AN EVOLUTIONARY ALGORITHM DETERMINING A DEFUZZYFICATION FUNCTIONAL

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Abstract: Order fuzzy numbers are defined that make it possible to deal with fuzzy inputs quantitatively, exactly in the same way as with real numbers, together with four algebraic operations. An approximation formula is given for a defuzzyfication functional that plays the main role when dealing with fuzzy controllers and fuzzy inference systems. A dedicated evolutionary algorithm is presented in order to determine the form of a functional when a training set is given. The form of a genotype composed of three types of chromosomes and the fitness function are given and Genetic operators are proposed.

Keywords: ordered fuzzy numbers, defuzzyfication, genetic algorithm

1. Introduction

Fuzzy numbers [1–3] are very special fuzzy sets defined on the universe of all real numbers, $\mathbb{R}$. They are of great importance in fuzzy systems. Triangular and trapezoidal (or triangularly and trapezoidally shaped) fuzzy numbers are usually used in applications or the so-called (L,R) numbers with two shape functions, L and R, proposed by Dubois and Prade [4] in 1978 as a restricted class of membership functions.

As long as one works with fuzzy numbers that possess continuous membership functions, the extension principle and the α-cut and interval arithmetic method produce the same results (*cf.* [5]). However, approximations of fuzzy functions and operations are required if one wants to follow the extension principle and remain with (L,R) numbers. This is a source of certain disadvantages, as well as unexpected and uncontrollable results of repeatedly applied operations [6, 7].

In most cases, the membership function of a fuzzy number, A, is assumed to satisfy convexity assumptions. Nguyen [8], when introducing his convex fuzzy numbers, required from all α cuts to be convex subsets of reals $\mathbb{R}$; he required the same from the support of A. Let us recall these notions: if μ_A is the membership function of A, then the α cut of A is a (classical) set, $A[\alpha] = \{x \in \mathbb{R} : \mu_A(x) \geq \alpha\}$, for each $\alpha \in [0,1]$, and the support of A is the (classical) set $\operatorname{supp} A = \{x \in \mathbb{R} : \ \mu_A(x) > 0\}$. It is additionally assumed [5, 6, 8–12] that the convex fuzzy number A has its core, *i.e.* the (classical) set of those $x \in \mathbb{R}$ for which its membership function $\mu_A(x) = 1$, which is not empty and its support is bounded.

However, even when subject to these restrictions, results of multiple operations on convex fuzzy numbers increase fuzziness significantly and depend on the order of operations since the distributive law, which involves the interaction of addition and multiplication, does not hold there.

This and other disadvantages have forced us to consider a generalization. Our main observation made in [13] was that a kind of quasi-invertibility of membership functions was crucial and arithmetic operations on their inverse parts needed to be defined for them to be in agreement with operations on crisp real numbers.

Following a series of papers [13–20], a generalization of the classical concept of fuzzy numbers has been made to define ordered fuzzy numbers and their algebra.

The main aim of the present paper is to deal with defuzzyfication operators (functionals) on ordered fuzzy numbers (see Section 3). A general approximation formula for such operators is presented in Section 4. An evolutionary algorithm enabling its determination is presented.

2. Ordered fuzzy numbers

If μ_A is a membership function of a convex fuzzy number, A, two functions, a_1, a_2 on $[0,1]$, can be defined that give the lower and upper bounds of each α cut of the membership function, μ_A, of number A:

$$A[\alpha] := \{x \in \mathbb{R} : \mu_A(x) \geq \alpha\} = [a_1(\alpha), a_2(\alpha)], \tag{1}$$

where boundary points are given for each $\alpha \in [0,1]$ by:

$$a_1(\alpha) = \mu_A|_{incr}^{-1}(\alpha) \ \text{ and } \ a_2(\alpha) = \mu_A|_{decr}^{-1}(\alpha)\ . \tag{2}$$

In Equation (2), the $\mu_A|_{incr}^{-1}$ symbol denotes the inverse function of the increasing part of the membership function, $\mu_A|_{incr}$, while the other symbol refers to the decreasing part, $\mu_A|_{decr}$, of μ. Then, we can see that the membership function μ_A of A is completely defined by two functions, $a_1 : [0,1] \to \mathbb{R}$ and $a_2 : [0,1] \to \mathbb{R}$. All arithmetic operations on the set of convex fuzzy numbers can be defined in their terms.

