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DIRICHLET/DIRICHLET AND DIRICHLET/DIRICHLET-NEUMANN/NEUMANN NON-OVERLAPPING ITERATIVE DOMAIN DECOMPOSITION METHODS

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Abstract: A new iterative non-overlapping domain decomposition method is proposed for solving the one- and two-dimensional Helmholtz equation on parallel computers. The spectral collocation method is applied to solve the Helmholtz equation in each subdomain based on the Chebyshev approximation, while the patching conditions are imposed at the interfaces between subdomains through a correction, being a linear function of the space coordinates. Convergence analysis is performed for two applications of the proposed method (DDLC and DDNNLC algorithms – the meaning of these abbreviations is explained below) based on the works of Zanolli and Funaro *et al.*

Numerical tests have been performed and results obtained using the proposed method and other iterative algorithms have been compared. Parallel performance of the multi-domain algorithms has been analyzed by decomposing the two-dimensional domain into a number of subdomains in one spatial direction.

For the one–dimensional problem, convergence of the iteration process was quickly obtained using the proposed method, setting a small value of the σ constant in the Helmholtz equation. Another application of the proposed method may be an alternative to other iterative schemes when solving the two-dimensional Helmholtz equation.

Keywords: non-overlapping domain decomposition method, parallel computing, spectral methods, Helmholtz equation

1. Introduction

The concept of domain decomposition constitutes the basis of implementation of numerical methods on parallel computers. The computational geometry can be subdivided into a given number of subdomains, Ω_m, $m = 1,\ldots,N_{\mathrm{el}}$. The differential

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equation is solved independently in each subdomain, Ω_m, while suitable transmission conditions are specified at the subdomains' interfaces.

Multi-domain algorithms allows us to extend the applicability of spectral methods to problems of more complex geometry, where the original computational domain can be decomposed into smaller subdomains, easily transformable into rectangular geometries in computational space. The solution's accuracy can be also improved in the case of stiff or singular problems [1]. In the former, the local size of subdomains and the degree of the polynomial is adjustable to the local level of stiffness, while the latter are easily handled by shifting the singularity points to the subdomains' corners.

Multi-domain methods can be generally classified as overlapping or non-overlapping. The alternating Schwarz method is an example of an overlapping domain decomposition iterative scheme. It was applied in computations with parallel computers by Rodrigue and Simon [2], Rodrigue and Saylor [3], Ortega and Voigt [4], *et al.* The alternating Schwarz method was first applied with the spectral method by Morchoisne [5]. However, the overlapping approach has not been widely used in spectral approximations.

In 1980, Orszag [6] proposed the patching-collocation method of solving the second-order differential equation where the patching conditions (*i.e.* continuity of the solution and of the first-order derivative) were imposed at the interfaces between subdomains. The domain decomposition methods of this type can generally be classified as direct [7–9], iterative [6, 10–12] and spectral-element methods [13]. The present work is focused on analysis of non-overlapping iterative methods.

Early applications of the direct multi–domain method in spectral solutions of one–dimensional Helmholtz and Stokes problems included Pulicani [8] and Lacroix *et al.* [9], where the influence matrix method was used to impose boundary conditions and continuity of the solution and its first-order derivative at the interfaces. The application of the multi–domain technique was demonstrated to enable accurate resolution of the inner layers appearing in the analyzed problem. Another example of the direct method application can be found in Macaraeg and Streett [7], where the global flux condition was specified instead of continuity of the first-order derivative at the interface. Exponential convergence of the spectral multi–domain method was obtained for solutions of the Burgers equation and the two–dimensional Laplace equation with discontinuity at the physical boundaries of the computational domain.

The iterative domain decomposition algorithm for spectral approximations was proposed by Zanolli [10] and Funaro *et al.* [11]. This method is based on solving two problems in two stages of an iterative step. In stage one the first problem is solved in one subdomain, specifying the Dirichlet condition at the interface, while the other problem is solved in stage two on the other subdomain, with the Neumann condition applied at the interface. As shown analytically by Funaro *et al.* [11], the method converges in two iterations splitting the computational domain into two subdomains. Modifications of the Zanolli algorithm were proposed in the nineties. The algorithm of Louchart *et al.* [12] may be applied where the patching conditions are satisfied through sequential solution of the two problems. In the first step, the Dirichlet boundary condition is specified at the interface between the two subdomains where the Neumann condition is imposed in the second. An application of this method to the solution of

the tall differentially heated cavity problem and the flow in the inverted Bridgman configuration was studied in [12].

The spectral-element method, proposed by Patera [13], is based on variational formulation of the problem with the trial functions continuous across the elements where the continuity of flux is obtained during the convergence process. In [13], the spectral-element method was applied to the solution of the Helmholtz equation where the unknown variables were represented as Lagrangian interpolants using the Chebyshev collocation points and applied to solve the advection-diffusion problem and the expansion of flow in a channel. Later modifications can be found in Maday and Patera [14] and Karniadakis and Henderson [15], including an extension to the Legendre polynomials and application to the Navier-Stokes equations.

In the current paper, a new iterative domain decomposition method is proposed, wherein the patching conditions at the interfaces are imposed through a correction, being a linear function of the space coordinates. A convergence analysis of the proposed method is shown by splitting the computational domain into two subdomains [16].

2. One-dimensional Helmholtz equation

The solution of the one-dimensional Helmholtz equation:

$$\begin{aligned} -u_{xx}+\sigma u &= f \quad \text{in } \Omega=(-a,b),\\ u(-a)=g_- &\quad u(b)=g_+, \end{aligned} \tag{1}$$

can be shown in a split form by dividing the computational domain, Ω, into two non-overlapping subdomains, $\Omega_1=(-a,\Gamma)$ and $\Omega_2=(\Gamma,b)$; σ is considered a positive constant.

$$\begin{aligned} Lu_m^n &= f_m \quad \text{in } \Omega_m,\\ u_m^n &= g_{-/+} \quad \text{on } -a,b. \end{aligned}$$

In the above, L is the second-order differential operator,[1] while the following patching conditions are specified at the interface, Γ the continuity of the function and continuity of its first-order derivative [6, 18]:

$$u_1(\Gamma)=u_2(\Gamma), \qquad \frac{du_1}{dx}(\Gamma)=\frac{du_2}{dx}(\Gamma). \tag{2}$$

The proposed iterative procedure consists in solving the Helmholtz equation in each subdomain $\Omega_m, m=1,2$ for $n\geq 1$ by imposing the Dirichlet boundary conditions at the Γ interface:

$$\begin{aligned} Lu_m^n &= f_m \quad \text{in } \Omega_m,\\ u_m^n &= g_{-/+} \quad \text{on } -a,b,\\ u_m^n &= \xi_m^n \quad \text{on } \Gamma, \end{aligned} \tag{3}$$

where

$$\xi_m^{n+1}=u_m^n(\Gamma)+\theta\lambda_m^n(\Gamma) \quad \text{for } m=1,2, \tag{4}$$

1. It should be stressed that the L operator is not written here in its discrete form. As mentioned in [17], its discrete form can be formulated – at each iterative step n and for each subdomain m – as matrix A_m consisting of three elements: matrix A_{Ω_m}, representing the interior collocation points, element $A_{-a,b}$, representing the true boundary conditions, and element A_Γ, representing the conditions at the interface between subdomains.

