ANALYTIC-NUMERICAL MODEL OF A CONVECTIVE BOUNDARY LAYER AND HEAT TRANSFER ON A HORIZONTAL CONE

SERGEY LEBLE AND WERONIKA PELC

Gdansk University of Technology,
Narutowicza 11/12, 80-233 Gdansk, Poland
leble@mif.pg.gda.pl

(Received 21 October 2008)

Abstract: An approximate analytical solution of a two dimensional problem for stationary Navier-Stokes, continuity and Fourier-Kirchhoff equations describing a free convective heat transfer from an isothermal cone is presented. The problem formulation is based on assumptions typical for natural convection: non-compressibility and the Boussinesq approximation. The solution is based on Frobenius expansions at the vicinities of two points: the initial point and the singular point of the boundary layer equation. Numerical matching of the expansions and Nusselt number evaluations are traced.

Keywords: natural convection, Fourier-Kirchhoff equations, boundary layer equation, isothermal cone, matching of Frobenius expansions

1. Introduction

The results of a theoretical and experimental study of free convective flows from heating objects are widely published and they are very useful for engineers and designers to determine convective heat losses from apparatus, devices, pipes in industrial or power installations, electronic equipment, architectonic objects, *etc.*

There are many publications on flat (vertical, horizontal and inclined) as well as cylindrical and spherical surfaces. It is obvious from an analysis of the literature data that only several papers can be found for heating surface conical configurations, for example [1–5] and [6]. It is only four out of about 120 results that are concerned with conical vertical surfaces in the review of Churchill's paper [7]. As we have found only one paper concerning the horizontal cone, written by Oosthuizen [6], we have decided to continue those investigations and extend this subject [8, 9]. Moreover, the only experimental studies have been performed in [6].

The presented paper is devoted to a general theoretical study of the problem of describing a boundary layer near an isothermal surface.

A horizontal position of the cone does not permit to apply the above mentioned approach because of a symmetry break by the gravity field. The proposed physical model of the flow is based on boundary layer approximations made in momentum and energy equations which permit only a convective tangential heat and momentum transfer. Two kinds of coordinates: a cylindrical coordinate system coupled with the cone surface and a special local system are used to apply such model for the investigated geometry related to the gravity field direction (horizontal cone) [9]. The local coordinate system selection is motivated by the physical model in which the coordinate curves are connected with stream lines. This statement is in good agreement with a picture of the pattern flow that we have observed directly [8]. The curve's tangent vector is directed along the total of the forces of surface interaction and gravitation.

An approximate analytical solution of a simplified convective flow induced by an isothermal conical body with a horizontal axis of symmetry is considered. The selected coordinate system within the frame of simplifications typical for laminar natural convection and for $\mathrm{Pr} \approx 1$ makes it possible to decrease the number of basic equations. Following the transformations, the basic set of two equations for the thermal boundary layer results in an ordinary differential equation of second order with a singularity [9]. The solution method used is the power series expansion near the point of singularity of the basic equation for the boundary layer thickness as a function of local coordinates.

The approach of [9] is continued and developed in this article. Our considerations are based on the parameterization of power series expansion at the singularity point by the boundary layer thickness at this (singularity) point and the expansion is matched to other expansion in the vicinity of the conventional starting point of the boundary layer. The principal feature of this expansion is that its complicated structure arises from the nonlinearity of the basic equation for the boundary layer thickness and geometry of the cone.

Two possible ways of behavior are observed when analyzing the Frobenius solution at this starting point, one of which prevails in some cone angle values range. The solutions are matched at the point and the parameters of expansion are found numerically. The correspondent Nusselt numbers are evaluated and tabulated basing on the selected parameters.

In Section 2, the cone geometry and physical model are presented and a derivation of the equation describing the boundary layer thickness is reviewed. In Section 3, the boundary layer thickness asymptotic expansion is found at the starting point $\varepsilon = -\varepsilon_m$ when analyzing a Frobenius-like solution in the vicinity of this point. In Section 4, this solution is matched with a solution in the vicinity of point $\varepsilon = 0$ and the Nusselt number is evaluated numerically.

The theoretical considerations are compared with the experiment results basing on the Nusselt number values.