However, when the classical denotation for independent and dependent variables of membership functions, x and y, is used, we put $y = \alpha$ and use x to denote values of functions a_1 and a_2.

New, so-called ordered fuzzy numbers are defined in this representation, which can be identified with pairs of continuous functions of y of the $[0,1]$ interval (*cf.* Equation (2)) with values x in $\mathbb{R}$.

Definition 1. By an ordered fuzzy number A we mean an ordered pair (f,g) of functions such that $f,g:[0,1]\rightarrow\mathbb{R}$ are continuous.

Please note that our definition does not require that two continuous functions f and g are inverse functions of a membership function. Moreover, membership function corresponding to A may not exist.

We call the corresponding elements: f – the up-part and g – the down-part of fuzzy number A. The continuity of both of these parts implies that their images are bounded intervals, say UP and $DOWN$, respectively (Figure 1a). We have used symbols to mark boundaries for $UP=[l_A,1_A^-]$ and $DOWN=[1_A^+,p_A]$.

In general, f, g functions need not to be invertible as functions of y; only their continuity is required. However, if we assume that: 1) they are monotonous, *i.e.* f increases and g decreases, and 2) $f\leq g$ (point-wise), we may define the membership function as: $\mu(x)=f^{-1}(x)$, if $x\in[f(0),f(1)]=[l_A,1_A^-]$, and $\mu(x)=g^{-1}(x)$, if $x\in[g(1),g(0)]=[1_A^+,p_A]$ and $\mu(x)=1$ when $x\in[1_A^-,1_A^+]$.

Please note that, in general, $f(1)$ is not necessarily less than $g(1)$. Thus, we can reach improper intervals, already discussed in the framework of the extended interval arithmetic by Kaucher in [21].

Notably, a large class of ordered fuzzy numbers represents the whole class of convex fuzzy numbers with continuous membership functions.

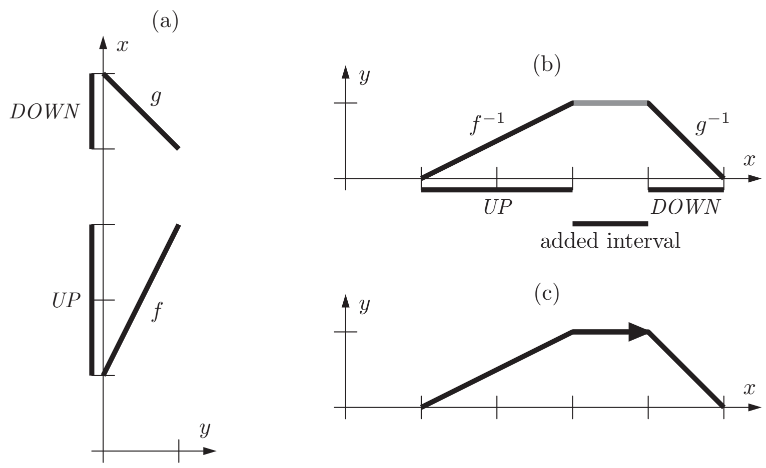

Figure 1. (a) Ordered fuzzy number; (b) ordered fuzzy number with membership function; (c) arrow denotes the order and orientation of inverted functions

In Figure 1c, a membership function of a convex fuzzy number corresponds to an ordered pair of two continuous functions f and g, (here, just two affine functions), with an additional arrow denoting the orientation of the closed curve formed below. The arrow marks an ordered pair of functions and we have appointed an additional feature, orientation, to the convex fuzzy number in Figure 1 and all ordered fuzzy numbers.

Please note that, if some of the conditions 1) or 2) for f and g formulated above are not satisfied, it is impossible to construct the classical membership function.

However, Prokopowicz [22] has introduced a *'corresponding'* membership function which can be defined by the following formula:

$$\mu(x)=\begin{cases}\max\arg\{f(y)=x,g(y)=x\} \text{ if } x\in\text{Range}(f)\cup\text{Range}(g),\\ 1 \text{ if } x\in[f(1),g(1)]\cup[g(1),f(1)] \text{ and}\\ 0 \text{ otherwise,}\end{cases} \tag{3}$$

where $[f(1),g(1)]$ or $[g(1),f(1)]$ may be empty, depending on the sign of $f(1)-g(1)$ (if it is less than zero, the other is empty).

