APPLICATION OF PIES AND RECTANGULAR BÉZIER SURFACES TO COMPLEX AND NON-HOMOGENEOUS PROBLEMS OF ELASTICITY WITH BODY FORCES

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Abstract: This paper presents a variety of applications of an effective way to solve boundary value problems of 2D elasticity with body forces. An overview of the approach is presented, its numerical implementation, as well as a number of applications, ranging from problems defined on elementary shapes to complex problems, *e.g.* with non-homogeneous material. The results obtained by the parametric integral equation system (PIES) were compared with the analytical and numerical solutions obtained by other computer methods, confirming the effectiveness of the method and its applicability to a variety of problems.

Keywords: boundary problems, elasticity, body forces, non-homogeneous materials, Bézier surfaces, parametric integral equation system (PIES)

1. Introduction

When problems of elasticity with body forces are solved, it is necessary to calculate integrals over the area. By using the well-known boundary element method (BEM) [1, 2], the aforementioned problem is often solved by transforming the domain integrals into equivalent boundary integrals. However, this is only possible in specific cases, *e.g.* in the case of gravitational forces, centrifugal forces or thermal loading [3–5]. In general, the approach is quite different, involving the division of the area of integration into smaller sub-areas, called cells, and the calculation of integrals over these areas, which contribute to the value of the final integral. This process is very similar to the discretization typical of the finite element method (FEM) [6].

Our study has been concerned with the development of an approach that could constitute an all-purpose alternative to the well-known numerical methods. The main advantage of this method should be versatility, which means that regardless of the case under study, this method would be an effective and simple tool to solve the aforementioned problems. The efficiency and simplicity of this method consist in its ability to avoid discretization when modeling the areas and in the manner in which the integration is carried out over a suitably defined area while using standard numerical procedures. Such an approach should be based on the parametric integral equation system (PIES), widely tested on other cases (such as those modeled by the Laplace [7] or Helmholtz [8] equations and elasticity without body forces [9]), and developed as an analytically modified boundary integral equation (BIE) (2) in [7].

In order to achieve this, the PIES was generalized to a form which could be applied to plane elasticity problems with body forces [10]. We propose to use in its definition the rectangular Bézier surfaces of the first and third degree, depending on the shape of the modeled area (polygonal vs. curved). We began testing the proposed approach on elementary rectangular shapes [10], where we proposed a technique for the global modeling of an area and the global integration over this area in conjunction with the PIES. This idea has been tested on model cases with basic types of body forces in rectangular areas modeled by one surface. Promising results encouraged us to generalize the method and test its effectiveness in the case of more complex problems, involving areas that were non-rectangular in shape. This raises the problem of modeling an area with more than one surface, as well as the situations in which we are dealing with more complex conditions, such as a variable density of the material. The question arises as to whether the method pre-tested on elementary cases remains applicable and effective even under such circumstances. The following problems arise: the need to combine multiple surfaces to model the area; the need for integration over modified areas, using the standard quadrature for rectangles; and the accuracy of the results obtained by the presented approach for cases with more complex boundary conditions and body forces.

The main aim of this paper is to apply the PIES and the effective domain modeling in order to solve various problems of elasticity with body forces. We focus on the application of the approach to bodies of more complex shapes, for the modeling of which a few patches should be applied, but also those, where we observe the heterogeneity of the material (*e.g.* its variable density). We test a number of cases and compare numerical results with analytical methods and results obtained by other known methods. Moreover, we demonstrate the effectiveness of the proposed approach to solving the aforementioned problems, as well as its accuracy and the reliability of the obtained results by comparing them with analytical and other numerical results.