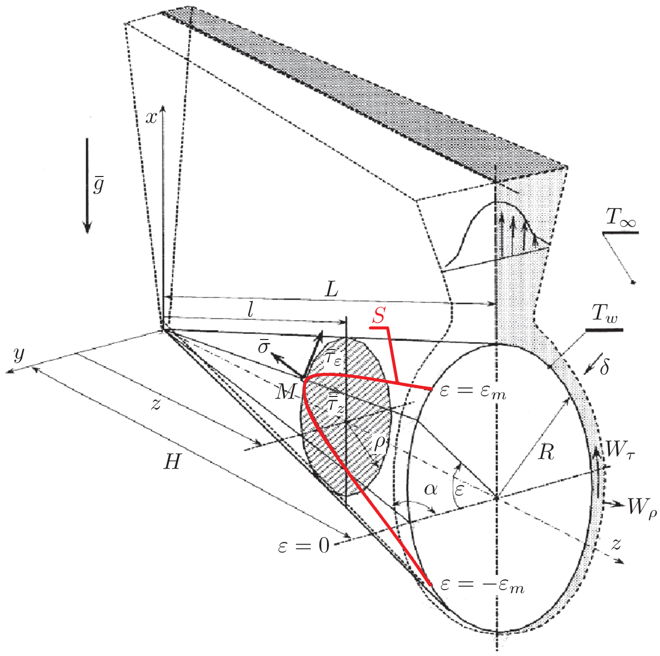

Figure 1. A coordinate system of an isothermal horizontal conical surface transferring heat by free convection [9]

2. A free convective boundary layer on an isothermal horizontal cone

2.1. Geometry

A cone with radius R and height H with a coordinate system is shown in Figure 1. The cone is horizontal and it is in a gravity field which is described by the gravity vector $\bar{g}$.

A free convective boundary layer on the horizontal conical surface has been investigated in this study. a cone horizontally immersed in a fluid at the temperature of T_w has been examined. The following parameters have been used to describe conical surface:

1. α – the angle between the base and lateral surface of the cone, it can obtain values from $\alpha = 0$ (the vertical round plate case) to $\alpha = \frac{\pi}{2}$ (horizontal cylinder case);
2. the red curve S shown in Figure 1 is a stream line with molecules of the fluid mean movement alongside. The angle $\varepsilon = -\varepsilon_m$ and $\varepsilon = \varepsilon_m$ is the beginning and the end of the boundary layer;
3. ρ_0 – expressed by the base and the angle ε_m as shown in Figure 2 (red curve).

The following vectors may be distinguished at an arbitrary point M on the cone's surface (see Figure 1):

- normal vector $\bar{\sigma}$ to the surface:

$$\bar{\sigma} = \bar{\imath}\sin\alpha\sin\varepsilon + \bar{\jmath}\sin\alpha\sin\varepsilon - \bar{k}\cos\alpha; \tag{1}$$

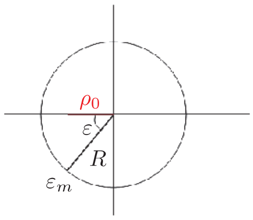

Figure 2. The cone base plot with the quantity ρ_0

- tangent $\bar{\tau}$ to the surface:

$$\bar{\tau} = \frac{\bar{S}}{S} = \frac{-\bar{\imath}(1-\sin^2\alpha\sin^2\varepsilon)+\bar{\jmath}sin^2\alpha\sin\varepsilon\cos\varepsilon-\bar{k}\cos\alpha\sin\alpha\sin\varepsilon}{\sqrt{1-\sin^2\alpha\sin^2\varepsilon}}. \tag{2}$$

$\bar{S}$ is defined as:

$$\bar{S} = \bar{g} - (\bar{g},\bar{\sigma})\bar{\sigma}, \tag{3}$$

where $\bar{g}$ is the gravitation acceleration vector.

2.2. The physical model

Natural convection which has been of interest in previous studies has made it possible to assume incompressibility and a laminar flow of the fluid. Based on former observations, inertia forces are considered to be negligibly small, hence, they can be ignored in our model. On the contrary, the viscosity forces play a significant role in our considerations, influencing future calculations. To create the Navier-Stokes equations, the authors have assumed that the thermal and hydraulic boundary layer thicknesses are equal. This allowed us to use the basic Navier-Stokes equations in two directions:

- tangent

$$\nu\frac{\partial^2 W\tau}{\partial\sigma^2} - g_\tau\beta(T-T_\infty) - \frac{1}{\rho_f}\frac{\partial p}{\partial\tau} = 0; \tag{4}$$

- normal

$$-g_\sigma\beta(T-T_\infty) - \frac{1}{\rho_f}\frac{\partial p}{\partial\sigma} = 0, \tag{5}$$

where:

$$g_\sigma = \bar{\sigma}\cdot\bar{g} \tag{6}$$

$$g_\tau = \bar{\tau}\cdot\bar{g}. \tag{7}$$

Assuming that the temperature distribution depends mainly on the distance from the cone's surface, the Fourier-Kirchhoff equation which confirms the former assumption has been used:

$$\Theta = \frac{T-T_\infty}{T_w-T_\infty} = \left(1-\frac{\sigma}{\delta}\right)^2. \tag{8}$$

The δ is the boundary layer thickness, T_∞ is the constant temperature of the fluid in infinity and T_w is the cone temperature. Both of them are shown in Figure 1.