The original definition of ordered fuzzy numbers [16–18] has been recently generalized [23] by allowing the (f,g) pair to be functions of bounded variation [24]. Thus, the case of convex fuzzy numbers with piecewise constant membership functions can be also described by the present approach, since functions of bounded variations may possess jumps of discontinuity at countable number of points, if at all. These discontinuity points are *of the first order*, *i.e.* one-sided limits of the functions exist at each such point, but they may be different. Then, each jump of discontinuity in the y variable corresponds to a constancy subinterval in the x variable (*cf.* [23] for more details).

In what follows we will apply our first definition of an ordered fuzzy number as a pair of continuous functions.

3. Operations and structure of OFN

Most naturally, the operation of addition of two pairs of such functions has been defined (*cf.* our main definition from [20]) as the pair-wise addition of their elements, *i.e.* if (f_1,g_1) and (f_2,g_2) are two ordered fuzzy numbers, then (f_1+f_2,g_1+g_2) will simply be their sum. This is exactly the same as the operation defined on convex fuzzy numbers in terms of α cuts of A and B.

DEFINITION 2. Let $A=(f_A,g_A),B=(f_B,g_B)$ and $C=(f_C,g_C)$ be mathematical objects called ordered fuzzy numbers. The sum $C=A+B$, subtraction $C=A-B$, product $C=A\cdot B$, and division $C=A\div B$ are defined by the formula:

$$f_C(y)=f_A(y)\star f_B(y) \quad \wedge \quad g_C(y)=g_A(y)\star g_B(y), \tag{4}$$

where '$\star$' stands for '$+$', '$-$', '$\cdot$' or '$\div$', and $A\div B$ is defined if functions $|f_B|$ and $|g_B|$ are greater than zero.

Please note that, as long as we are adding ordered fuzzy numbers which have their classical counterparts in the form of membership functions and the same orientation, the results of addition are in agreement with the α cut and the interval arithmetic. However, this does not generally hold if the numbers have opposite orientations, for the result of addition may lead to improper intervals for some α cuts. We are thus close to the Kaucher arithmetic [21] with improper or directed intervals, as he preferred to call them, *i.e.* such $[n,m]$ where n may be greater than m.

However, thanks to this definition, we will have $A-A=0$ for any order fuzzy number A, where 0 is crisp zero.

Let $\mathscr{R}$ be a universe of all OFN's. This set is composed of all pairs of continuous functions defined on closed interval $I=[0,1]$ and can be identified with the linear space of real 2D vector-valued functions defined on I with the norm of $\mathscr{R}$ as follows:

$$\|A\|=\max(\sup_{s\in I}|f_A(s)|,\sup_{s\in I}|g_A(s)|) \text{ if } A=(f_A,g_A). \tag{5}$$

Hence, $\mathscr{R}$ is a Banach space. The neutral element of addition in $\mathscr{R}$ is a pair of constant functions equal to crisp zero. It is also a Banach algebra with unity: the multiplication has a neutral element – the pair of two constant functions equal to crisp one.

4. Defuzzyfication of ordered fuzzy numbers

Defuzzyfication is the main operation of fuzzy controllers and fuzzy inference systems [1, 2, 25] where fuzzy inference rules appear. If the consequent parts of fuzzy rules are fuzzy, a defuzzyfication process is required, in the course of which real numbers are attached to membership functions. A number of defuzzyfication procedures for convex fuzzy numbers can be found in the literature (*cf.* [2, 9]), and some of these defuzzyfication procedures are indeed applicable to ordered fuzzy numbers when the ordered fuzzy number is a *proper* one, *i.e.* when its membership relation is a function. However, when the number is non-proper, *i.e.* the relation is by no means of the functional type, the situation is quite different.

A general representation of the linear and continuous functional on $\mathscr{R}$ can be obtained in the Banach space $\mathscr{R}$ equal to $C([0,1])\times C([0,1])$ with the Banach-Kakutami-Riesz representation theorem [26], which states that any linear and continuous functional $\bar{\phi}$ on a Banach space, $C(S)$, of continuous functions defined on a compact topological space, S, is uniquely determined by a Radon (*i.e.* signed Borel) measure ν on S such that:

$$\bar{\phi}(f)=\int_S f(s)\nu(ds) \text{ where } f\in C(S). \tag{6}$$

A Radon measure is a regular signed Borel measure, or a difference of two positive Borel measures. A Borel measure is a measure defined on a σ-additive family of subsets of S which contains all open subsets.