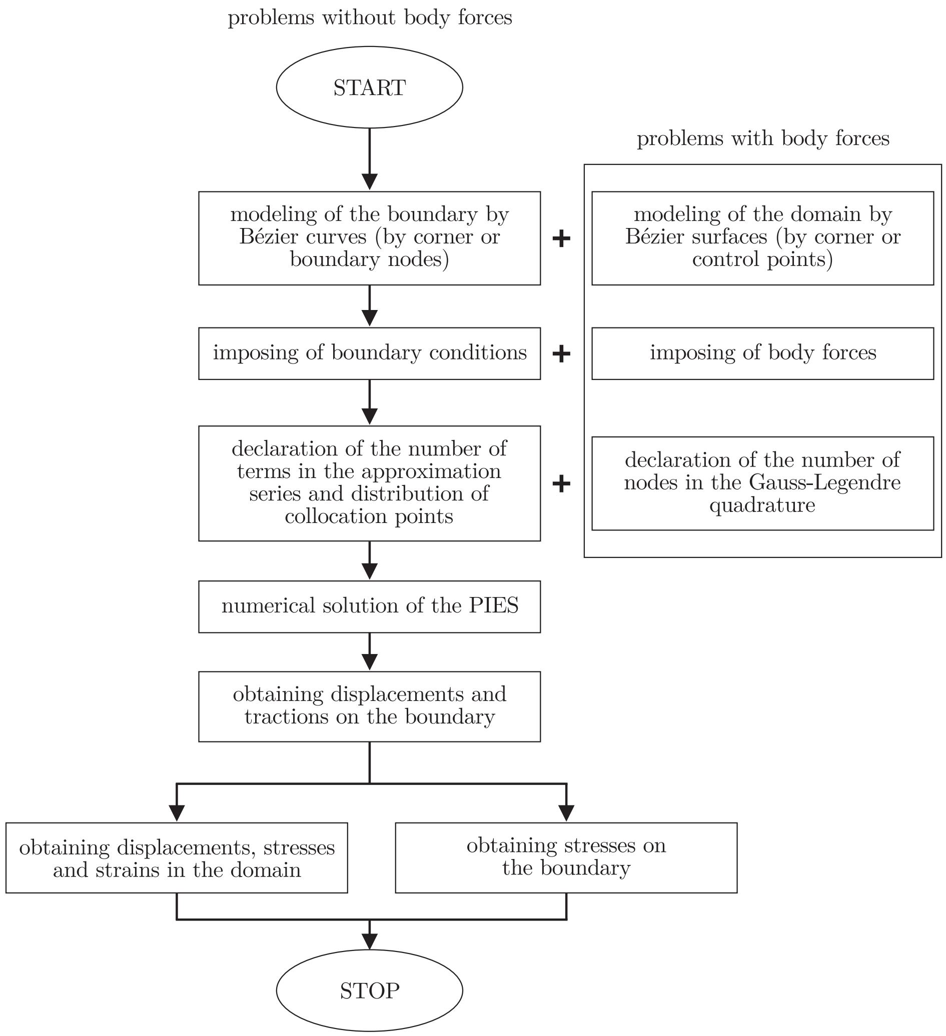

Figure 1. The process of the numerical solution of boundary problems of elasticity by the PIES

2. Overview of the proposed approach

The process of solving boundary-value problems using computer methods consists of several subsequent stages. The main steps involved in solving problems of elasticity using the proposed approach are presented in Figure 1.

The detailed description of the presented stages of the solution of elasticity problems without body forces (presented on the left side of Figure 1) is given in many works, *e.g.* [9, 11, 12]. However, in order to tackle problems with body forces, it is necessary to develop additional strategies for modeling the area where the problem is defined, and calculating integrals over this area (Figure 1 – right part). For this reason, the main aim of this paper is to present an effective approach for the accurate resolution of such problems. Further sections of this paper are devoted to a generalization of the PIES and its application to 2D elasticity problems with body forces. These sections also provide a description of all stages of solving these problems (outlined with a gray rectangle in Figure 1), which are additional to those already published.

3. PIES with body forces

Using the strategy of modifying the BIE outlined in [8, 9], the following generalized form of the PIES for the Navier-Lamé equation with body forces was obtained [10]:

$$0.5\boldsymbol{u}_p(s_1)=\sum_{r=1}^{n}\int_{s_{r-1}}^{s_r}\left\{\overline{\boldsymbol{U}}^*_{pr}(s_1,s)\boldsymbol{p}_r(s)-\overline{\boldsymbol{P}}^*_{pr}(s_1,s)\boldsymbol{u}_r(s)\right\}J_r(s)\mathrm{d}s+ \\ +\int_{\Omega}\overline{\overline{\boldsymbol{U}}}^*_p(s_1,\boldsymbol{x})\boldsymbol{b}(\boldsymbol{x})J(\boldsymbol{x})\mathrm{d}\Omega(\boldsymbol{x}) \tag{1}$$

where $J_r(s)=\left[\left(\frac{\partial\Gamma_r^{(1)}(s)}{\partial s}\right)^2+\left(\frac{\partial\Gamma_r^{(2)}(s)}{\partial s}\right)^2\right]^{0.5}$, $p=1,2,\dots n$, n is the number of segments, $s_{p-1}\le s_1\le s_p$, $s_{r-1}\le s\le s_r$, $\boldsymbol{x}=\{v,w\}$, $\boldsymbol{b}(\boldsymbol{x})$ is the vector of body forces, $J(\boldsymbol{x})=\frac{\partial B^{(1)}(\boldsymbol{x})}{\partial w}\frac{\partial B^{(2)}(\boldsymbol{x})}{\partial v}-\frac{\partial B^{(1)}(\boldsymbol{x})}{\partial v}\frac{\partial B^{(2)}(\boldsymbol{x})}{\partial w}$.