In the article [9] the authors solved the Navier-Stokes and the Fourier-Kirchhoff equation deriving the equation which describes the dimensionless boundary layer thickness [10, 11]:

$$y^4(\varepsilon)E\frac{\partial^2 y(\varepsilon)}{\partial\varepsilon^2}+3y^3(\varepsilon)E\left(\frac{\partial y(\varepsilon)}{\partial\varepsilon}\right)^2+y^3(\varepsilon)\frac{\partial y(\varepsilon)}{\partial\varepsilon}G+y^5(\varepsilon)H+y^4(\varepsilon)F= \\ =r^2(1-\sin^2\alpha\sin^2\varepsilon)\cos^{-2\cos^2\alpha}\varepsilon. \tag{9}$$

The coefficients in Equation (9) are defined by:

$$E=\frac{5}{9}(\cos^{(2+\cos^2\alpha)}\varepsilon)\sin\alpha\sin\varepsilon, \tag{10}$$

$$G=3(\cos^{(1-\cos^2\alpha)}\varepsilon)r(\cos^2\varepsilon+\cos^2\alpha-\cos^2\varepsilon\cos^2\alpha)+ \\ +\frac{8}{9}(\cos^{(3+\cos^2\alpha)}\varepsilon\sin\alpha)y(\varepsilon), \tag{11}$$

$$H=\frac{2}{9}\frac{\sin\varepsilon\sin\alpha}{\sin^2\alpha\sin^2\varepsilon-1}\cos^{(2+\cos^2\alpha)}\varepsilon(\sin^2\varepsilon\cos^4\alpha+3\cos^2\alpha+\cos^2\varepsilon), \tag{12}$$

$$F=\frac{r\sin^2\alpha(1-\sin^2\alpha\sin^2\varepsilon)}{\cos^{(\cos^2\alpha)}\varepsilon}\sin\varepsilon. \tag{13}$$

In Equation (9) the authors use new variables: the dimensionless boundary layer thickness $y(\varepsilon)$ and the dimensionless radius r which are defined by:

$$y(\varepsilon)=\delta K^{1/3}, \tag{14}$$

$$r=\rho_0 K^{1/3}, \tag{15}$$

where:

$$K=\frac{\mathrm{Ra}}{240R^3}. \tag{16}$$

Equation (9) is the final nonlinear ordinary differential equation describing the boundary layer thickness.

3. Solution in the vicinity of point $\varepsilon=-\varepsilon_m$

The main aim of this paper has been to find the solution of an equation in the vicinity of point $\varepsilon=-\varepsilon_m$ which is also the beginning of the boundary layer. It is followed by matching two solutions; in the vicinity of point $\varepsilon=0$ and point $\varepsilon=-\varepsilon_m$, respectively. In [9], owing to the solution of Equation (9) it has been possible to obtain an asymptotic solution like the Taylor series in the vicinity of point $\varepsilon=0$:

$$y(\varepsilon)=y(0)+g\varepsilon+f\varepsilon^2/2+\dots \tag{17}$$

where, approximately (we left three terms in the series):

$$g=\frac{1}{3}\frac{r}{y^3(0)} \tag{18}$$

$$f=-\frac{1}{3}\frac{r^2}{y^7(0)} \tag{19}$$

$$y(0)=\sqrt[4]{\frac{1+\sqrt{7}}{12}r\left[\pi-2\arcsin\left(\frac{\rho}{R}\right)\right]} \tag{20}$$

Equation (17) is the dimensionless boundary layer thickness. It is known that:

$$\delta = y(\varepsilon)K^{-\frac{1}{3}}. \tag{21}$$

Thus, the boundary layer thickness at the point $\varepsilon = 0$ is:

$$\delta(\varepsilon) = \sqrt[4]{\frac{240\rho_0 R^3}{\mathrm{Ra}}}\left(\sqrt[4]{\frac{1+\sqrt{7}}{12}\left[\pi - 2\arcsin\left(\frac{\rho_0}{R}\right)\right]} + \right.$$
$$\left. + \frac{\varepsilon}{3\left(\frac{1+\sqrt{7}}{12}\left[\pi - 2\arcsin\left(\frac{\rho_0}{R}\right)\right]\right)^{\frac{3}{4}}} - \frac{\varepsilon^2}{6\left(\frac{1+\sqrt{7}}{12}\left[\pi - 2\arcsin\left(\frac{\rho_0}{R}\right)\right]\right)^{\frac{7}{4}}}\right). \tag{22}$$