In the case when space S is a $[0,1]$ interval, each Radon measure is represented by a Stieltjes integral [24, 26] with respect to a function of a bounded variation, *i.e.* for any continuous, linear functional $\bar{\phi}$ on $C([0,1])$ there is a function of bounded variation, h, such that:

$$\bar{\phi}(f)=\int_0^1 f(s)dh(s) \text{ where } f\in C([0,1]). \tag{7}$$

Consequently, in the space of ordered fuzzy numbers $\mathscr{R}$ each bounded linear functional is given by a sum of two bounded, linear functionals defined on the $C([0,1])$ space, *i.e.*:

$$\phi(x_{up},x_{down})=\int_0^1 x_{up}(s)dh_1(s)+\int_0^1 x_{down}(s)dh_2(s), \tag{8}$$

where the pair of continuous functions $(x_{up},x_{down})\in\mathscr{R}$ represents an ordered fuzzy number and $h_1(s),h_2(s)$ are two functions of the bounded variation defined on $[0,1]$.

Remark 1. Due to the general representation (Equation (8)) and the functional representation (Equation (7)) we can identify each linear and bounded functional on space $\mathscr{R}$ with a pair of functions (h_1, h_2) of the bounded variation.

An infinite number of defuzzyfication procedures can be defined from the above formula. The standard defuzzyfication procedure in terms of the area under membership relation can be realized by linear combinations of two Lebesgue measures of $[0,1]$. However, in the present case, the area is calculated in the y-variable, since the ordered fuzzy number is represented by a pair of continuous functions in the y variable. Moreover, a 'delta' (or atom) measure can be related to each point $s \in [0,1]$, such measure representing a linear and bounded functional which realizes the corresponding defuzzyfication procedure. Discussion of other linear functionals as well as their non-linear generalization can be found in [27].

5. Approximation of defuzzyfication functional

In a recent paper [27], we have stated and proved a uniform approximation theorem concerning defuzzyfication functionals. Let us recall this formulation here.

Let $\mathscr{A} \subset \mathscr{R}$ be a compact subset[1] of the space of all ordered fuzzy numbers $\mathscr{R}$. Let $\mathscr{G}$ denote the set of all multi-variant continuous functions defined on the appropriate Cartesian product of the set of real numbers. In other words, $F \in \mathscr{G}$ if there is a natural k such that $F : \mathbb{R}^k \to \mathbb{R}$, continuous in the natural norm of $\mathbb{R}^k$. $\mathscr{D}$ will be the set of all linear and continuous functionals defined on $\mathscr{A} \subset \mathscr{R}$. Here, we could identify the set $\mathscr{D}$ with the adjoint space $\mathscr{R}^*$, since each continuous (bounded) and linear functional on the whole space $\mathscr{R}$ is also a continuous, linear functional on each subspace, including subset $\mathscr{A}$. Moreover, each continuous, linear functional on a subspace $\mathscr{A} \subset \mathscr{R}$ can be extended to the whole space $\mathscr{R}$ thanks to the Hahn–Banach theorem [26].

If a function F of k variables is from $\mathscr{G}$ and $\varphi_1, \varphi_2, \ldots, \varphi_k \in \mathscr{D}$, then their superposition $F \circ (\varphi_1, \varphi_2, \ldots, \varphi_k)$ is a function from $\mathscr{D}$ into $\mathbb{R}$, *i.e.* the functional

$$F \circ (\varphi_1, \varphi_2, ..., \varphi_k) : \mathscr{D} \to \mathbb{R}, \text{ with } F \in \mathscr{G}, \ \varphi_1, \varphi_2, \ldots, \varphi_k \in \mathscr{D} \tag{9}$$

is a defuzzyfication functional, generally nonlinear.

Theorem 1. Let $\mathscr{A} \subset \mathscr{R}$ be a compact subset of the space of all ordered fuzzy numbers $\mathscr{R}$. Then the set $\mathscr{H}$ composed of all possible compositions (superpositions) of type (Equation (9)), where F is from $\mathscr{G}$ and $\varphi_1, \varphi_2, \ldots, \varphi_k$ are from $\mathscr{D}$, with arbitrary k, is dense in the space of all continuous functionals from $\mathscr{R}$ into reals $\mathbb{R}$.