The integrands $\overline{\boldsymbol{U}}^*_{pr}(s_1,s)$ and $\overline{\boldsymbol{P}}^*_{pr}(s_1,s)$ in Equation (1) are presented in explicit forms in [10], whilst the function $\overline{\overline{\boldsymbol{U}}}^*_p(s_1,\boldsymbol{x})$ in the second integral (over the domain) can be expressed as follows:

$$\overline{\overline{\boldsymbol{U}}}^*_p(s_1,\boldsymbol{x})=-\frac{1}{8\pi(1-v)\mu}\begin{bmatrix}(3-4v)\ln(\overline{\boldsymbol{\eta}})-\frac{\overline{\eta}_1^2}{\overline{\boldsymbol{\eta}}^2} & -\frac{\overline{\eta}_1\overline{\eta}_2}{\overline{\boldsymbol{\eta}}^2}\\ -\frac{\overline{\eta}_1\overline{\eta}_2}{\overline{\boldsymbol{\eta}}^2} & (3-4v)\ln(\overline{\boldsymbol{\eta}})-\frac{\overline{\eta}_2^2}{\overline{\boldsymbol{\eta}}^2}\end{bmatrix} \tag{2}$$

$$p=1,2,\dots n$$

where $\overline{\boldsymbol{\eta}}=[\overline{\eta}_1^2+\overline{\eta}_2^2]^{0.5}$, $\overline{\eta}_1=B^{(1)}(\boldsymbol{x})-\Gamma_p^{(1)}(s_1)$ and $\overline{\eta}_2=B^{(2)}(\boldsymbol{x})-\Gamma_p^{(2)}(s_1)$.

The shapes of the boundary and the domain are directly included in the integrands in (1) and are defined by the parametric curves $\Gamma(s)=\left\{\Gamma^{(1)},\Gamma^{(2)}\right\}$ and the parametric surfaces $B(\boldsymbol{x})=\left\{B^{(1)},B^{(2)}\right\}$. Since only 2D problems are considered, the third dimension can be omitted from the definition of the surfaces.

4. Bézier surfaces in PIES

The way in which the boundary and the domain are modeled is crucial for assessing the effectiveness of the method. It is known that in the case of the so-called element methods it leads to discretization. With the BEM and the specific cases that require integration over the area, the modeling process requires defining the boundary by boundary elements and the area by using the so-called cells, which is a time-consuming and cumbersome process, because it requires the user applying the method to have at least minimal experience.

In the case of the PIES, we propose to use rectangular surfaces, successfully used in computer graphics [13, 14]. Polygons can be defined by Bézier surfaces of the first degree, and areas with curved edges – by Bézier surfaces of the third degree. In the first case, modeling is performed only by the coordinates of the corner points (Figure 2a). It is more efficient in comparison with dividing the area into cells in the BEM and specifying the coordinates of all nodes that define cells. The areas with curved edges require 16 control points which define the third-degree Bézier surface. Twelve of them (denoted with circles in Figure 2b) are responsible for the shape of the boundary, while the remaining four (denoted with crosses in Figure 2b) are responsible for the shape of the area in the third dimension. Since only plane problems are considered here, these four points are on the surface, but their location does not affect the final results.

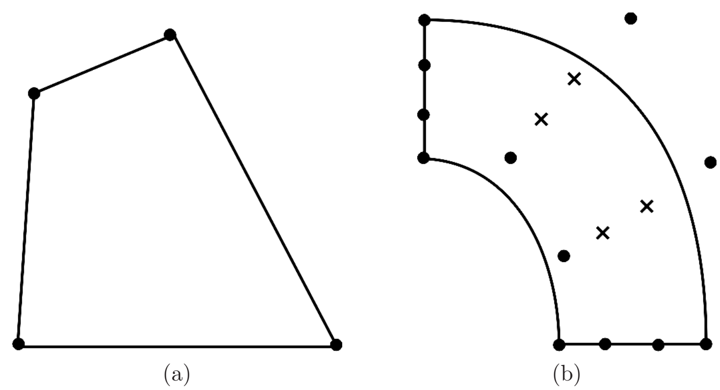

Figure 2. Modeling (a) polygonal and (b) curved shapes;
• – control points corresponding to the shape of the edge,
× – control points corresponding to the shape of the domain only in the third dimension