The solution (17) of Equation (9) in the vicinity of point $\varepsilon = -\varepsilon_m$ is not a satisfactory approximation. Obtaining asymptotic solution of Equation (9) in the vicinity of the point $\varepsilon = -\varepsilon_m$, hence, having found a matching point we have been able to introduce corrections to the Equation (22). To achieve the foregoing, the solution in the vicinity of the point $\varepsilon = -\varepsilon_m$ to be expressed via series the Frobenius factor $(\varepsilon + \varepsilon_m)^\mu$ is taken into account:

$$y = (\varepsilon + \varepsilon_m)^\mu (a_0 + a_1(\varepsilon + \varepsilon_m) + a_2(\varepsilon + \varepsilon_m)^2 + \ldots). \tag{23}$$

A substitution of the boundary layer thickness (9) into the equation yields two possibilities. To find a coefficient μ in Equation (23) all coefficients in Equation (9) are assumed to be of a Taylor series expansion form. The terms of the same power are collected and the coefficients by these powers are equalized to zero. The values of coefficient μ for a general conical surface case are found to be equal to $\frac{2}{5}$ and $\mu = \frac{1}{4}$. In the limit $\alpha \to 0$ (the vertical round plate case) we get conincidence with the known result (the only contribution $\mu = \frac{1}{4}$ survives). In further studies the round plate contribution is suppressed, hence rather big values of α are considered.

For an approximate solution in the vicinity of the point $\varepsilon = -\varepsilon_m$ we left the only first term in Equation (23):

$$y(\varepsilon) = a_0(\varepsilon + \varepsilon_m)^{\frac{2}{5}}. \tag{24}$$

The approximate solution (24) together with Equation (9) can be used to find coefficient a_0. Assuming that:

$$(\varepsilon + \varepsilon_m) = z \tag{25}$$

and also that:

$$y(z) = a_0 z^{\frac{2}{5}} \tag{26}$$

using Equation (9) the following equation is obtained:

$$a_0^4 z^{\frac{8}{5}} E\left(-\frac{6}{25}\frac{a_0}{z^{\frac{8}{5}}}\right) + 3a_0^3 z^{\frac{6}{5}} E \frac{4}{25}\frac{a_0^2}{z^{\frac{6}{5}}} + a_0^4 z^{\frac{6}{5}} \frac{2}{5}\frac{1}{z^{\frac{3}{5}}} G + a_0^5 z^2 H + a_0^4 z^{\frac{8}{5}} F =$$
$$= r^2(1 - \sin^2\alpha \sin^2\varepsilon)\cos^{-2\cos^2\alpha}\varepsilon. \tag{27}$$

Comparing expressions by the term z^0, the result is:

$$\frac{6}{25} a_0^5 E = r^2(1 - \sin^2\alpha \sin^2\varepsilon_m)\cos^{-2\cos^2\alpha}\varepsilon_m. \tag{28}$$

At the point $\varepsilon = -\varepsilon_m = \arccos(\rho_0/R)$ the following solution is provided:

$$\frac{6}{25}a_0^5 E_0 = r^2\left(1-\sin^2\alpha\left(1-\frac{\rho_0^2}{R^2}\right)\right)\left(\frac{\rho_0}{R}\right)^{-2\cos^2\alpha}, \tag{29}$$

where E_0 is defined by:

$$E_0 = E_{\varepsilon=-\varepsilon_m} = \frac{5}{9}(\sin\alpha)\left(\frac{\rho_0}{R}\right)^{\cos^2\alpha+2}\sqrt{1-\frac{\rho_0^2}{R^2}}. \tag{30}$$

Thus, Equation (29) results in the expression on a_0:

$$a_0 = \sqrt[5]{\frac{15r^2}{2(\sin\alpha)\left(\frac{\rho_0}{R}\right)^{3\cos^2\alpha+2}\sqrt{1-\frac{\rho_0^2}{R^2}}}\left(1-(\sin^2\alpha)\left(1-\frac{\rho_0^2}{R^2}\right)\right)}. \tag{31}$$

Therefore, our final solution is the following:

$$g(\varepsilon) = \left[\sqrt[5]{\frac{15r^2}{2(\sin\alpha)\left(\frac{\rho_0}{R}\right)^{3\cos^2\alpha+2}\sqrt{1-\frac{\rho_0^2}{R^2}}}\left(1-(\sin^2\alpha)\left(1-\frac{\rho_0^2}{R^2}\right)\right)}\right](\varepsilon+\varepsilon_m)^{\frac{2}{5}}. \tag{32}$$

A change of the notation from $y(\varepsilon)$ to $g(\varepsilon)$ allows us to distinguish easily both solutions as two different points. The solution $g(\varepsilon)$ is the boundary layer thickness at point $\varepsilon = -\varepsilon_m$, while $-\varepsilon_m$ is the beginning of the boundary layer.