Let us now see how this theorem can help us in the following determination problem.

Problem A. Let values of an unknown continuous defuzzyfication functional, $r_1, r_2, \ldots, r_N$, be attached to a given finite family of ordered fuzzy numbers, composed of N numbers: $A_1, A_2, \ldots, A_N$. Find the form of this functional.

1. It follows from the theorem of Ascoli-Arzelà [26] that a subset of $C([0,1])$ is compact if its elements are equi-continuous and equi-bounded.

The problem has no general solution. However, we can look for its 'weak' solution in the approximate sense. In other words, we can reformulate it according to Theorem 1.

PROBLEM B. Let a finite set of training data be given in the form of N pairs of an ordered fuzzy number and a value (of action) of a defuzzyfication functional thereon, *i.e.*:

$$\mathsf{TRE} = \{(A_1, r_1), (A_2, r_2), \ldots, (A_N, r_N)\} \ . \tag{10}$$

For a given small ϵ, find a continuous functional $\Psi : \mathscr{R} \to \mathbb{R}$ which approximates the values of the TRE set with an error less than ϵ. In other words, find Ψ defined on $\mathscr{R}$ such that:

$$\max_{1 \le p \le N} |\Psi(A_p) - r_p| \le \epsilon, \ \text{where} \ (A_p, r_p) \in \mathsf{TRE} \ . \tag{11}$$

Problem B may have several solutions, but we can look for one of them with the help of Theorem 1 and the classical result of the approximation theory known as the Weierstrass theorem. The latter theorem states that each continuous function (of many variables) defined on a compact set can be approximated with a given accuracy by a polynomial (of many variables) of an appropriate, *i.e.* sufficiently high, order.

In what follows, we have assumed that the training set is not trivial, i.e. that it possesses at least two pairs different in both positions. We are going to propose a method of finding a solution to Problem B in the form of a superposition of a polynomial (of many variables) with a number of linear functionals from $\mathscr{D}$ with the use of a dedicated evolutionary algorithm.

The role of the polynomial is to approximate a nonlinear function F from the $\mathscr{G}$ set. Unfortunately, Theorem 1 does not determine the number of independent variables of function F. However, the cardinality of the TRE family gives the natural upper bound on the number of independent variables of the unknown function F and, consequently, of the polynomial. Since the cardinality of the family is N, the upper bound on the polynomial's order is $N-1$ if it is a polynomial of a single variable, since N values of r_p, $p = 1, 2, \ldots, N$, can be used to determine N coefficients, $c_0, c_1, \ldots, c_{N-1}$, standing in front of the corresponding powers of the variable. However, if it is a polynomial of k variables, its order should be less, $n < N$, since then the maximal number M of its coefficients is given by the following combinatoric formula:

$$M = \binom{n+k}{k}, \tag{12}$$

and it should not exceed N. The formula follows from the fact that the number of different solutions in the form of nonnegative integers of inequalities:

$$x_1 + x_2 + \cdots + x_k \le n \tag{13}$$

is equal to M given by Equation (12) (see [28], p. 31). These integers are possible exponents accompanying k independent variables of the polynomial of order n. Notably, two solutions that differ by the order must be regarded as different. Since $M = (n+k)!\{k!n!\}^{-1}$, the strong upper bound for the order of the polynomial will be $n + 1 \le \{k!N\}^{1/k}$.

6. An evolutionary algorithm to solve Problem B

With so many unknowns, we can suggest the following hierarchical optimization procedure to determine the form of the function to be used to approximate the required functional Ψ with a given accuracy, ϵ, on the TRE set:

1. find two natural numbers, k and n, both smaller than N, and
2. find M real-valued coefficients[2], $c_0, c_1, \ldots, c_{M-1}$, with M given by Equation (12), of a polynomial $W(z_1, z_2, \ldots, z_k)$ of k variables of order n, and
3. find $M-1$ aggregates of nonnegative integer exponents $\{m_{1j}, m_{2j}, \ldots, m_{kj}\}$, satisfying the inequality $m_{1j} + m_{2j} + \cdots + m_{kj} \le n$, with $j = 1, 2, \ldots, M-1$, each of them giving the power of the corresponding k variables, $z_1, z_2, \ldots, z_k$, and hence appearing in the following representation of polynomial W of order n:

$$W(z_1, z_2, \ldots, z_k) = c_0 + \sum_{j=1}^{M-1} c_j z_1^{m_{1j}} z_2^{m_{2j}} \ldots z_k^{m_{kj}}, \tag{14}$$