When the shape of the area is complex, it may be necessary to use more than one Bézier surface. Modeling by using many surfaces can be done in two ways. In the first approach, one can use the points identifying the boundary, which have already been defined, as corner or control points. This is particularly effective in the case of polygonal areas, where, in practice, the modeling of the boundary and of the domain is performed simultaneously. The second approach allows one to define surfaces independently of the points which define the shape of the boundary. This may be necessary when one needs to impose specific, discontinuous body forces, or when the first method cannot be applied. Modeling the domain by using both techniques is presented in Figure 3.

As shown in Figure 3a, the modeling of the polygonal area was performed in the same way as in problems without body forces and it leads to the imposing of only six corner points. In the case of the area presented in Figure 3b, an additional point (×) was introduced. This point lies on the boundary and defines the corner of one of the surfaces. Such a definition of the area can be very useful when one considers bodies made of a heterogeneous material.

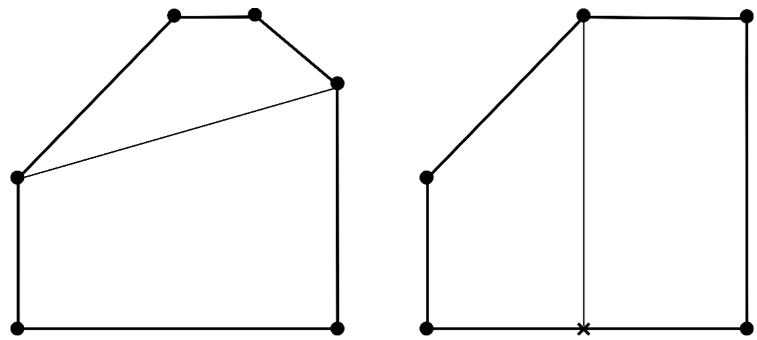

Figure 3. Modeling (a) dependent on, (b) independent of the boundary definition

5. Numerical strategy of the calculation of integrals

The difference between the numerical solution of boundary value problems of elasticity with body forces and those without body forces comes down to the necessity of the calculation of domain integrals of the following form:

$$\int_{\Omega} \overline{\overline{U}}_p^*(s_1, \boldsymbol{x})\boldsymbol{b}(\boldsymbol{x})J(\boldsymbol{x})\mathrm{d}\Omega(\boldsymbol{x}) \tag{3}$$

where $\boldsymbol{b}(\boldsymbol{x})$ is the vector of body forces, whilst the integrands in (3) were described in Section 3 of this paper. The domain Ω was modeled using Bézier surface of the first or third degree, depending on the shape of the boundary. The body forces $\boldsymbol{b}(\boldsymbol{x})$ were imposed in a continuous way over the entire surface. The next step is to select a numerical quadrature suitable for the integration with the appropriate number of nodes.

In this paper, we use the standard Gauss-Legendre quadrature [15] intended for the integration over rectangular areas. Such a strategy is also adopted in the case of the BEM, however, in this case the integration is performed locally over sub-areas. In order to achieve this, we apply quadratures with a small number of nodes. In the case of the PIES, the situation is radically different, because of the global modeling of the entire area of integration. Such an approach necessitates the calculation of the integrals over the whole area, as well as requiring more quadrature nodes than in the case of the BEM. In the end, by disposing of the discretization problem of both the boundary and the domain, we arrive at an approach characterized by global modeling and integration. However, the question arises as to whether the Gaussian quadrature used for the integration over the rectangular areas would be applicable in the case of modified shapes. By modifying the base shape of a rectangle, we also modify the distribution of the nodes where the weights are calculated. The mapping of this distribution to the new shape of the area is performed automatically, but there arises a question of reliability and accuracy of the obtained results. An example of mapping the arrangement of nodes is presented in Figure 4. Subsequently, we give several examples serving to determine whether the strategies applied to modeling and integration are versatile.

