of 103.85Hz (Figure 6) only the second tuned bladed disc vibrated predominantly.

There is a similar influence of uncoupled mistuned bladed discs' bending modes of 109.89Hz, 110.78Hz, 110.82Hz and 110.88Hz on the coupled natural frequencies of the shaft with two mistuned bladed discs of (109.87Hz and 109.88Hz), (110.78Hz, 110.78Hz), (110.82Hz, 110.82Hz) and (110.88Hz, 110.88Hz). In these cases, two mistuned bladed discs vibrate and the shaft motion is not noticeable (colour red). This is the difference between the mistuned and tuned system, where for the corresponding mode of two nodal diameters (see Figure 6) only the first or second tuned bladed disc vibrated predominantly.

These frequencies are connected with a tuned bladed disc-shaft system with a blade length of $L = 0.250$m (see Figure 6). The splitting of frequencies in the mistuned system is very small in comparison to the corresponding frequencies of the tuned bladed-disc-shaft system, so the influence of mistuning is negligible.

The uncoupled mistuned bladed disc's natural frequency of the zero nodal diameter mode from the second series is 178.98Hz (colour red); the coupled natural frequencies of the mistuned bladed discs on the shaft with zero nodal diameters split into values 113.66Hz and 139.79Hz (*cf.* Figure 11).

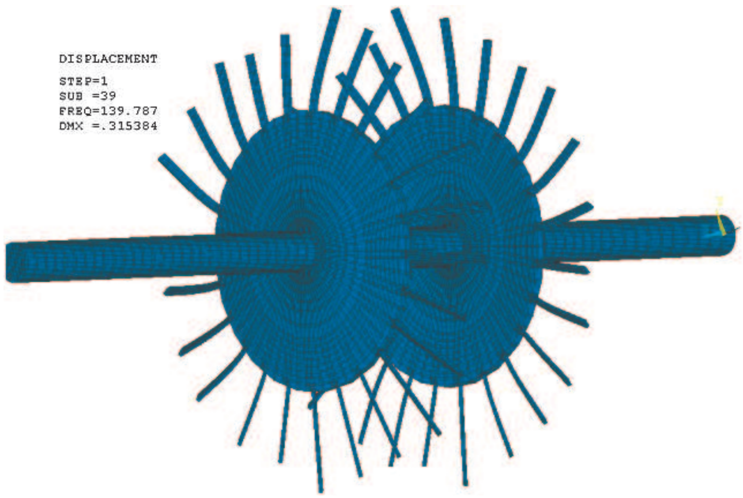

Figure 11. Mode shape at 139.79Hz

In the 113.66Hz and 139.79Hz modes, the torsion-bending motion of the blades-disc-shaft system is noticeable (colours blue and green). In these modes, the torsion-bending motion of the shaft is caused by mistuning. In the case of the tuned system for frequencies of 116.09Hz and 134.04Hz (see Figure 6), the shaft performed only the torsional motion.

The following coupled mistuned frequencies are connected with the tuned the bladed disc-shaft system with a blade length of $L = 0.225$m (*cf.* Figure 8).

The next uncoupled mistuned bladed discs' bending mode is 123.74Hz and the coupled natural frequencies of the shaft with two mistuned bladed discs are 120.42Hz and 120.90 Hz (both in blue). The shaft's motion is noticeable in these modes (colour blue). For the tuned system's modes corresponding to one nodal diameters (124.35 Hz, see Figure 8), only the first tuned bladed disc vibrated.

The second uncoupled mistuned bladed discs' bending mode is 125.41 Hz and the coupled natural frequencies of the shaft with two mistuned bladed discs are 121.85 Hz and 122.57 Hz (both in blue). In this case, two bladed discs vibrate and shaft motion is noticeable. There is a difference between the mistuned and tuned frequency equal to 124.35 Hz (see Figure 8), corresponding to one nodal diameters mode where the shaft is not vibrating.

The uncoupled mistuned bladed discs' bending mode is 130.59 Hz and the coupled natural frequencies of the shaft with two mistuned bladed discs are 129.13 Hz and 135.38 Hz (both in light blue). In this case two bladed discs vibrate and shaft motion is slightly noticeable (marked light blue). In the tuned mode of 133.67 Hz (*cf.* Figure 8), only the first tuned bladed disc vibrated predominantly and the shaft was reactionless. The mistuning causes an interaction between the bladed discs' vibration and the shaft. Similar behaviour can be observed for mistuned bladed discs-shaft frequencies of 131.29–132.67 Hz.