ξ_1^1 and ξ_2^1 can be set to arbitrary values at the beginning of the iterative process, θ is the relaxation factor and λ^n is the correction function.

In order to satisfy the patching conditions (2), the solutions (3) are corrected at the Γ interface using the λ^n function, which is a linear function of the space coordinate $x : \lambda_m^n(x) = a_m^n x + b_m^n,\ m = 1,2$. Denoting by $(\cdot)^c$ the solution after correction and considering that $u_m^c(\Gamma) = u_m^n(\Gamma) + a_m^n\Gamma + b_m^n$ and $du_m^c/dx(\Gamma) = du_m^n/dx(\Gamma) + a_m^n$ for $m = 1,2$, the patching requirements (2) can be expressed as follows:

$$\begin{aligned} u_1^n(\Gamma) + a_1^n\Gamma + b_1^n &= u_2^n(\Gamma) + a_2^n\Gamma + b_2^n, \\ \frac{du_1^n}{dx}(\Gamma) + a_1^n &= \frac{du_2^n}{dx}(\Gamma) + a_2^n, \end{aligned} \tag{5}$$

where at the $-a$ and b boundaries of the Ω_1 and Ω_2 subdomains the correction is equal to zero and the following equations can be specified:

$$-a_1^n a + b_1^n = 0, \qquad a_2^n b + b_2^n = 0. \tag{6}$$

Using Equations (5) and (6), correction coefficients a_1^n, b_1^n, a_2^n and b_2^n can be obtained and ξ_1^{n+1} and ξ_2^{n+1} are computed using Equation (4). The iterative scheme (Equations (3)–(4)) is repeated until convergence is obtained. The algorithm shown above will be hereinafter referred to as the Dirichlet/Dirichlet Linear Correction method (DDLC for short).

Our next example concerns application of the proposed method in solving the one-dimensional Helmholtz equation imposing the Dirichlet boundary conditions at the Γ interface in the first step and the Neumann boundary conditions in the second step. Therefore, this algorithm will be hereinafter referred to as the Dirichlet/Dirichlet-Neumann/Neumann Linear Correction method, or DDNNLC.

The iterative procedure of solving the problem (1) is shown below. In the first step, the problem's solution (3) is obtained independently by assuming the initial values of ξ_1^1 and ξ_2^1 at the Γ interface. Next, the correction coefficients are computed using Equations (5) and (6), enabling evaluation of the new function derivatives, ζ_1^{n+1} and ζ_2^{n+1}, at the Γ interface:

$$\zeta_m^{n+1} = \frac{du_m^n}{dx}(\Gamma) + a_m^n \quad \text{for } m = 1,2. \tag{7}$$

In the next iterative step $(n+1)$, the solution of the following problem is considered in each subdomain, Ω_1 and Ω_2,

$$\begin{aligned} Lu_m^{n+1} &= f_m \quad \text{in } \Omega_m, \\ u_m^{n+1} &= g_{-/+} \quad \text{on } -a, b, \\ \frac{du_m^{n+1}}{dx} &= \zeta_m^{n+1} \quad \text{on } \Gamma, \end{aligned}$$

allowing us to evaluate the new functions' values at the Γ interface,

$$\xi_m^{n+2} = u_m^{n+1}(\Gamma) + \theta\lambda_m^{n+1}(\Gamma),$$

where $\lambda_m^{n+1}(\Gamma) = a_m^{n+1}\Gamma + b_m^{n+1}$ for $m = 1,2$ and the unknown correction coefficients, a_m^{n+1} and b_m^{n+1}, are evaluated using relations (5) and (6). The Helmholtz equations are

then solved in subdomains Ω_1 and Ω_2 assuming ξ_1^{n+2} and ξ_2^{n+2}. The iterative process is repeated until convergence is obtained.

2.1. Convergence analysis – solution of the one-dimensional Helmholtz equation

Defining the error function for $n \geq 1$:

$$e_m^n = u_m^n - u_m \quad \text{for } m = 1,2, \tag{8}$$

where u_m denotes the exact solution, and applying the scheme (3) to Equation (8), we obtain the following for $m = 1,2$:

$$\begin{aligned} Le_m^n &= 0 \quad \text{in } \Omega_m, \\ e_m^n &= 0 \quad \text{on } -a,b, \\ e_m^n &= \delta_m^n \quad \text{on } \Gamma, \end{aligned} \tag{9}$$

where $\delta_m^n = \xi_m^n - u_m(\Gamma)$, $m = 1,2$ can be expressed using (4) and (8) as follows:

$$\delta_m^{n+1} = e_m^n(\Gamma) + \theta\lambda_m^n(\Gamma) \quad \text{for } m = 1,2. \tag{10}$$

Assuming $e_m^n(\Gamma) = \delta_m^n = \delta^n$, $m = 1,2$ and $\Gamma = 0$, the corresponding solutions to (9) are as follows:

$$e_1^n(x) = \delta^n \frac{\sinh\left(\sqrt{\sigma}(x+a)\right)}{\sinh(\sqrt{\sigma}a)}, \tag{11}$$

$$e_2^n(x) = \delta^n \frac{\sinh\left(\sqrt{\sigma}(b-x)\right)}{\sinh(\sqrt{\sigma}b)}. \tag{12}$$

The correction coefficients can be computed by applying Equation (5) to (8) and using Equation (6). Then, using Equation (10), the following is obtained:

$$\delta^{n+1} = \delta^n \left[1 - \theta\frac{ab}{a+b}\sqrt{\sigma}\left(\coth(\sqrt{\sigma}b) + \coth(\sqrt{\sigma}a)\right)\right] \quad \text{for } n \geq 1, \tag{13}$$

which can be written as follows:

$$\delta^{n+1} = \delta^n c_\theta \quad \text{for } n \geq 1.$$

For $n \to \infty$, convergence of the iterative process can be obtained if $|c_\theta| < 1$ [11]. Having set $a = b = 1$ in Equation (13), $\lim_{\sigma\to 0}\left(\sqrt{\sigma}\coth(\sqrt{\sigma})\right) = 1$, showing that for small values of σ in the Helmholtz equation, very quick convergence of the iterative procedure can be obtained if $\theta \cong 1$. If $\sigma > 0$, quick convergence can be obtained by choosing $\theta \cong 1/\left(\sqrt{\sigma}\coth(\sqrt{\sigma})\right)$.

Convergence analysis will now be given for the other analyzed iterative scheme, where, in the first step of the iterative process, the Helmholtz equations are solved in each subdomain, Ω_1 and Ω_2, by imposing the Dirichlet boundary conditions at the Γ interface, while the Neumann boundary conditions are applied in the second step. The corresponding solutions obtained in the first iterative step are given by Equations (11) and (12). In order to satisfy the (2) patching conditions and set λ^n at the $-a$ and b boundaries at zero, the correction coefficients are computed using Equations (5) and (6).

In the subsequent step, the φ^{n+1} error can be computed by using Equation (7) and applying (8):

$$\varphi_m^{n+1} = \frac{de_m^n}{dx}(\Gamma) + a_m^n \quad \text{for } m = 1,2.$$

Assuming $e_m^n(\Gamma) = \delta_m^n = \delta^n$, $m = 1,2$ and $\Gamma = 0$, φ_1^{n+1} can be obtained as follows:

$$\varphi_1^{n+1} = \delta^n \frac{\sqrt{\sigma}}{a+b}\big(b\coth(\sqrt{\sigma}a) - a\coth(\sqrt{\sigma}b)\big). \tag{14}$$

Notably, $\varphi_1^{n+1} = 0$ for $a = b$ in Equation (14), a proof that convergence has been obtained after the second step of the iterative process.