This leads to an evaluation of the two solutions in the vicinity of two points: $\varepsilon = 0$ (Equation (17)) and $\varepsilon = -\varepsilon_m$ (Equation (32)). The purpose of this work is to compare these two solutions and obtain a matching point. one cone with the following values of coefficients is considered to achieve this purpose:

1. $\alpha = \frac{\pi}{4}$ is chosen as this is a case between the vertical plate case and the horizontal cylinder case;
2. the coefficient $\rho_0 = 0.7$ m value is taken from the experiment [8];
3. to simplify the calculations, it has been assumed that the cone base radius is equal to one;
4. the following coefficients depend on α, ρ_0 and R which permits to estimate their values: $-\varepsilon_m = 1.5586$, $a_0 = 2.9716$, $y_0 = 1.2344$.

It is not possible to find a matching point using the above mentioned values. Nevertheless, it can be easily done when some parameters are changed. The matching point is not obtained by changing the μ value from $\frac{2}{5}$ to $\frac{1}{4}$. On the contrary, it can be obtained by changing the a_0 parameter. The most essential results are obtained by changing the y_0 coefficient (the dimensionless boundary layer thickness at point $\varepsilon = 0$). See Table 1.

Table 1. Parameter y_0 dependence on the matching point value and the boundary layer thickness

y_0	matching point	boundary layer thickness
1.49	−1.4762	0.9943
1.45	−1.4766	0.8492
1.39	−1.4771	0.6558
1.35	−1.4838	0.5107
1.25	−1.5344	0.0271

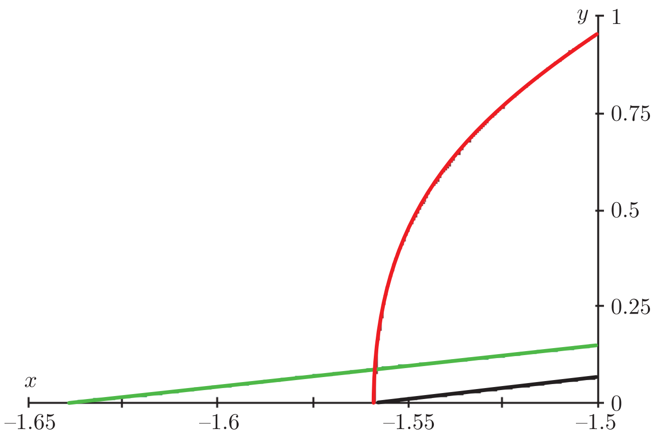

Figure 3. Comparison of the boundary layer plots for $y_0 = 1.25\,\mathrm{cm}$

Moreover, the matching point is obtained by even a small change in parameter $y(0)$ (Figure 3).

Owing to that, it is possible to provide an estimation of new Nusselt number values whereby theoretical considerations can be compared with experimental results.

4. Nusselt number

A new variable $y(0)$ at point $\varepsilon = \varepsilon_s$ is needed to calculate the Nusselt number, where ε_s is the matching point. The following equation is considered:

$$y(\varepsilon) = y(0) + \frac{1}{3}\frac{r\varepsilon}{y^3(0)} - \frac{1}{6}\frac{r^2\varepsilon^2}{y^7(0)}. \tag{33}$$

In this situation $\varepsilon = \varepsilon_s$.

The expression described by Equation (33) and the expression on the boundary layer in Equation (32) meet at the matching point, thus they can be compared:

$$g(\varepsilon) = a_0\left(\varepsilon_s + \arccos\left(\frac{\rho_0}{R}\right)\right)^{\frac{2}{5}}. \tag{34}$$

In the end, the following solution is obtained:

$$a_0\left(\varepsilon_s + \arccos\left(\frac{\rho_0}{R}\right)\right)^{\frac{2}{5}} = y(0) + \frac{1}{3}\frac{r\varepsilon_s}{y^3(0)} - \frac{1}{6}\frac{r^2\varepsilon_s^2}{y^7(0)}, \tag{35}$$

where:

$$a_0 = \sqrt[5]{\frac{15r^2}{2(\sin\alpha)\left(\frac{\rho_0}{R}\right)^{2\cos^2\alpha}\sqrt{1-\frac{\rho_0^2}{R^2}}\left(\frac{\rho_0}{R}\right)^{\cos^2\alpha+2}}\left(1-\left(\sin^2\alpha\right)\left(1-\frac{\rho_0^2}{R^2}\right)\right)}. \tag{36}$$