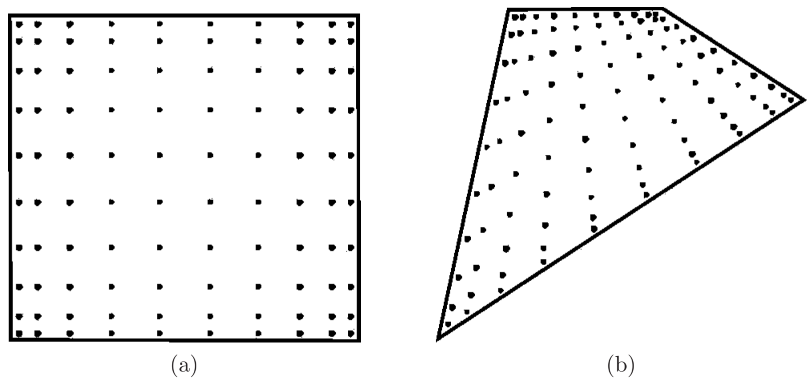

Figure 4. The arrangement of nodes of the Gauss-Legendre quadrature in (a) the basic square shape and (b) the modified shape

In case of areas with more complex shapes for which more than one rectangular surface is required, the integral (3) is the sum of integrals calculated over the sub-areas represented by individual surfaces.

6. Numerical analysis

6.1. Verification of the proposed approach

As the first example, a problem involving a square with a side of length $a=5$ (Figure 5) is presented. The body is fixed along all edges and is subjected to gravitational force $b_y=-1$. The material constants adopted for the calculations are: $E=2$ and $v=0.2$.

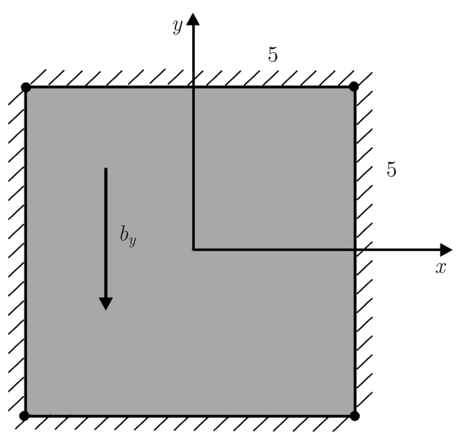

Figure 5. The plane body under gravitational force

This example comes from the work [16], where the problem was solved by using the analog equation method (AEM) and the FEM. Modeling in both methods was performed taking into account the different number of elements: in the AEM, the edge was defined using 60 constant boundary elements, whilst the area was defined using 100 cells (160 nodes), whereas in the FEM 400 four-node finite elements (441 nodes) were used. In the proposed approach only one Bézier surface of the first degree was used to model the area. The surface was defined by the coordinates of only four corner points, which is the minimum number of points required for an accurate mapping of the shape. This method is very effective in comparison with the abovementioned methods, taking into account not only the modeling of the shape, but also its modification. The important thing is that the modeling of the shape of the area is performed in the same intuitive and simple way, just like in the absence of body forces.

The values of displacements were analyzed in two vertical cross-sections: $y=0.25$ and $y=1.25$. Results were compared with those obtained from the AEM and the FEM [16].

Table 1. The numerical results obtained by various methods

y	x	$100u$			v		
		FEM	AEM	PIES	FEM	AEM	PIES
0.25	0.25	−0.480	−0.462	−0.474	−1.274	−1.266	−1.270
	0.75	−1.335	−1.289	−1.295	−1.198	−1.192	−1.195
	1.25	−1.852	−1.798	−1.810	−1.034	−1.028	−1.031
	1.75	−1.766	−1.730	−1.741	−0.749	−0.746	−0.747
	2.25	−0.833	−0.827	−0.829	−0.301	−0.301	−0.301
1.25	0.25	−1.975	−1.917	−1.922	−0.966	−0.961	−0.963
	0.75	−5.600	−5.435	−5.479	−0.915	−0.910	−0.912
	1.25	−8.090	−7.839	−7.905	−0.801	−0.796	−0.798
	1.75	−8.110	−7.891	−7.961	−0.595	−0.592	−0.593
	2.25	−3.990	−3.938	−3.964	−0.248	−0.247	−0.247

It is apparent from Table 1 that the results obtained by the PIES are very close to those obtained by the AEM and the FEM. It should be emphasized that they were obtained without any discretization of the boundary or of the domain, but rather by using global modeling and integration over the area.

6.2. Versatility of the procedure for numerical integration

The next step in the study of the proposed approach is to examine whether it can be successfully applied in the case of a more complex shape of the area. For this reason, a dam under hydrostatic pressure and gravity, presented in Figure 6, was considered. The following values have been used for the material constants and other parameters: $E=25\,\text{GPa}$, $v=0.2$, $\rho_w=1000\,\text{kg/m}^3$, $\rho_m=2400\,\text{kg/m}^3$.