3. Two-dimensional Helmholtz equation

The solution of the two-dimensional Helmholtz equation:

$$\begin{aligned} -\Delta u + \sigma u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega, \end{aligned} \tag{15}$$

where $\sigma \geq 0$ can be written in the split form dividing the $\Omega = (-a,b)\times(-c,d)$ computational domain into two non–overlapping subdomains in the x direction, $\Omega_1 = (-a,\Gamma)\times(-c,d)$ and $\Omega_2 = (\Gamma,b)\times(-c,d)$,

$$\begin{aligned} -\Delta u_m + \sigma u_m &= f_m \quad \text{in } \Omega_m, \\ u_m &= g \quad \text{on } \partial\Omega_m \backslash \Gamma, \end{aligned}$$

while the following patching conditions are specified at the Γ interface:

$$u_1(\Gamma,y) = u_2(\Gamma,y), \quad \frac{\partial u_1}{\partial x}(\Gamma,y) = \frac{\partial u_2}{\partial x}(\Gamma,y). \tag{16}$$

For $n \geq 1$, the DDLC method consists in solving the Helmholtz equation in each subdomain Ω_m for $m = 1,2$:

$$\begin{aligned} -\Delta u_m^n + \sigma u_m^n &= f_m \quad \text{in } \Omega_m, \\ u_m^n &= g \quad \text{on } \partial\Omega_m \backslash \Gamma, \\ u_m^n &= \xi_m^n \quad \text{on } \Gamma, \end{aligned} \tag{17}$$

where

$$\xi_m^{n+1} = u_m^n(\Gamma,y) + \theta\lambda_m^n(\Gamma,y) \quad \text{for } m = 1,2, \tag{18}$$

while ξ_1^1 and ξ_2^1 can be set to arbitrary values at the initial stage of the iterative process and $\lambda_m^n(x,y) = a_m^n(y)x + b_m^n(y)$ for $m = 1,2$.

By denoting by $(\cdot)^c$ the solution after correction $u_m^c(\Gamma,y) = u_m^n(\Gamma,y) + a_m^n(y)\Gamma + b_m^n(y)$ and $\partial u_m^c/\partial x(\Gamma,y) = \partial u_m^n/\partial x(\Gamma,y) + a_m^n(y)$ for $m = 1,2$ and including these relations into patching conditions (16), the following relations can be formulated:

$$\begin{aligned} u_1^n(\Gamma,y) + a_1^n(y)\Gamma + b_1^n(y) &= u_2^n(\Gamma,y) + a_2^n(y)\Gamma + b_2^n(y), \\ \frac{\partial u_1^n}{\partial x}(\Gamma,y) + a_1^n(y) &= \frac{\partial u_2^n}{\partial x}(\Gamma,y) + a_2^n(y), \end{aligned} \tag{19}$$

while at the $-a$ and b boundaries of the computational domain the $\lambda_1^n(-a,y)$ and $\lambda_2^n(b,y)$ correction it is required to be set to zero:

$$-a_1^n(y)a + b_1^n(y) = 0, \quad a_2^n(y)b + b_2^n(y) = 0. \tag{20}$$

Solving problems (19) and (20) enables evaluation of $a_m^n(y)$ and $b_m^n(y)$ for $m = 1,2$ and new values of ξ_m^{n+1}, $m = 1,2$ can be computed from Equation (18). The iterative procedure (17)–(18) is repeated until convergence is obtained.

Remark: By splitting computational domain Ω in the x direction into the N_{el} subdomains and applying the spectral collocation method to solve problem (17)–(18), the $M+1$ Gauss-Lobatto nodes will be defined in the y direction: $y_j = \cos(\pi j)/M$ for $j = 0,\ldots,M$. Conditions (19) and (20) will then be defined for each y_j, $j = 1,\ldots,M-1$, yielding the total number of correction coefficients a^n, b^n equal to $2\times(M-1)$ in each subdomain Ω_m, $m = 1, N_{\mathrm{el}}$.

When using the DDNNLC method, the solution of problem (17) is followed by determination of new values of function derivatives ζ_m^{n+1}, $m = 1,2$, at the Γ interface:

$$\zeta_m^{n+1} = \frac{\partial u_m^n}{\partial x}(\Gamma, y) + a_m^n(y) \quad \text{for } m = 1,2, \tag{21}$$

where functions $a_m^n(y)$, $m = 1,2$ can be obtained from the solution of problems (19) and (20).

In the next step $(n+1)$, the following problems are solved in subdomains Ω_m, $m = 1,2$:

$$\begin{aligned} -\Delta u_m^{n+1} + \sigma u_m^{n+1} &= f_m \quad \text{in } \Omega_m, \\ u_m^{n+1} &= g \quad \text{on } \partial\Omega_m \backslash \Gamma, \\ \frac{\partial u_m^{n+1}}{\partial x} &= \zeta_m^{n+1} \quad \text{on } \Gamma, \end{aligned} \tag{22}$$

evaluating the new values of the functions at the Γ interface,

$$\xi_m^{n+2} = u_m^{n+1}(\Gamma, y) + \theta\lambda_m^{n+1}(\Gamma, y) \quad \text{for } m = 1,2. \tag{23}$$

The iterative procedure consists in solving problems (17), evaluating Equation (21) and then solving problems (22)–(23). The procedure is repeated until convergence is obtained.

3.1. Convergence analysis: solution of the two-dimensional Helmholtz equation

The following error equations are obtained for $n \geq 1$ and $m = 1,2$ by putting error function (8) into (17):

$$\begin{aligned} -\Delta e_m^n + \sigma e_m^n &= 0 \quad \text{in } \Omega_m, \\ e_m^n &= 0 \quad \text{on } \partial\Omega_m \backslash \Gamma, \\ e_m^n &= \delta_m^n \quad \text{on } \Gamma, \end{aligned}$$

where

$$\delta_m^{n+1} = e_m^n(\Gamma, y) + \theta\lambda_m^n(\Gamma, y) \quad \text{for } m = 1,2. \tag{24}$$

For $n \geq 1$, functions e_1^n and e_2^n can be expressed as $e_1^n = \phi(x)\psi(y)$ and $e_2^n = \chi(x)\psi(y)$. Taking $c = d = 1$ in each subdomain Ω_1 and Ω_2, the solution of the following problem:

$$\begin{gathered} \psi''(y) + \tau\psi(y) = 0, \quad -1 < y < 1, \\ \psi(-1) = 0, \quad \psi(1) = 0, \end{gathered}$$

is expressed as a series of eigenfunctions ψ_k,

$$\psi_k(y) = \sin\left(\frac{k\pi}{2}\right)(y+1) \quad \text{for } k \geq 1, \tag{25}$$

where $\tau = \tau_k = \left((k\pi)/2\right)^2$ are eigenvalues of the considered problem.

For $\Gamma = 0$, the solution of problems $\{\phi_k(x), k \geq 1\}$ in subdomain Ω_1,

$$\phi_k''(x) - \sigma_k \phi_k(x) = 0, \quad -a < y < 0,$$
$$\phi_k(-a) = 0, \quad \phi_k(0) = 1,$$

is given as follows:

$$\phi_k(x) = \frac{\sinh\left(\sqrt{\sigma_k}(x+a)\right)}{\sinh\left(\sqrt{\sigma_k}a\right)} \quad \text{for } k \geq 1, \tag{26}$$

where $\sigma_k = \sigma + \tau_k$.