If the matching point (ε_s) is very close to point $-\varepsilon_m$ as is shown in Figure 3, a new boundary layer thickness at $\varepsilon = 0$ point is changing slightly, but in a significant way. According to that, the following expression is presented:

$$y(0) = y^0(0) + \xi y^1(0) = \sqrt[4]{\frac{1+\sqrt{7}}{12}r\left[\pi - 2\arcsin\left(\frac{\rho_0}{R}\right)\right]} + \xi y^1(0), \tag{37}$$

where ξ is very small:

$$\xi \ll 1. \tag{38}$$

Now, the Taylor expansion of Equation (37) can be used and put to Equation (35). As $y^1(0)$ is very small, $\xi = 1$ can be selected and the term: $\xi y^1(0)$ will remain small and Equation (35) can be written as:

$$a_0\left(\varepsilon_s + \arccos\left(\frac{\rho_0}{R}\right)\right)^{\frac{2}{5}} = y_0 + \frac{1}{3}\frac{r\varepsilon_s}{y_0^3} - \frac{1}{6}\frac{r^2\varepsilon_s^2}{y_0^7} + y_1 - \frac{r}{y_0^4}y_1\varepsilon_s + \frac{7}{6}\frac{r^2}{y_0^8}y_1\varepsilon_s^2, \tag{39}$$

where the variables y_0 and y_1 are defined as:

$$\begin{aligned} y^0(0) &= y_0, \\ y^1(0) &= y_1. \end{aligned} \tag{40}$$

Let us introduce a known variable y_0 which gives:

$$\begin{aligned} a_0\left(\varepsilon_s + \arccos\left(\frac{\rho_0}{R}\right)\right)^{\frac{2}{5}} &= \sqrt[4]{\frac{1+\sqrt{7}}{12}r\left[\pi - 2\arcsin\left(\frac{\rho_0}{R}\right)\right]} + \\ &+\frac{1}{3}\frac{r\varepsilon_s}{\left[\sqrt[4]{\frac{1+\sqrt{7}}{12}r\left(\pi - 2\arcsin\left(\frac{\rho_0}{R}\right)\right)}\right]^{\frac{3}{4}}} - \frac{1}{6}\frac{r^2\varepsilon_s^2}{\left[\sqrt[4]{\frac{1+\sqrt{7}}{12}r\left(\pi - 2\arcsin\left(\frac{\rho_0}{R}\right)\right)}\right]^{\frac{7}{4}}} + \\ &+ y_1 - \frac{r\varepsilon_s y_1}{\frac{1+\sqrt{7}}{12}r\left(\pi - 2\arcsin\left(\frac{\rho_0}{R}\right)\right)} + \frac{7}{6}\frac{r^2\varepsilon_s^2 y_1}{\left[\frac{1+\sqrt{7}}{12}r\left(\pi - 2\arcsin\left(\frac{\rho_0}{R}\right)\right)\right]^2}. \end{aligned} \tag{41}$$

This linear equation having been solved, a new variable y_1 is obtained which is equal to:

$$y_1 = \frac{D}{2\dfrac{\varepsilon_s}{\left(\pi - 2\arcsin\dfrac{\rho_0}{R}\right)\dfrac{1+\sqrt{7}}{6}} - \dfrac{14}{3}\dfrac{\varepsilon_s^2}{\left(\pi - 2\arcsin\dfrac{\rho_0}{R}\right)^2\left(\dfrac{1+\sqrt{7}}{6}\right)^2} - 1}. \tag{42}$$

And the new boundary layer thickness value at point $\varepsilon = 0$ is equal to:

$$\begin{aligned} y(0) = &\sqrt[4]{\frac{1+\sqrt{7}}{12}r\left[\pi - 2\arcsin\left(\frac{\rho_0}{R}\right)\right]} + \\ &+ \xi\frac{D}{2\dfrac{\varepsilon_s}{\left(\pi - 2\arcsin\dfrac{\rho_0}{R}\right)\dfrac{1+\sqrt{7}}{6}} - \dfrac{14}{3}\dfrac{\varepsilon_s^2}{\left(\pi - 2\arcsin\dfrac{\rho_0}{R}\right)^2\left(\dfrac{1+\sqrt{7}}{6}\right)^2} - 1}, \end{aligned} \tag{43}$$

where:

$$\begin{aligned} D = &-a_0\left(\varepsilon_s + \arccos\frac{\rho_0}{R}\right)^{\frac{2}{5}} + \sqrt[4]{\frac{1}{2}r\left(\pi - 2\arcsin\frac{\rho_0}{R}\right)\frac{1+\sqrt{7}}{6}} + \\ &+\frac{1}{3}r\frac{\varepsilon_s}{\sqrt[4]{\frac{1}{2}r\left(\pi - 2\arcsin\frac{\rho_0}{R}\right)\frac{1+\sqrt{7}}{6}}^{\frac{3}{4}}} - \frac{1}{6}r^2\frac{\varepsilon_s^2}{\sqrt[4]{\frac{1}{2}r\left(\pi - 2\arcsin\frac{\rho_0}{R}\right)\frac{1+\sqrt{7}}{6}}^{\frac{7}{4}}} \end{aligned} \tag{44}$$