Since the cross-section of the dam is not a quadrangle, one Bézier surface does not suffice to model it. Two different strategies can be adopted. The first one is based on the definition of surfaces according to which their corner points coincide with the corner points of the polygon boundary (Figure 6a). The other strategy allows any definition of the corner points of the surfaces regardless of the corner points of the boundary (Figure 6b), which have already been defined. It was decided to examine both strategies and verify whether the effectiveness of the method and its accuracy depended on the definition of the surfaces.

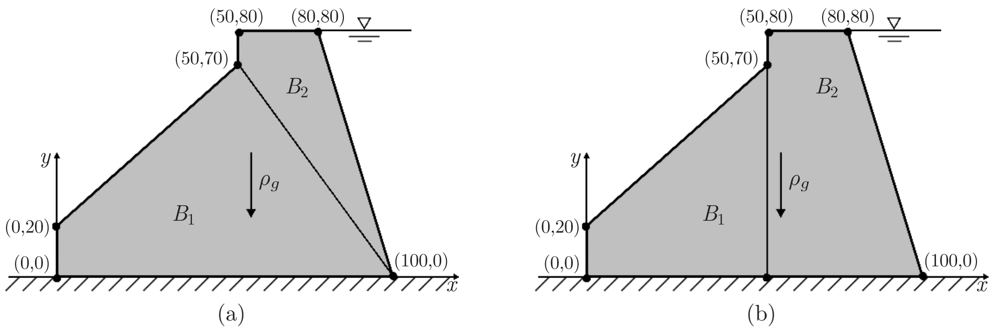

Figure 6. A dam modeled by two surfaces

We analyzed all components of the displacements and stresses in the cross-section $y = 20$m for the two strategies of modeling. The numerical solutions obtained using the proposed approach were compared with those obtained by the BEM, as implemented in the program BEASY [17]. All solutions were characterized by high accuracy in comparison with the BEM. In Table 2 we present only a selection from the large number of results that we obtained.

Table 2. The numerical results obtained by the BEM and the PIES

x	y	BEASY (BEM)		PIES			
				Figure 6a		Figure 6b	
		v	σ_x	v	σ_x	v	σ_x
10	20	−0.6974	−0.0599	−0.6965	−0.0557	−0.6965	−0.0604
20	20	−0.7985	−0.1195	−0.7983	−0.1186	−0.7983	−0.1189
30	20	−0.8948	−0.1615	−0.8954	−0.1618	−0.8954	−0.1594
40	20	−0.9652	−0.1951	−0.9664	−0.1915	−0.9664	−0.1910
50	20	−1.0044	−0.2302	−1.0053	−0.2286	−1.0053	−0.2281
60	20	−1.0091	−0.2742	−1.0090	−0.2737	−1.0090	−0.2720
70	20	−0.9702	−0.3374	−0.9691	−0.3350	−0.9691	−0.3353
80	20	−0.8615	−0.4383	−0.8599	−0.4369	−0.8599	−0.4363
90	20	−0.6170	−0.5912	−0.6167	−0.5871	−0.6167	−0.5869

As can be inferred from Table 2, the obtained results are satisfactory, regardless of the definition of the Bézier surfaces. In the case of modeling the area presented in Figure 6a, the modification of the shape of both surfaces (as compared with the shape of a rectangle) was significant, but the desired accuracy of results was achieved. It is worth emphasizing that this procedure for numerical integration is also universal for problems with more complex domains.

6.3. Potential applications

Problem 1

Here we demonstrate the application of the proposed approach to the solution of problems, where the density of the material varies. We consider a square

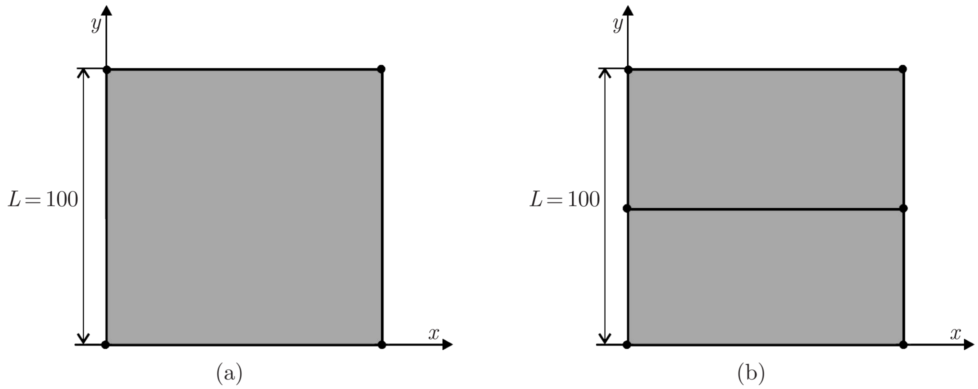

Figure 7. The shape of plate under consideration and the two methods to model it: (a) using one surface, (b) using two surfaces

plate rotating about the x-axis with an angular velocity of $\omega = 100$ rad/s, as shown in Figure 7.