The solution of problems $\{\chi_k(x), k \geq 1\}$ in subdomain Ω_2,

$$\chi_k''(x) - \sigma_k \chi_k(x) = 0, \quad 0 < y < b,$$
$$\chi_k(0) = 1, \quad \chi_k(b) = 0,$$

is given by:

$$\chi_k(x) = \frac{\sinh\left(\sqrt{\sigma_k}(b-x)\right)}{\sinh(\sqrt{\sigma_k}b)}, \quad \text{for } k \geq 1. \tag{27}$$

Evaluation of functions $a_m^n(y), b_m^n(y)$, $m = 1,2$ from Equations (19) and (20) and putting them into Equation (24) yields:

$$\delta_1^{n+1} = e_1^n(0,y) + \theta \frac{ab}{a+b}\left[\frac{\partial e_2^n}{\partial x}(0,y) - \frac{\partial e_1^n}{\partial x}(0,y)\right],$$

for subdomain Ω_1.

Using the method of variable separation allows us to write solutions e_1^n and e_2^n in the following form [11]:

$$e_1^n(x,y) = \sum_{k=1}^{\infty} \beta_k^n \phi_k(x) \psi_k(y), \tag{28}$$

$$e_2^n(x,y) = \sum_{k=1}^{\infty} \gamma_k^n \chi_k(x) \psi_k(y), \tag{29}$$

where β_k^n and γ_k^n are respectively the expansion coefficients of δ_1^n and δ_2^n on Γ.

It should be noted that:

$$\int_{-1}^{1} e_\Gamma^n \psi_k(y) dy = \beta_k^n = \gamma_k^n, \tag{30}$$

which is due to the orthonormality of $\psi_k(y)$ and $\psi_k(0) = \chi_k(0)$ as well as the assumption of $e_1^n(0,y) = e_2^n(0,y) = e_\Gamma^n$.

The following expression is obtained by applying relations (28)–(29) and (30) and the orthonormality condition:

$$\beta_k^{n+1} = \beta_k^n \left[1 - \theta \frac{ab}{a+b}\sqrt{\sigma_k}\left(\coth(\sqrt{\sigma_k}b) + \coth(\sqrt{\sigma_k}a)\right)\right] = \beta_k^n c_k(\theta) \quad \text{for } k \geq 1. \tag{31}$$

As can be seen from Equation (31), the absolute value of the reduction factor, $c_k(\theta)$, of the k^{th} frequency error at the Γ interface can increase substantially as $k \to \infty$. The proposed method will perform poorly when imposing the Dirichlet boundary conditions at the Γ interface during the iterative process and divergence of the iterative procedure will be observed, specifying higher values of relaxation factor θ.

The convergence analysis performed for the proposed method of imposing the Dirichlet boundary conditions at the interface (DDLC method) has shown that the algorithm is unstable for solutions of the two-dimensional Helmholtz equation. Therefore, similar analysis will be performed of imposing the Dirichlet boundary conditions at the Γ interface at the first iterative step and the Neumann conditions at the second (DDNNLC method). As the first iterative step in the DDNNLC method is similar to the iterative step of the DDLC algorithm, solutions e_1^n and e_2^n are expressed by Equations (28) and (29), where $\psi(y), \phi(x)$ and $\chi(x)$ are respectively given by Equations (25)–(26) and (27).

Considering $\varphi_m^n = \zeta_m^n - \partial u_m/\partial x(\Gamma, y)$ and applying Equations (8) and (21), the following error function can be defined at the Γ interface for $n \geq 1$:

$$\varphi_m^{n+1} = \frac{\partial e_m^n}{\partial x}(\Gamma, y) + a_m^n(y) \quad \text{for } m = 1,2.$$

It should be noted that, when using Equation (30), the following relation is valid for $a = b$ at the $\Gamma = 0$ interface:

$$\frac{\partial e_1^n}{\partial x}(0,y) = -\frac{\partial e_2^n}{\partial x}(0,y). \tag{32}$$

The following relation can be obtained for subdomain Ω_1 by putting Equations (19) and (20) into Equation (8) and applying (32):

$$\varphi_1^{n+1} = \frac{\partial e_1^n}{\partial x}(0,y) - \frac{2b}{a+b}\frac{\partial e_1^n}{\partial x}(0,y). \tag{33}$$

Expressing the $\partial e_1^n/\partial x(0,y)$ term in Equation (33) as the following series:

$$\frac{\partial e_1^n}{\partial x}(0,y) = \sum_{k=1}^{\infty} \beta_k^n \phi_k'(0)\psi_k(y),$$

and by orthonormality of ψ_k^n, Equation (33) becomes:

$$\beta_k^{n+1} = \beta_k^n \frac{(a-b)}{ab}\sqrt{\sigma_k}\coth\left(\sqrt{\sigma_k}\right) = \beta_k^n c_k(\theta). \tag{34}$$

As can be seen from Equation (34) for $a = b$, the method converges in two iterations as the β_k^{n+1}, $k \to \infty$ error functions of the k^{th} frequency error at the Γ interface are equal to zero.

4. Numerical results

The application of the methods introduced above will be shown for a solution of the Helmholtz equation using the spectral collocation method based on the Chebyshev approximation [18]. The results will be compared to those of Zanolli [10], Louchart *et al.* (LR) [12] and the Neumann-Neumann (NN) algorithms [19].

4.1. One-dimensional problems

Numerical results will first be presented for a solution of the one-dimensional problem (1) assuming $a = b = 1$. The exact solution was given by $u(x) = \cos(\pi x/2)$. In each subdomain $\Omega_m, m = 1, N_{\text{el}}$, the solution of the local system of equations was obtained using the diagonalization method [20, 21] setting the number of Chebyshev collocation points at $N = 20$. The initial values of ξ at the interfaces between

subdomains were set to zero and the solution was assumed to converge when the maximum error measured at the interfaces between subdomains was less than $1 \cdot 10^{-8}$ in two subsequent iterations.

As has been mentioned above, correction $\lambda_m^n(x) = a_m^n x + b_m^n$ (*i.e.* correction coefficients a_m^n, b_m^n, $m = 1,\ldots,N_{\mathrm{el}}$) is computed after solution of the Helmholtz equation in order to satisfy patching requirements (Equation (2)) using the proposed method by allowing to set new Dirichlet or Neumann conditions for the next iteration step. In order to reduce the computational time required, we recommend application of the recurrence relation shown in Appendix. (The correction coefficients can be obtained on the master processor). Notably, this relation is directly applicable in the analysis of two-dimensional problems by decomposing the computational domain into N_{el} subdomains following a single spatial direction.

The number of iterations required to obtain a converged solution of the Helmholtz equation using the Zanolli [10], Louchart *et al.* [12], Neumann-Neumann [19] algorithms and the proposed methods (DDLC or DDNNLC) is shown in Table 1. The numbers in brackets denote the specified optimum values of the relaxation factor, θ.