As a result, a local solution at the $\varepsilon = \varepsilon_s$ point, called a matching point, has been obtained. It is also known that both $y(\varepsilon) = \delta K^{\frac{1}{3}}$ and the boundary layer thickness and radius r equations are:

$$\delta = y(\varepsilon) K^{-\frac{1}{3}}, \tag{45}$$

$$r = \rho_0 K^{\frac{1}{3}}, \tag{46}$$

where:

$$K = \frac{\mathrm{Ra}}{240R^3}. \tag{47}$$

The Nusselt number mean value for whole conic surface is defined by the following equation:

$$\mathrm{Nu}_m = \frac{2R}{S} \int_0^{\pi/2} \int_{-\varepsilon_m}^{\varepsilon_m} \frac{1}{\delta} dA_k, \tag{48}$$

where dA_k as described in paper [9], is equal to:

$$dA_k = \cos\alpha \cdot (\cos\varepsilon)^{-2\cos^2\alpha} \cdot R^2 \sin\varepsilon_m (\cos\varepsilon_m)^{2\cos^2\alpha - 1} d\varepsilon_m d\varepsilon. \tag{49}$$

It is known that our boundary layer thickness changes for a small value of ξy_1. Thus, it can be written as:

$$\delta(\varepsilon) = \left[y^0(0) + \xi y^1(0) + \frac{1}{3} \frac{r\varepsilon}{\left[y^0(0) + \xi y^1(0)\right]^3} - \frac{1}{6} \frac{r^2\varepsilon^2}{\left[y^0(0) + \xi y^1(0)\right]^7} \right] \left(\frac{240R^3}{\mathrm{Ra}} \right)^{\frac{1}{3}}. \tag{50}$$

The boundary layer thickness value is obtained using the Taylor expansion of the Equation (39):

$$\delta(\varepsilon) = \left(\frac{240R^3}{\mathrm{Ra}} \right)^{\frac{1}{3}} \left(y_0 + \frac{1}{3} \frac{r\varepsilon}{y_0^3} - \frac{1}{6} \frac{r^2\varepsilon^2}{y_0^7} + y_1 - \frac{r}{y_0^4} y_1 \varepsilon + \frac{7}{6} \frac{r^2}{y_0^8} y_1 \varepsilon^2 \right), \tag{51}$$

where:

$$\delta_0(\varepsilon) = \left(\frac{240R^3}{\mathrm{Ra}} \right)^{\frac{1}{3}} \left(y_0 + \frac{1}{3} \frac{r\varepsilon}{y_0^3} - \frac{1}{6} \frac{r^2\varepsilon^2}{y_0^7} \right) \tag{52}$$

is the boundary layer thickness value which has been calculated in [9]; and where:

$$\delta_1(\varepsilon) = \left(\frac{240R^3}{\mathrm{Ra}} \right)^{\frac{1}{3}} \left(y_1 - \frac{r}{y_0^4} y_1 \varepsilon + \frac{7}{6} \frac{r^2}{y_0^8} y_1 \varepsilon^2 \right) \tag{53}$$

is the boundary layer thickness correction. The value $\delta_1(\varepsilon)$ contains the parameter ε_s, which approximate value is taken as ε_m in further evaluations.

Equation (52) and Equation (53) having been included in Equation (48) and assuming that the boundary layer thickness is changing slightly, the following solution is obtained:

$$\mathrm{Nu}_m = \int_0^{\pi/2} \int_{-\varepsilon_m}^{\varepsilon_m} \frac{1}{\delta_0(\varepsilon) + \xi \delta_1(\varepsilon)} dA_k. \tag{54}$$

Assuming that $\xi = 1$ and using a Taylor expansion inside the integral:

$$\frac{1}{\delta_0(\varepsilon) + \xi\delta_1(\varepsilon)} = \frac{1}{\delta_0(\varepsilon)} + \frac{\partial}{\partial \xi} \left[\frac{1}{\delta_0(\varepsilon) + \xi\delta_1(\varepsilon)} \right]_{\xi=0} + \ldots = \frac{1}{\delta_0(\varepsilon)} + \frac{-\delta_1(\varepsilon)}{\left(\delta_0(\varepsilon)\right)^2}, \tag{55}$$