The following values of the parameters are assumed: $E = 210$ GPa, $v = 0.3$. The density distribution has the following discontinuity [18]:

$$\begin{aligned} \rho_1 &= 1 \cdot 10^{-6}\,\mathrm{kg/mm^3}, \quad 0 \le y \le 50 \\ \rho_2 &= 2 \cdot 10^{-6}\,\mathrm{kg/mm^3}, \quad 50 < y \le 100 \end{aligned} \tag{4}$$

The problem was solved in two ways. In the first case, the area was modeled by a single rectangular Bézier surface of the first degree and the density varied according to (4), *cf.* Figure 7a. In the second case, two Bézier surfaces were defined, in such a way that the corresponding density value coincided with the shape of the sub-areas. In other words, two surfaces modeled the areas of different densities (Figure 7b).

The analytical solution is known [18] and has the following form:

$$\begin{aligned} \sigma_y &= \tfrac{\rho_1 \omega^2}{8}\left[2L(L-2y) - (L-2y)^2\right] + \rho_2 \omega^2 \tfrac{3L^2}{8}, && 0 \le y \le 50 \\ \sigma_y &= \rho_2 \omega^2 \left[L(L-y) - \tfrac{(L-y)^2}{2}\right], && 50 < y \le 100 \end{aligned} \tag{5}$$

The values of the normal stresses along the y direction in two vertical cross-sections at $x = 50$ mm and $x = 90$ mm were analyzed for both cases. The results obtained in both cases were almost identical and consistent with the analytical solutions (Figure 8).

Problem 2

The final problem concerns gravitational force acting on a material of varying density [19]. The value of the density depends on the parameter r, which specifies the distance from the center of a square body of the side, 200 mm in

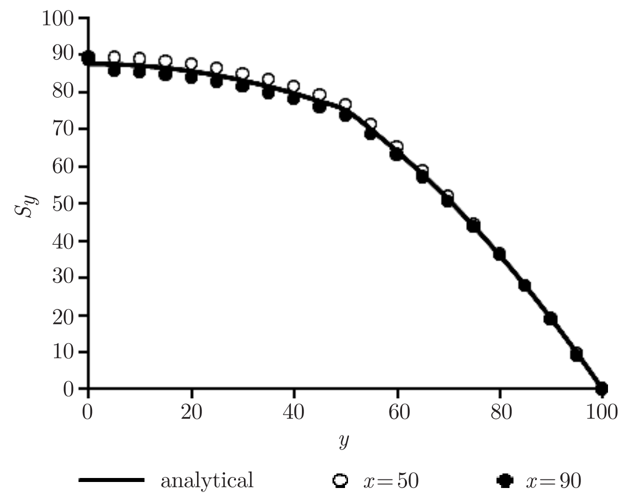

Figure 8. The distribution of stress in the square plate with variable density

length (Figure 9). The distribution of the density is described by the following expression:

$$\rho(r)=\begin{cases}\rho_0\left[0.1+\frac{\sqrt{r_0^2-r^2}}{r_0}\right], & r<r_0\\ 0.1\rho_0, & r\geq r_0\end{cases} \tag{6}$$

where $r_0=50$mm. The acceleration due to gravity is denoted as g and $g\rho_0=1\cdot 10^{-4}\,\mathrm{mm}^{-3}$. The following values were adopted for the material constants: $E=210$GPa and $v=0.3$.

Defining the body forces in the presented form is uncomplicated in the case of PIES . Since the entire area can be defined by one Bézier surface, it is also for the entire area that one should specify a function describing the variation of the density. In the case of the BEM, the body force should be defined taking into account the individual cells into which the area of the integration was divided.

There are no analytical results for the problem posed above. Figure 10 presents the stress distribution σ_y obtained by the proposed approach in two cross-sections of the domain under consideration, namely, $x=110$mm and $x=170$mm.