All modes of the mistuned system from 135.85 Hz to 136.56 Hz where shaft motion is absent (colour red) are influenced by the shaft with two tuned bladed discs modes of 133.67–136.79 Hz (*cf.* Figure 8).

The next group of mistuned bladed discs-shaft frequencies are connected with the tuned bladed disc of blade length $L = 0.2\,\mathrm{m}$ (see Figure 9). For example, the uncoupled mistuned bladed discs' bending mode has a frequency of 147.07 Hz and the coupled natural frequencies of the shaft with two mistuned bladed discs are 142.04 Hz and 142.24 Hz (both marked light blue). In this case, two mistuned bladed discs vibrate and shaft motion is slightly noticeable (marked light blue). In the tuned system, for the corresponding frequency of 148.38 Hz (see Figure 9, $L = 0.2\,\mathrm{m}$) of the two nodal diameter mode, only the first tuned bladed disc vibrated predominantly.

A general conclusion can be drawn that mistuning of bladed discs placed on a shaft causes coupling of bladed disc with the shaft's bending motion, a torsion-bending motion and splitting of frequencies to an extent greater than in tuned systems.

6. Mistuned structure at 3000 RPM

Similar calculations have been performed using the ANSYS code for rotating mistuned bladed discs on a shaft. A splitting of the nodal diameter modes of the shaft-disc assembly into forward and backward whirl modes, due to the gyroscopic effect, has been observed. The results for forward whirl modes are presented in Figure 12.

Generally, rotation does not alter the behaviour of mistuned bladed discs on a shaft in comparison with non-rotating mistuned systems.

7. Conclusions

The natural frequencies of cantilever blades, bladed discs, discs on a shaft and two tuned and mistuned bladed discs on a shaft have been calculated in this paper in order to discover their resonance conditions and possible coupling.

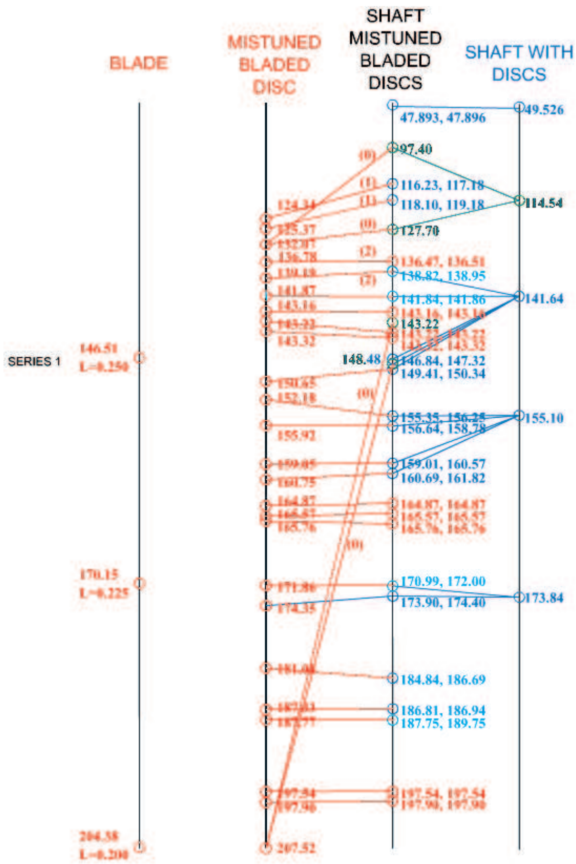

Figure 12. Natural frequencies of rotating mistuning bladed discs on a shaft for rotational speed 3000RPM

It has been found that zero nodal diameters of uncoupled bladed disc modes split for coupled modes of tuned bladed discs-shaft systems due to the closeness of the torsion modes of discs on a shaft with zero nodal diameters and the bladed disc modes with zero nodal diameters. The value of splitting of the one and two nodal diameter modes depend on the proximity of the modes of two discs on a shaft with one and two nodal diameters to the corresponding bladed disc modes.

The effect of mistuning has been shown on the free vibration of rotating and non-rotating flexible bladed discs on a flexible shaft.

Generally, mistuned blades-disc-shaft modes are affected by shaft flexibility. A frequency split occurs in mistuned systems, absent from tuned bladed disc-shaft assemblies. The mistuning causes a coupling between the torsion and bending modes and between individual stages. Shaft motion is noticeable in some bladed disc-dominated modes, which is not the case for tuned systems.

The results presented here depend on the mistuning pattern. The study relies on numerical prediction and its frequencies and mode shapes should in the future be verified experimentally.

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