Table 1. The number of iterations required to obtain a converged solution of the one-dimensional Helmholtz equation by applying various iterative algorithms decomposing the computational domain into two subdomains. The numbers in brackets denote optimum values of relaxation factor θ

	$\sigma = 0.1$	$\sigma = 1$	$\sigma = 10$	$\sigma = 100$
Zanolli/LR	2 (0.500)	2 (0.500)	2 (0.500)	2 (0.500)
NN	2 (0.250)	2 (0.250)	2 (0.250)	2 (0.250)
DDLC	4 (0.970)	4 (0.760)	3 (0.315)	2 (0.100)
DDNNLC	2	2	2	2

As has been demonstrated in [11], for two domain decomposition the Zanolli algorithm converges in two iterations, setting $\theta = 0.5$. Similar convergence of the iterative process can also be observed using the Louchart *et al.* [12] and Neumann-Neumann [19] methods, specifying $\theta = 0.5$ for the former and $\theta = 0.25$ for the latter. The smallest number of iterations was required to obtain a converged solution of the considered problem using the proposed DDLC method when $\theta \cong 1/\sqrt{\sigma}$ was set for $\sigma > 0$. For $\sigma = 0$, the algorithm converged in two iterations when setting $\theta = 1$. Using DDNNLC, convergence of the iterative process was obtained in the second iterative step. The results of numerical tests performed for the proposed methods (DDLC and DDNNLC) confirm those of the convergence analysis.

The number of iterations required to obtain a converged solution of the Helmholtz equation by splitting the computational domain into four subdomains is shown in Table 2. The number of iterations (and optimum values of the relaxation factor) required to obtain a converged solution of the one-dimensional problem using the Zanolli algorithm [10] and its modification proposed by Louchart *et al.* [12] are exactly the same. Notably, when parallelization of the numerical algorithm is taken into account, using the Louchart *et al.* algorithm enables avoidance of some of the synchronization problems with the Zanolli method, but the number of iterations

required to obtain a converged solution remains the same for both methods. It is observed that for small values of the σ constant, numerous iterations were necessary to obtain a converged solution of the considered problem using the Zanolli [10], Louchart *et al.* [12] and Neumann-Neumann [19] algorithms, while using the DDLC and DDNNLC algorithms produced convergence of the iterative procedure in $N_{\text{iter}} = 4$ and $N_{\text{iter}} = 10$, respectively. Setting higher values of the σ constant in the Helmholtz equation results in a substantial reduction in the number of iterations required to obtain a converged solution of the one-dimensional problem using the Zanolli, Louchart *et al.* and Neumann-Neumann methods; a similar number of iterations is required while applying the DDLC algorithm.

Table 2. The number of iterations required to obtain a converged solution of the one-dimensional Helmholtz equation by applying various iterative algorithms decomposing the computational domain into four subdomains. The numbers in brackets denote optimum values of relaxation factor θ

	$\sigma = 0.1$	$\sigma = 1$	$\sigma = 10$	$\sigma = 100$
Zanolli/LR	594 (0.023)	78 (0.170)	16 (0.450)	5 (0.500)
NN	158 (0.022)	27 (0.120)	9 (0.250)	3 (0.250)
DDLC	4 (0.965)	4 (0.725)	5 (0.235)	6 (0.058)
DDNNLC	10 (0.960)	10 (0.780)	10 (0.370)	10 (0.250)

For smaller values of the σ constant in the Helmholtz equation, convergence of the iterative process was obtained in much less iteration steps when the proposed methods are used instead of the Zanolli, Louchart *et al.* or Neumann-Neumann methods. For higher values of the constant, convergence was obtained in almost the same number of iteration steps using the DDLC method and the Zanolli, Louchart *et al.* and Neumann-Neumann algorithms.

4.2. Two-dimensional problems

We will now consider the solution of the two-dimensional Helmholtz Equation (15) by splitting the $\Omega = (-1,1) \times (-1,1)$ computational domain in the x direction into a number of subdomains. The exact solution of the problem (15) was given by $u(x,y) = \cos(\pi x/2)\sin(\pi y)$. The complete diagonalization technique [20, 21] was used to solve the local problems, setting the number of Chebyshev collocation points at $N = M = 20$ in the x and y directions. The initial values of ξ were set to zero and the iterative procedure was assumed to converge when the maximum value of error measured at the interfaces between subdomains was less than $1 \cdot 10^{-8}$ in two subsequent steps.

Table 3 shows the number of iterations required to obtain a converged solution of the two-dimensional Helmholtz equation by splitting the computational domain into two subdomains.

As has been demonstrated in [11], convergence of the iterative procedure for two-domain decomposition using the Zanolli method can be obtained in two iteration steps. The same result is obtained using the Louchart *et al.* [12] and Neumann-Neumann [19] algorithms. Notably, numerous iteration steps are necessary to obtain a converged solution of the two-dimensional Helmholtz equation using the DDLC

Table 3. The number of iterations required to obtain a converged solution of the two-dimensional Helmholtz equation by applying various iterative schemes decomposing the computational domain into two subdomains. The numbers in brackets denote optimum values of relaxation factor θ

	$\sigma = 0.1$	$\sigma = 1$	$\sigma = 10$	$\sigma = 100$
Zanolli/LR	2 (0.500)	2 (0.500)	2 (0.500)	2 (0.500)
NN	2 (0.250)	2 (0.250)	2 (0.250)	2 (0.250)
DDLC	339 (0.014)	325 (0.014)	244 (0.014)	98 (0.015)
DDNNLC	2	2	2	2

proposed method. This confirms the results of convergence analysis performed for the proposed iterative scheme and is due to the fact that the reduction factor of the k^{th} frequency error at the Γ interface can be amplified as much as $k \to \infty$ and divergence of the iterative procedure may occur. The $e_m^n(\Gamma, y)$, $m = 1,2$ error converges slowly to zero when the relaxation factor, θ, is set to very small values. For the other proposed iterative algorithm (DDNNLC method) convergence of the iterative process is obtained in two iteration steps.

The number of iterations required to obtain a converged solution of the two-dimensional Helmholtz equation using the analyzed iterative algorithms for splitting the computational domain into four subdomains in the x direction is shown in Table 4. As demonstrated in [11], setting higher values of the σ constant in the Helmholtz equation ($\sigma = 100$) allows to obtain a converged solution with the Zanolli method in less iteration steps than for $\sigma = 0.1$. The same applies for the Louchart *et al.* [12] and Neumann-Neumann [19] algorithms; only three iterations are necessary to get convergent solution with the latter and $\sigma = 100$. The number of iterations obtained using the DDLC method is again very large, which is a consequence of setting the relaxation factor to very small values ($\theta = 0.0075$). Applying the DDNNLC method enables obtaining a converged solution in $N_{\text{iter}} = 10$ for $\sigma \leq 10$ and in $N_{\text{iter}} = 12$ for $\sigma = 100$.

