TORSIONAL BUCKLING OF RESTRAINED THIN-WALLED BARS OF OPEN BISYMMETRIC CROSS-SECTION

MARCIN KUJAWA

Gdansk University of Technology,
Faculty of Civil and Environmental Engineering,
Department of Structural Mechanics and Bridges,
Narutowicza 11/12, 80-233 Gdansk, Poland
markuj@pg.gda.pl

(Received 3 October 2012; revised manuscript received 19 October 2012)

Abstract: The problem of torsional buckling of restrained thin-walled bars of an open constant bisymmetric cross-section was solved using the minimum total stationary elastic energy condition (J. S. Przemieniecki 1968 *Theory of Matrix Structural Analysis* McGraw-Hill, New York) and the Newton-Raphson method (S. C. Chapra and R. P. Canale 1998 *Numerical Methods for Engineers* McGraw-Hill Book Company). The consideration was restricted to the elastic structures. The example presented in the paper helps to assess the correctness of the proposed solution. This article is an addition to the author's considerations contained in M. Kujawa 2012 *TASK Quart.* **16** (1/2) 5.

Keywords: thin-walled bars, torsional buckling, energetic approach, Newton-Raphson's method

1. Introduction

The advances in the computer methods have brought a significant development of the theory and methods of structural stability analysis. This is due to the growing needs and requirements of the engineering practice, and tends to a more thorough description and detailed interpretation of the phenomena that occur during the loss of stability of a structure. A classic problem of initial stability, which occurs with rod buckling, is usually solved by conventional analysis. Then, we split the elastic stiffness matrix and geometric stiffness matrix, solving the eigenproblem and finding the eigenvalues [1]. With the development of numerical methods, much attention was also paid to the base of the nonlinear theory of elastic stability of structures, subjected to conservative loads. In the analysis of stability on exposure to conservative loads it is the limit state criterion that is commonly used due to the zero determinant of a stiffness matrix ($\det(\boldsymbol{K})$)

(Figure 2) taking into account the boundary conditions of the problem. This is due to the ease of calculating the value of the determinant. It is also relatively easy to perform an analysis of incremental search limits [2]. In the article, the iterative method of Newton-Raphson (NR) was used to solve the problem [3]. The analysis presented in this article has been described generally in the literature: [4–6] and was performed with the use of the commercial computing packages: Mathematica [7], MATLAB [8] and ABAQUS [9].

2. Solving the problem

Let us consider an element ik cut from a thin-walled bar of an open cross-section. The element has length l, constant cross-section A and is made of a homogeneous material of Young's module E. Let us define, for element ik loaded with axial force P, the function of the torsion angle $\theta(x)$ from the known differential equation of the torsional buckling in the case of the bar of a bisymmetric open cross-section [10]:

$$EJ_\omega \frac{d^4\theta}{dx^4} + (Pr^2 - GJ_d)\frac{d^2\theta}{dx^2} = 0 \tag{1}$$

where the square radius of inertia of the cross-section relative to the coordinate system beginning was determined by $r^2 = J_o/A$. The general solution of Equation (1) is:

$$\theta(x) = -\frac{cos(\overline{\kappa}x)}{\overline{\kappa}^2}\overline{C}_1 - \frac{sin(\overline{\kappa}x)}{\overline{\kappa}^2}\overline{C}_2 + xC_3 + C_4 \tag{2}$$

where $\overline{\kappa} = \sqrt{\frac{Pr^2 - GJ_d}{EJ_\omega}}$ is a characteristic number which depends on the pure torsional rigidity ratio GJ_d, understood in the sense of Sait-Venant's theory, and the sectorial rigidity EJ_ω. Adopting the new variables C_1 and C_2 instead of $-\frac{\overline{C}_1}{\overline{\kappa}^2}$ and $-\frac{\overline{C}_2}{\overline{\kappa}^2}$ we obtain the final form of the equation of the angle of torsion:

$$\theta(x) = \cos(\overline{\kappa}x)C_1 + \sin(\overline{\kappa}x)C_2 + xC_3 + C_4 \tag{3}$$

Substituting the boundary conditions of the form:

$$\begin{aligned} &\theta = \theta_i,\ \theta' = \theta'_i \text{ for } x = 0 \\ &\theta = \theta_k,\ \theta' = \theta'_k \text{ for } x = l \end{aligned} \tag{4}$$

at both ends of the bar ik into Equation (3), we will set the constants: C_1, C_2, C_3, C_4. Let us write the equation of the angle of torsion bar (3) in a matrix form:

$$\theta(x) = \boldsymbol{\Phi}^T \boldsymbol{C}\boldsymbol{q} \tag{5}$$

where $\boldsymbol{\Phi}$ and $\boldsymbol{q}$ are vectors of the form:

$$\begin{aligned} &\boldsymbol{\Phi}^T = \{\cos(\overline{\kappa}x), sin(\overline{\kappa}x), x, 1\} \\ &\boldsymbol{q}^T = \{\theta_i, \theta'_i, \theta_k, \theta'_k\} \end{aligned} \tag{6}$$

and $\boldsymbol{C}$ is a matrix of elements dependent of the characteristic number $\overline{\kappa}$ and the length of the bar l (see Appendix).