4. find k continuous and linear functionals $\varphi_1, \varphi_2, \ldots, \varphi_k \in \mathscr{D}$ on $\mathscr{R}$,
5. always subject to the following condition (*cf.* Equation (11)):

$$\max_{1 \le p \le N} |W \circ \Phi(A_p) - r_p| \le \epsilon, \text{ where } (A_p, r_p) \in \mathsf{TRE}, \tag{15}$$

where $W \circ \Phi$ denotes the composition of polynomial W with k functionals, $\varphi_1, \varphi_2, \ldots, \varphi_k$, *i.e.* functional φ_i substitutes variable z_i in representation (14), with $i = 1, 2, \ldots, k$.

In order to solve Problem B, we have designed the evolutionary algorithm presented below. Considering items 1–5 above, we can see that finding three types of objects subject to constraint 5, *viz.* those of 2, 3 and 4, may be sufficient if we additionally require additional constraints on numbers M, k and n. Thus, the genotype encoding searching solution (shown in Figure 2) is composed of three types of chromosomes:

1. a g chromosome, consisting of N real numbers,
2. an m chromosome, composed of $N-1$ nonnegative integers, and
3. a ϕ chromosome, being a pair of real value functions of the bounded variation defined on $I = [0,1]$.

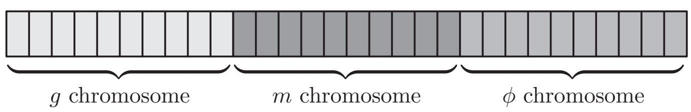

Figure 2. The schema of the genotype

More precisely, the g chromosome represents coefficients $c_0, c_1, \ldots, c_{M-1}$ (where M is given by formula (12)) of the polynomial $W(z_1, z_2, ..., z_k)$ from (14) of k variables of order n. This chromosome is schematically shown in Figure 3. The first gene refers

2. As a matter of course, some of them may equal zero.

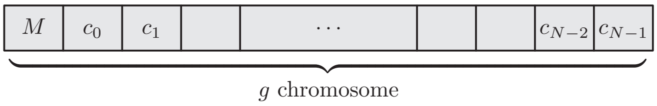

Figure 3. Schema of the g chromosome

to the value of M which is less or equal to N and causes that only M genes are active in creating the polynomial when the chromosome is decoded.

The next chromosome is presented in Figure 4. It is composed of genes encoding nonnegative integer exponents, $\{m_{1j}, m_{2j}, \ldots, m_{kj}\}$, satisfying the $m_{1j} + m_{2j} + \cdots + m_{kj} \leq n$ inequality, with $j = 1, 2, \ldots, M-1$, each of them giving the power of the corresponding k variables, $z_1, z_2, \ldots, z_k$.

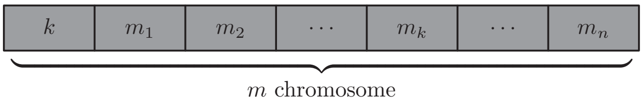

Figure 4. Schema of the m chromosome

As in the g chromosome, the length of the m chromosome is equal to N, but – after its decoding – the value of the first gene informs that only k exponents are active. It also assigns the number of genes active in the ϕ chromosome, which is shown in the Figure 5.

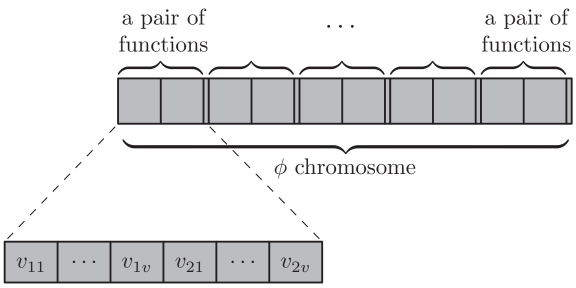

Figure 5. Schema of the ϕ chromosome

Here, the ϕ chromosome after decoding gives k continuous and linear functionals. Functional φ_i is substituted for variable z_i in representation (14).