Table 4. The number of iterations required to obtain a converged solution of the two-dimensional Helmholtz equation by applying various iterative schemes decomposing the computational domain into four subdomains. The numbers in brackets denote optimum values of relaxation factor θ

	$\sigma = 0.1$	$\sigma = 1$	$\sigma = 10$	$\sigma = 100$
Zanolli/LR	16 (0.45)	11 (0.48)	10 (0.49)	5 (0.50)
NN	8 (0.23)	7 (0.24)	6 (0.25)	3 (0.25)
DDLC	466 (0.0075)	442 (0.0075)	304 0.0075)	109 (0.0075)
DDNNLC	10 (0.38)	10 (0.37)	10 (0.32)	12 (0.25)

The Zanolli and Louchart *et al.* algorithms and the proposed methods were subsequently applied to solve the Helmholtz equation for the L-shaped configuration (see Figure 1). The exact solution was given by $u(x,y) = \sin(\pi x)\sin(\pi y)$. The initial values of ξ at the interfaces between subdomains were set to unity and convergence of the iterative process persisted as long as the maximum difference in the function's values at the interfaces between subdomains was less than $1 \cdot 10^{-4}$ in two subsequent

steps. In the first stage of the iteration step using the Zanolli method, the Helmholtz equation was solved in the Ω_2 subdomain, while the problems in subdomains Ω_1 and Ω_3 were solved in the second stage. Using the proposed methods at each iteration step, the Helmholtz equations were solved independently in each subdomain and the correction was evaluated in order to satisfy the patching conditions at the interfaces. The number of iterations obtained has been collected in Table 5, showing the Zanolli method to be superior to the other algorithms as far as the L-shaped configuration is considered.

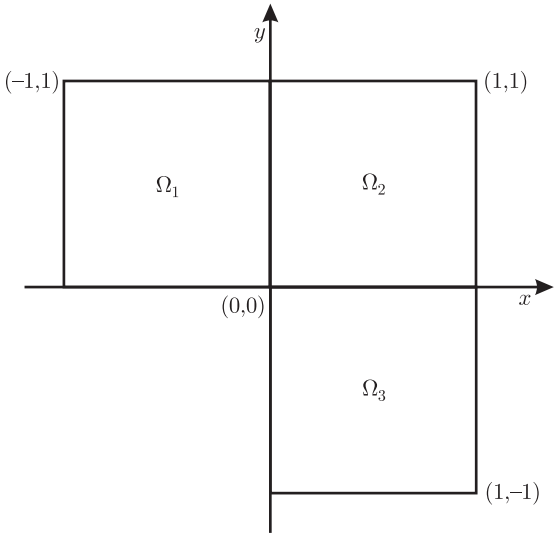

Figure 1. An L-shaped domain subdivided into three subdomains

Table 5. The number of iterations required to obtain a converged solution of the two-dimensional Helmholtz equation for an L-shaped domain. The numbers in brackets denote optimum values of relaxation factor θ

	$\sigma = 0.1$	$\sigma = 1$	$\sigma = 10$	$\sigma = 100$
Zanolli	9 (0.65)	9 (0.65)	9 (0.65)	8 (0.64)
LR	85 (0.10)	76 (0.11)	58 (0.14)	38 (0.20)
DDLC	446 (0.009)	422 (0.009)	295 (0.009)	109 (0.01)
DDNNLC	56 (0.39)	52 (0.40)	36 (0.45)	22 (0.55)

The results of numerical tests obtained using the proposed DDLC and DDNNLC algorithms confirm the results of convergence analysis. Attribution of the Dirichlet conditions at the interfaces between subdomains during the iteration process (DDLC algorithm) results in instability of the proposed iterative algorithm. Using the DDNNLC method enables obtaining convergence of the iteration process in two steps when splitting the computational domain into two subdomains. The number of iterations obtained using the DDNNLC method for decomposition into four subdomains following a single spatial direction is similar to those obtained using the other algorithms. When the L-shaped configuration is applied, the Zanolli method converges in less iteration steps than the Louchart *et al.* and the proposed algorithms.

4.3. Results of parallelization

Results of the multi-domain algorithms' parallelization will be analyzed by measuring the computational time of solving the two-dimensional Helmholtz equation and analyzing the speed-up obtained by running the iterative algorithms on parallel computers. The tests were performed on a PC cluster with a Linux system and a Fast Ethernet network. Communication between processors was established using MPI libraries.

A comparison of the computational time required to obtain a converged solution of the two-dimensional Helmholtz equation using the proposed DDNNLC algorithm and the other iterative schemes when splitting the computational domain into two subdomains is shown in Figure 2. The number of processors equaled the number of subdomains. Only two iterations were required to obtain a converged solution of the problem for all the considered methods (the number of iterations is shown in brackets).

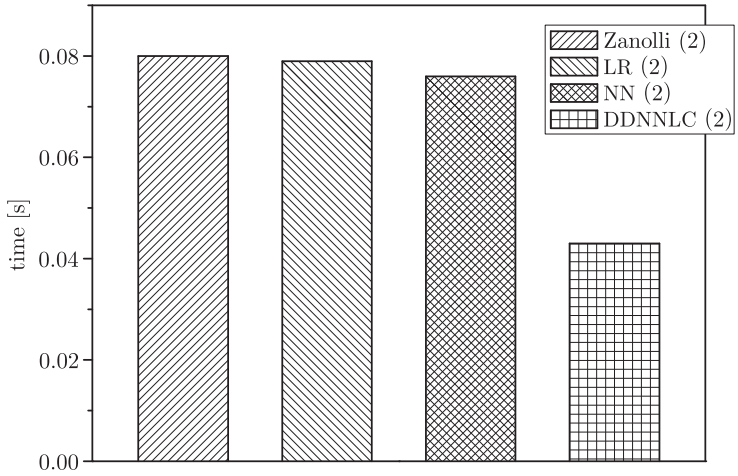

Figure 2. The execution time for solving the two-dimensional Helmholtz equation on parallel computers by splitting the computational domain into two subdomains. The values in brackets denote the number of iterations required to obtain a converged solution of the problem

The speed-up obtained by running the multi-domain algorithms on parallel computers is shown in Figure 4. It is defined as a ratio between computational times of running the parallel algorithm on a single and N_{p} processors ($S = t_{\mathrm{s}}/t_{\mathrm{p}}$). In order to measure the computational time on a single processor, the multi-domain algorithms were written in their sequential form, with no communication between processors. The number of subdomains was equal to the number of processors.

As the Zanolli [10] method consists of solving the Helmholtz equation alternatively with Dirichlet and Neumann conditions at the interfaces between subdomains, the computational time required for one iteration step is similar to the time required using the Louchart *et al.* [12] and Neumann-Neumann [19] algorithms. However, the Louchart *et al.* [12] and Neumann-Neumann [19] methods require solving two Helmholtz equations in each subdomain at each iteration step. As has been shown above, the iteration step of the proposed method consists of solving one Helmholtz