7. Conclusions

In this paper we presented an efficient way of modeling and solving boundary value problems of elasticity with body forces. In order to demonstrate the advantages of the method, we focused on problems of various shapes, which we were able to model using rectangular surfaces and areas of heterogeneous material. We also solved several problems employing a global method of modeling and integration. The obtained results, as compared to the analytical and numerical solutions obtained using *i.a.* the BEM and the FEM, proved to be highly satisfactory, and the complexity of the method was significantly reduced, as evidenced by the shorter time necessary to model the area.

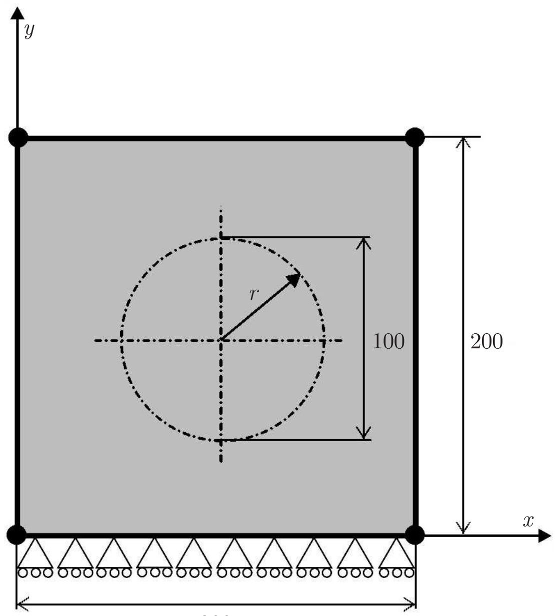

Figure 9. The square body with variable density

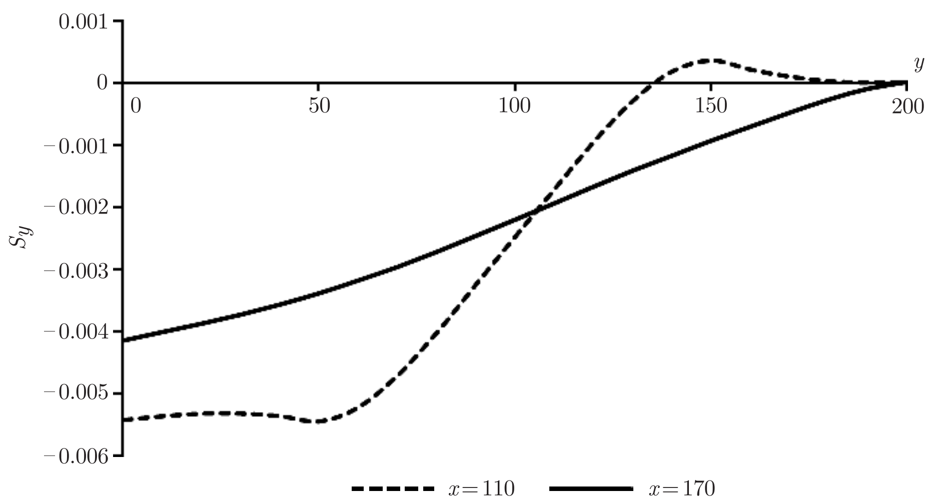

Figure 10. The stress distribution σ_y in two cross-sections of the domain

The accuracy of the obtained results legitimizes the application of the standard Gauss-Legendre quadrature with a large number of nodes to the global integration over an area defined by a Bézier surface in the PIES. It turned out that the integration is possible even in the case of highly modified surfaces and the obtained results are satisfactory.

Certain problems presented in this paper allowed us to test the PIES with the proposed concept of global numerical integration over areas modeled by combined surfaces. The main aim of this paper was to combine this approach with the previously studied PIES, where only the boundary value problems which do not require integration over the area were studied.

Until now, only very preliminary studies of the concept of integration over polygonal areas [10] have been carried out. In this paper this concept was generalized to areas modeled by connected surfaces. Only elementary problems having analytical solutions were studied in this paper in order to verify whether the proposed concept of integration over combined areas has practical application to the solution of problems with more complex areas. Taking into account the results obtained for the problems under study, the method can be considered an alternative to the FEM and the BEM. The practical advantages of this approach are particularly visible when solving boundary problems which require an iterative procedure (such as optimization, identification and nonlinear problems), where multiple discretization is required when using classical methods.

We believe that the presented results encourage further research on the development of this approach. This method can be applied to other problems which require integration over the area, including problems modeled by the Poisson equation, nonlinear problems and three-dimensional problems. However, this should be preceded by implementing and testing Bézier triangular surfaces, which may allow a more efficient modeling of arbitrary shapes of the domains.

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