This chromosome has a more complex structure, a consequence of the assumed representation of linear and bounded functionals (8). Considering the general properties of a function of bounded variation (*cf.* [23, 24]), according to which it may possess a countable number of jumps, we have assumed that there is a maximal number of such jumps situated at points of the function's discontinuity in the $I = [0,1]$ interval equal to v. Hence, for the pair of such functions (h_1, h_2) those points are given by s_{q1}, $s_{q2}, \ldots, s_{qv}$, $q = 1, 2$. They may differ for different functions.

Two values should be given at each discontinuity point s_{qi}, $1 \leq i \leq v$: a left-hand limit value, $h_q(s_{qi}^-)$, and a right-hand limit value, $h_q(s_{qi}^+)$, of the corresponding function h_q. Assuming, for the sake of simplicity, that between each pair of the consequent discontinuity points, *i.e.* between s_{qi} and $s_{q(i+1)}$, each function h_q is affine, *i.e.* of the form $a_i s + b_i$, with $i \leq v-1$ and two reals a_i and b_i, where s denotes the independent variable varying in [0,1], one can uniquely determine the pair of constants a_i and b_i from those limit values. Obviously, in the general case the constants are different for different i. Consequently, a pair of functions is represented in a typical ϕ chromosome by the $\{v_{11}, v_{12}, \ldots, v_{1v}, v_{21}, v_{22}, \ldots, v_{2v}\}$ collection, where the typical entry v_{qi}, $q = 1, 2$, $1 \leq i \leq v$, forms three real numbers $(s_{qi}, h_q(s^-_{qi}), h_q(s^+_{qi}))$ in Figure 5.

The two main operations performed on genotypes are mutation and crossover. Uniform crossover is chosen where information is exchanged between corresponding chromosomes, which means that the cutting points are only allowed between chromosomes. Mutation is defined separately, depending on the type of gene it refers to. For genes being natural numbers it is realized as a substitution of the current value with a new one, assigned randomly. For genes containing real number x, its new value is defined by a randomly chosen small real value Δ_x, positive or negative:

$$x_{new} = x_{cur} + \Delta_x. \tag{16}$$

In order to complete our description of genetic operations, it is necessary to introduce the repairing operation. When new values of genes (received after mutation) are outside the domain, they must be repaired. New values are defined by random choice, but it must be verified once again whether the relevant conditions are satisfied. In order to apply an evolutionary algorithm, the individuals in each generation have to be evaluated; in the proposed approach they are evaluated on the level of the phenotype decoding the genotype. Then, the value of the fitness function is calculated, forming a basis for the last genetic operation, *viz.* selection, *e.g.* proportional selection.

The fitness function is a natural consequence of condition (11) which should be satisfied in order to obtain a solution. It can be formulated as follows:

$$eval(W_{g,m}, \Phi) = \{1 + \sum_{p=1}^{N} |W_{g,m} \circ \Phi(A_p) - r_p|\}^{-1}. \tag{17}$$

After defining all necessary components of the proposed evolutionary algorithm, its performance can be described in steps typical for a classical genetic algorithm. First, the necessary parameters, such as crossover probability, p_{cross}, and mutation probability, p_{mu}, as well the number *card* of individuals in the population should be assigned by the user. Then, the algorithm proceeds as follows:

1. create randomly *card* individuals in the population;
2. evaluate them using the fitness function given by Equation (17);
3. while the stopping criteria is not satisfied (test the condition defined by Equation (11)).

   Begin

       select individuals to the new generation;

       perform genetic operations;

       calculate fitness function (17);

   End

   Stop

7. Conclusions

The obtained solution of Problem B should be checked with a different set of data, called TEST. The error on that set should not excessively exceed the primitive value of ϵ, $d\epsilon$, factor d of less than 2 being acceptable. The results of implementation of the presented algorithm will be reported in a separate paper.

References

[1] Czogała E and Pedrycz W 1985 *Elements and Methods of Fuzzy Set Theory*, PWN, Warsaw, Poland (in Polish)

[2] Piegat A 1999 *Fuzzy Modelling and Control*, Akademicka Oficyna Wydawnicza PLJ, Warsaw (in Polish)

[3] Zadeh L A 1965 *Information and Control* **8** (3) 338

[4] Dubois D H and Prade H 1978 *Int. J. Sys. Sci.* **9** (6) 613

[5] Buckley J J and Eslami E 2005 *An Introduction to Fuzzy Logic and Fuzzy Sets*, Physica-Verlag, A Springer-Verlag Company, Heidelberg