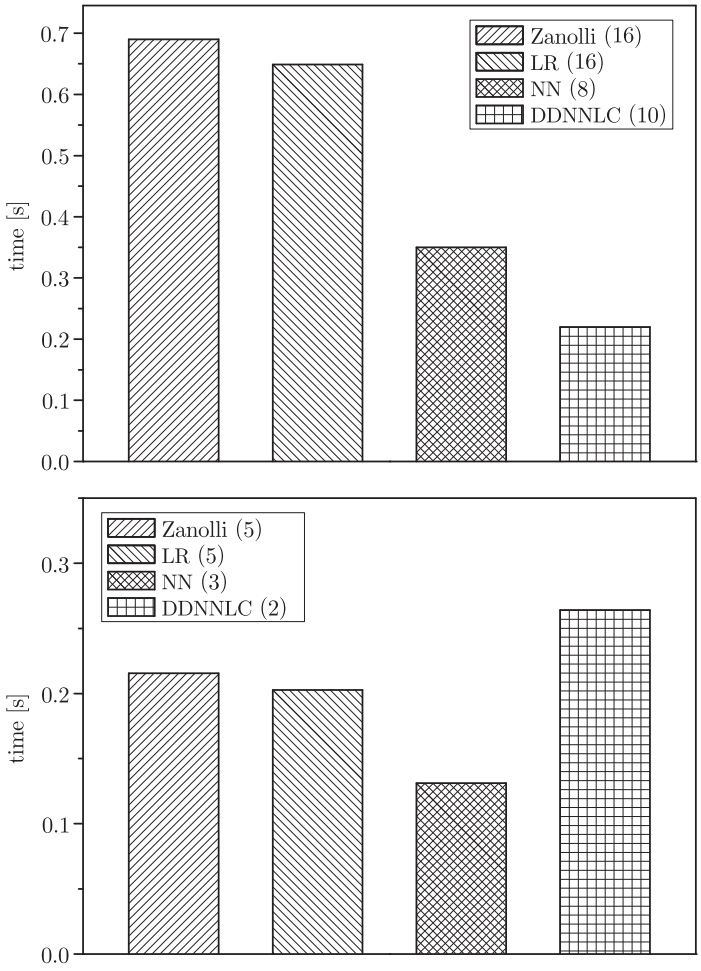

Figure 3. The execution time for solving the two-dimensional Helmholtz equation on parallel computers by splitting the computational domain into four subdomains: $\sigma = 0.1$ (top) and $\sigma = 100$ (bottom); the values in brackets denote the number of iterations required to obtain a converged solution of the problem

equation in each subdomain and the subsequent computation of correction in order to satisfy the patching conditions at the interfaces. For the sake of simplicity of implementation, the correction coefficients are computed on one processor and subsequently distributed to the other in order to modify the Dirichlet or Neumann data at the subdomain interfaces for the next iteration step. Thus, global (or collective) communication is applied in order to exchange data between processors using the proposed method while with the other algorithms data are transmitted from one processor to another.

As shown in Figure 2, the computational time required to obtain a converged solution of the Helmholtz equation using the Zanolli [10], Louchart *et al.* [12] or Neumann-Neumann [19] methods is about twice as the time required when using the proposed method. As the number of Helmholtz equations to be solved is smaller with

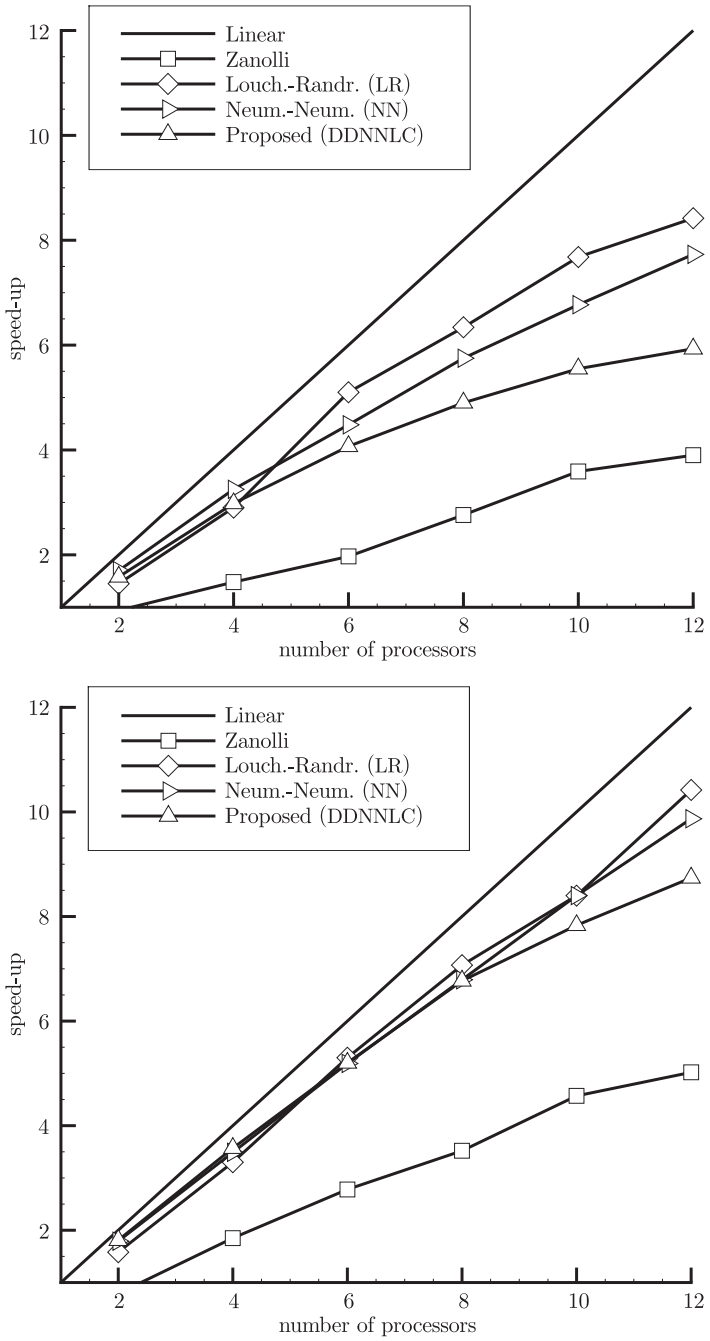

Figure 4. The computational speed-up obtained by running the multi-domain algorithms on parallel computers: $N = M = 40$ (top) and $N = M = 80$ (bottom)

the proposed method, advantages of using this method are expected for decomposition into more subdomains.

As a next stage, computational time was measured for solving the two-dimensional Helmholtz equation using various iterative algorithms and splitting the

computational domain into four subdomains following a single spatial direction and specifying various values of the σ constant in the Helmholtz equation. The results are shown in Figure 3, with $\sigma = 0.1$ (top) and $\sigma = 100$ (bottom). For $\sigma = 0.1$, the smallest computational time was obtained using the proposed method: only ten iteration steps were necessary to obtain convergence of the iterative process. The fastest convergence of the iterative process for $\sigma = 100$ was obtained with the Neumann-Neumann algorithm.

The speed-up obtained by running the Zanolli [10] algorithm on two processors was less than unity, which is a consequence of its sequential nature, while running the parallel algorithm on 12 processors produced $S \cong 3.5$ for $N = M = 40$ and $S \cong 5$ for $N = M = 80$. The speed-up obtained by running the Louchart *et al.* [12] and Neumann-Neumann [19] algorithms on 12 processors equaled $S = 7.5$–8 for $N = M = 40$, while it was close to $S = 10$ for $N = M = 80$. For $N = M = 40$, the performance of the proposed algorithm was poorer than that of the Louchart *et al.* and Neumann-Neumann methods. It can be explained by the advantage of using the interprocessor type of communication in the Louchart *et al.* and Neumann-Neumann methods over global communication applied in the proposed method (exchanging data between master and other processors). However, setting a higher number of modes in each subdomain, $N = M = 80$, enables obtaining almost the same performance of these algorithms.

5. Concluding remarks

A new iterative non-overlapping domain decomposition algorithm has been proposed in this paper for solving the one- and two-dimensional Helmholtz equation using the spectral method. The patching requirements of the proposed method were imposed through a correction, being a linear function of the space coordinates. The correction, λ^n, was evaluated at the interfaces between subdomains, allowing us to define new values of functions or their first order derivatives for the next iteration step after the solution of local problems in each subdomain.