Equation (54) takes the form:

$$\mathrm{Nu}_m = \int_0^{\pi/2} \int_{-\varepsilon_m}^{\varepsilon_m} \frac{1}{\delta_0(\varepsilon)} dA_k - \int_0^{\pi/2} \int_{-\varepsilon_m}^{\varepsilon_m} \frac{\delta_1(\varepsilon)}{\left(\delta_0(\varepsilon)\right)^2} dA_k. \tag{56}$$

The integral $\int_0^{\pi/2}\int_{-\varepsilon_m}^{\varepsilon_m}\frac{1}{\delta_0(\varepsilon)}dA_k$ is computed in paper [1], hence the following integral:

$$I=\int_0^{\pi/2}\int_{-\varepsilon_m}^{\varepsilon_m}\frac{\delta_1(\varepsilon)}{(\delta_0(\varepsilon))^2}dA_k \tag{57}$$

needs to be calculated, where $\delta_1(\varepsilon)$ and dA_k are given by Equation (53) and Equation (49), respectively.

The method described precisely in [8] has been used to solve the integral (57). The integral has been calculated numerically for the following numbers of integration steps: $n=300$, for the internal integral and $p=150$, for the external one.

Taking into account Equation (56) the Nusselt number can be calculated and compared with the result from [8]. The result of this calculation is shown in Table 2.

Table 2. Nusselt number correction based on the boundary layer thickness new value, Equation (50)

Angle	30°	45°	60°	75°
Nu_{sr} for $\delta=y^0(0)K^{-\frac{1}{3}}$	6.2698	6.0189	5.1944	2.5145
Correction of the Nusselt number	−0.1970	0.0006	−0.0099	−0.0034
Percentage change of Nusselt number	3.15	0.01	0.17	0.13
New Nusselt number	6.4668	6.0183	5.2043	2.5179

When the boundary layer thickness parameter $y^0(0)$ is corrected, the Nusselt number changes. The Nusselt number change tendency falls generally within the scope of a comparison between theory and experiment [8]. It follows from an evaluation of the Nusselt number that the correction is small which is in accordance with our assumptions (see conclusion (38)). It may be concluded that there are many reasons for the Nusselt number to account for a complete theory. It should be noted that approximations have been used to solve Equation (9). The account of possible discrepancies could be the subject of a future analysis.

5. Conclusions

The theory of convective heat transfer from a surface is based on boundary layer notion, its thickness dependence on a surface points allows to evaluate the Nusselt number integrating across the surface. In the case of horizontal conic surface we consider here the equation for the boundary layer thickness is rather complicated nonlinear ordinary differential equation (9) in ε with variable coefficients that also depend on other variable r as a parameter. A numerical solution of this equation could give the necessary dependence of the boundary layer thickness on ε, r for every value of basic angle α of the cone. Numeric approach however is not simple enough to analise the dependencies of the heat transfer properties on problem parameters (Ra number, *etc.*). The analytic approximate integration we develop here give results close to experimentsand allows to follow the parameter dependencies iin a rather simple way. We hope it could serve as an useful tool in engeneering. Note that the parameter ε_s (matching curve coordinate) may be chosen to fit better experimental data.

References

[1] Jaluria Y 1980 *Natural Convection Heat and Mass Transfer*, Pergamon Press, Oxford

[2] Lewandowski W M, Kubski P, Szymański S, Bieszk H, Wilczewski T, Radziemska E and Seramak T 1997 *5^{th} UK National Conference on Heat Transfer*, London, p. 5
[3] Stewart W E 1971 *Int. J. Heat Mass Transfer* **14** 1013
[4] Hering R G 1965 *Int. J. Heat Mass Transfer* **8** 1333
[5] Oosthuizen P H and Donaldson E 1972 *J. Heat Transfer Transitions of the ASME* **94** 330
[6] Oosthuizen P H 1973 *J. Heat Transfer Transitions of the ASME* **95** 409
[7] Churchill S W 1983 *Free Convection Around Immersed Bodies, 2.5 Single-Phase Convective Heat Transfer*, Hemisphere Publishing Corporation
[8] Lewandowski W M and Leble S 2003 *Int. J. Heat and Mass Transfer* **46** 4925
[9] Leble S and Lewandowski W M 2004 *Appl. Math. Model.* **28** 305
[10] Squire H B 1938 *Modern Developments in Fluid Dynamics* (Goldstein S, Ed.), Clarendon Press, Oxford (reprinted by Dover, New York, 1965)
[11] Eckert E R G 1959 *Heat and Mass Transfer*, McGraw-Hill, New York 312