[6] Wagenknecht M 2001 *Fuzzy Sets and their Applications* (Chojcan J and Łęski J, Eds), Wydawnictwo Politechniki Śląskiej, Gliwice, pp. 291–310 (in Polish)

[7] Wagenknecht M, Hampel R and Schneider V 2001 *Fuzzy Sets and Systems* **123** (1) 49

[8] Nguyen H T 1978 *J. Math. Anal. Appl.* **64** 369

[9] Chen Guanrong and Pham Trung Tat 2001 *Fuzzy Sets, Fuzzy Logic, and Fuzzy Control Systems*, CRS Press, Boca Raton, London, New York, Washington, D. C.

[10] Drewniak J 2001 *Fuzzy Sets and their Applications* (Chojcan J and Łęski J, Eds), WPS, Gliwice, Poland, pp. 103–129 (in Polish)

[11] Klir G J 1997 *Fuzzy Sets and Systems* **91** (2) 165

[12] Kaufman A and Gupta M M 1991 *Introduction to Fuzzy Arithmetic*, Van Nostrand Reinhold, New York

[13] Kosiński W, Piechór K, Prokopowicz P and Tyburek K 2001 *Methods of Artificial Intelligence in Mechanics and Mechanical Engineering*, Gliwice, Poland (Burczyński T and Cholewa W, Eds), PACM, pp. 95–98

[14] Kosiński W 2004 *Proc. 7th Int. Conf. Artificial Intelligence and Soft Computing – ICAISC 2004*, Zakopane, Poland (Rutkowski L, Siekmann J, Tadeusiewicz R and Zadeh L A, Eds), LNAI, Springer-Verlag, Berlin, Heidelberg, **3070**, pp. 326–331

[15] Kosiński W and Prokopowicz P 2004 *Matematyka Stosowana. Matematyka dla Społeczeństwa* **5** (46) 37 (in Polish)

[16] Kosiński W, Prokopowicz P and Ślęzak D 2002 *Proc. Intelligent Information Systems*, Sopot, Poland (Kłopotek M, Wierzchoń S T and Michalewicz M, Eds), Physica Verlag, Heidelberg, pp. 311–320

[17] Kosiński W, Prokopowicz P and Ślęzak D 2002 *Proc. Methods of Aritificial Intelligence*, Gliwice, Poland (Burczyński T, Cholewa W and Moczulski W, Eds), PACM, pp. 231–237

[18] Kosiński W, Prokopowicz P and Ślęzak D 2003 *Proc. 3rd Int. Conf. Intelligent Information Processing and Web Mining*, Zakopane, Poland (Kłopotek M, Wierzchoń S T and Trojanowski K, Eds), Physica Verlag, Heidelberg, pp. 353–362

[19] Koleśnik R, Prokopowicz P and Kosiński W 2004 *Proc. 7th Int. Conf. Artificial Intelligence and Soft Computing – ICAISC 2004*, Zakopane, Poland (Rutkowski L, Siekmann J, Tadeusiewicz R, Zadeh L A, Eds), LNAI, Springer-Verlag, Berlin, Heidelberg, **3070**, pp. 320–325

[20] Kosiński W, Prokopowicz P and Ślęzak D 2003 *Bulletin of the Polish Academy of Sciences, Sér. Sci. Math.* **51** (3) 327

[21] Kaucher E 1980 *Computing, Suppl* **2** 33

[22] Prokopowicz P 2005 *Algorithmization of Operations on Fuzzy Numbers and its Applications*, PhD Thesis, IPPT PAN (in Polish)

[23] Kosiński W 2006 *Int. J. Appl. Math. Comput. Sci.* **16** (1) 51

[24] Łojasiewicz S 1973 *Introduction to the Theory of Real Functions*, Biblioteka Matematyczna, PWN, Warsaw, **46** (in Polish)

[25] Kosiński W and Weigl M 1998 *Sys. Anal. Model. Simul.* **30** (1) 11

[26] Alexiewicz A 1969 *Functional Analysis*, Monografie Matematyczne, PWN, Warsaw, **49** (in Polish)
[27] Kosiński W 2006 *Defuzzyfication Functionals of Ordered Fuzzy Numbers* (submitted for publication)
[28] Mostowski A and Stark M 1965 *Elements of Higher Algebra*, PWN, Warsaw (in Polish)