In the Zanolli algorithm, the patching conditions were enforced by successive solution of the differential problems, imposing the Dirichlet and Neumann conditions at the subdomain interfaces in two stages of the iteration step. This resulted in some problems with the processes' synchronization when parallelization of the algorithm was considered. When using the other analyzed algorithms, the converged solution was obtained in two iteration steps for two-domain decomposition. However, each iteration of the Louchart *et al.* [12] and Neumann-Neumann [19] algorithms consisted of subsequent solution of the Helmholtz equation in each subdomain, imposing the Dirichlet conditions at their interface in the first stage of the iteration step and the Neumann conditions in the second.

Two applications of the proposed method were analyzed: one (DDLC) involved imposing the Dirichlet conditions at the interfaces during the iteration process, while the other (DDNNLC) – imposing the Dirichlet conditions in the first step and the Neumann conditions in the second.

The results of numerical tests performed for solving one- and two-dimensional problems by splitting the computational domain into two subdomains confirmed the results of the convergence analysis.

When solution of a one-dimensional problem involved splitting the computational domain into four subdomains, quick convergence of the iteration process was obtained with the proposed methods for a small value of σ in the Helmholtz equation. For higher values of σ, the number of iterations obtained using the DDLC method was similar to those of the Zanolli [10], Louchart *et al.* [12] and Neumann-Neumann [19] methods. The proposed algorithms are thus well-suited for solving one-dimensional problems.

As predicted theoretically, application of the DDLC method to solve the two-dimensional Helmholtz equation results in instability of the iterative scheme. Numerous iterations steps were necessary to obtain a converged solution using the DDLC method even when decomposition into two subdomains was considered.

The results of the numerical tests performed for solving the two-dimensional problems by subdividing domain Ω into four subdomains following the x direction have shown the computational time obtained using the DDNNLC to be less than those of the other iterative schemes with a small value of constant σ in the Helmholtz equation. For higher values of σ, more iteration steps are required in order to obtain a converged solution with the proposed method than when applying the other iterative algorithms.

If the L-shaped configuration is applied, the number of iterations obtained for the Zanolli method is substantially smaller than when using the Louchart *et al.* algorithm or the proposed methods.

The results obtained when using the proposed method for solving one-dimensional problems are encouraging. DDNNLC may also be regarded as a viable alternative to other iterative schemes for solving the two-dimensional Helmholtz equation as the computational time required to run the algorithm on parallel computers is in some cases less than than required by other methods.

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Appendix: The recurrence relation for evaluation of the correction coefficients

The recurrence relation allowing to evaluate correction coefficients a_m and b_m, $m = 1,\ldots,N_{\mathrm{el}}$ will be shown here, splitting the one-dimensional domain into N_{el} subdomains (Figure A1).

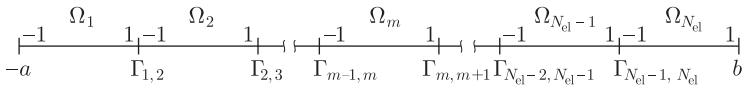

Figure A1. Decomposition of the domain Ω into N_{el} subdomains

By decomposing domain Ω into N_{el} subdomains, correction λ_m^n can be expressed for each subdomain $\Omega_m = (h_{m-1}, h_m)$ in the local coordinate system $\hat{x}_m = \hat{x} \in (-1,1)$:

$$\lambda_m^n = a_m \hat{x}_m + b_m, \tag{A1}$$

where $\hat{x}_m$ is related to the global coordinate system by the following relation:

$$\hat{x}_m = \frac{2}{l_m}(x - h_{m-1}) - 1, \tag{A2}$$

l_m being the length of the m-th subdomain $l_m = h_m - h_{m-1}$. It is further assumed later $l_m = l$.

The following expressions are defined for $m = 1, \ldots, N_{\mathrm{el}} - 1$:

$$\begin{aligned} C_m^n &= u_{m+1}^n(\Gamma_{m,m+1}) - u_m^n(\Gamma_{m,m+1}), \\ D_m^n &= \frac{du_{m+1}^n}{dx}(\Gamma_{m,m+1}) - \frac{du_m^n}{dx}(\Gamma_{m,m+1}), \end{aligned} \tag{A3}$$

where n is the consecutive number of an iteration step. Then, the following relations can be calculated using Equation (A3):

$$\begin{aligned} rhs_1 &= \sum_{m=1}^{N_{\mathrm{el}}-1} C_m^n \\ & \qquad \text{for } N_{\mathrm{el}} \geq 2, \\ rhs_2 &= \frac{1}{c_T} \sum_{m=1}^{N_{\mathrm{el}}-1} D_m^n \end{aligned} \tag{A4}$$

$$rhs_3 = \frac{1}{c_T} \sum_{m=1}^{N_{\mathrm{el}}-2} [(N_{\mathrm{el}} - m - 1) D_m^n] \quad \text{for } N_{\mathrm{el}} \geq 3, \tag{A5}$$

where $rhs_3 = 0$ for $N_{\mathrm{el}} = 2$ and c_T is the constant of transformation from the global to the local coordinate system, equal to $c_T = 2/l$.

Using Equations (A4) and (A5), coefficients a_1^n, b_1^n, $a_{N_{\mathrm{el}}}^n$ and $b_{N_{\mathrm{el}}}^n$ can be obtained as follows:

$$\begin{aligned} a_1^n &= (0.5 rhs_1 + rhs_2 + rhs_3)/N_{\mathrm{el}}, \\ b_1^n &= a_1^n, \\ a_{N_{\mathrm{el}}}^n &= a_1^n - rhs_2, \\ b_{N_{\mathrm{el}}}^n &= -a_{N_{\mathrm{el}}}^n. \end{aligned} \tag{A6}$$

Then, coefficients a_m^n and b_m^n are computed for $m = N_{\mathrm{el}} - 1, \ldots, 2$:

$$\begin{aligned} a_m^n &= D_m^n / c_T + a_{m+1}^n, \\ b_m^n &= C_m^n - a_m^n - a_{m+1}^n + b_{m+1}^n. \end{aligned} \tag{A7}$$

Evaluation of correction coefficients a_m^n, b_m^n, $m = 1, \ldots, N_{\mathrm{el}}$ and their distribution to the respective subdomains allows us to compute for $m = 2, \ldots, N_{\mathrm{el}}$ the new function's values or its derivatives at the left boundaries of subdomains Ω_m:

$$\begin{aligned} u^{n+1}(\Gamma_{m-1,m}) &= u^n(\Gamma_{m-1,m}) + \theta(-a_m + b_m), \\ \frac{du}{dx}^{n+1}(\Gamma_{m-1,m}) &= \frac{du}{dx}^{n}(\Gamma_{m-1,m}) + c_T a_m. \end{aligned} \tag{A8}$$

At the right boundaries of subdomains Ω_m, the new function's values or its derivatives can be computed for $m = 1, \ldots, N_{\mathrm{el}} - 1$ using the following relations:

$$\begin{aligned} u^{n+1}(\Gamma_{m,m+1}) &= u^n(\Gamma_{m,m+1}) + \theta(a_m + b_m), \\ \frac{du}{dx}^{n+1}(\Gamma_{m,m+1}) &= \frac{du}{dx}^{n}(\Gamma_{m,m+1}) + c_T\, a_m, \end{aligned} \tag{A9}$$

where θ is the relaxation factor.

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