The elastic energy is given by:

$$U=\frac{1}{2}\left\{\int_0^l\left[EJ_\omega(\theta'')^2+GJ_d(\theta')^2\right]dx-EJ_o\int_0^l\left[u_o'(\theta')^2\right]dx\right\} \tag{7}$$

where J_o is the polar moment of inertia relative to the center of gravity of the cross-section, and u_o' is:

$$u_o'=\frac{P}{EA} \tag{8}$$

where P is the axial force. Substituting the previously derived equation of the angle of torsion, in the matrix form (5) in the relation (7), we obtain:

$$\frac{1}{2}\boldsymbol{q}^T EJ_\omega\int_0^l\left[\left(\boldsymbol{C}^T\boldsymbol{\Phi}''\right)\left(\boldsymbol{\Phi}''\right)^T\boldsymbol{C}-\overline{\kappa}^2\left(\boldsymbol{C}^T\boldsymbol{\Phi}'\right)\left(\boldsymbol{\Phi}'\right)^T\boldsymbol{C}\right]dx\,\boldsymbol{q}=\frac{1}{2}\boldsymbol{q}^T\boldsymbol{K}\boldsymbol{q} \tag{9}$$

By implementing the operations from the formula (9), we get the searched stiffness matrix:

$$\boldsymbol{K}=EJ_\omega\begin{bmatrix}k_{11} & k_{12} & k_{13} & k_{14}\\ & k_{22} & k_{23} & k_{24}\\ & & k_{33} & k_{34}\\ \text{sym.} & & & k_{44}\end{bmatrix} \tag{10}$$

where:

$$[k_{11}]=\frac{\overline{\kappa}^3}{2\tan\left(\frac{\overline{\kappa}l}{2}\right)-\overline{\kappa}l},\qquad [k_{12}]=\frac{\overline{\kappa}^2}{2-\overline{\kappa}l\cot\left(\frac{\overline{\kappa}l}{2}\right)}$$

$$[k_{13}]=\frac{\overline{\kappa}^3}{\overline{\kappa}l-2\tan\left(\frac{\overline{\kappa}l}{2}\right)},\qquad [k_{14}]=\frac{\overline{\kappa}^2}{2-\overline{\kappa}l\cot\left(\frac{\overline{\kappa}l}{2}\right)}$$

$$[k_{22}]=\frac{\overline{\kappa}(\overline{\kappa}l\cos(\overline{\kappa}l)-\sin(\overline{\kappa}l))}{\overline{\kappa}l\sin(\overline{\kappa}l)+2\cos(\overline{\kappa}l)-2},\qquad [k_{23}]=\frac{\overline{\kappa}^2}{\overline{\kappa}l\cot\left(\frac{\overline{\kappa}l}{2}\right)-2}$$

$$[k_{24}]=\frac{\overline{\kappa}(\sin(\overline{\kappa}l)-\overline{\kappa}l)}{\overline{\kappa}l\sin(\overline{\kappa}l)+2\cos(\overline{\kappa}l)-2},\qquad [k_{33}]=\frac{\overline{\kappa}^3}{2\tan\left(\frac{\overline{\kappa}l}{2}\right)-\overline{\kappa}l}$$

$$[k_{34}]=\frac{\overline{\kappa}^2}{\overline{\kappa}l\cot\left(\frac{\overline{\kappa}l}{2}\right)-2},\qquad [k_{44}]=\frac{\overline{\kappa}(\overline{\kappa}l\cos(\overline{\kappa}l)-\sin(\overline{\kappa}l))}{\overline{\kappa}l\sin(\overline{\kappa}l)+2\cos(\overline{\kappa}l)-2}$$

3. Numerical example

Let us consider the I-section bar ($E=70\,\text{GPa}$, $v=0.33$) loaded by compressive concentrated force P (Figure 1). The numerical example has been analyzed in the articles [1, 11]. The critical force in the I-bar (Figure 1) is 3342.56 kN. The process of numerical analysis using the NR method is well known [3]. Therefore, it will not be described in detail here. During the study, the commercial computational package – MATLAB [8] was used. The critical force values in accordance with the initial graphics solution (Figure 2) were sought in the range from 0 to 4000 kN. The resulting critical value of the torsional force is consistent with the solutions given in the articles [1, 11]. The summary of the results of successive iterations is given in Table 1. The result of the analytical solution given in Table 1 is compared with the space solution – shell model (Figure 3) [9]. The difference between the solutions for the torsional buckling critical force does not exceed 0.5% (*cf.* Table 1 and Figure 3).

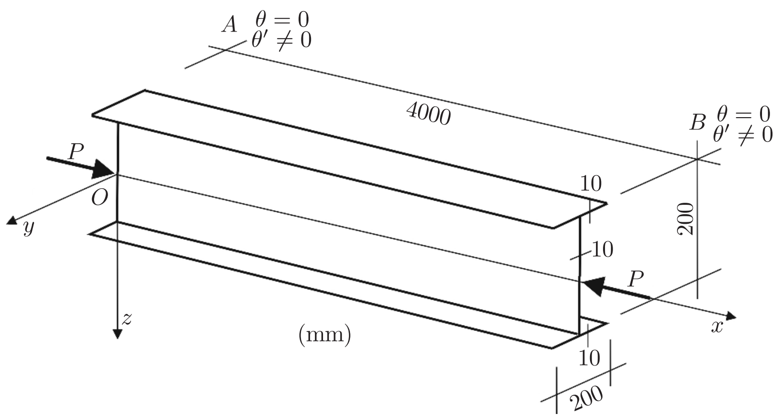

Figure 1. Schematic diagram

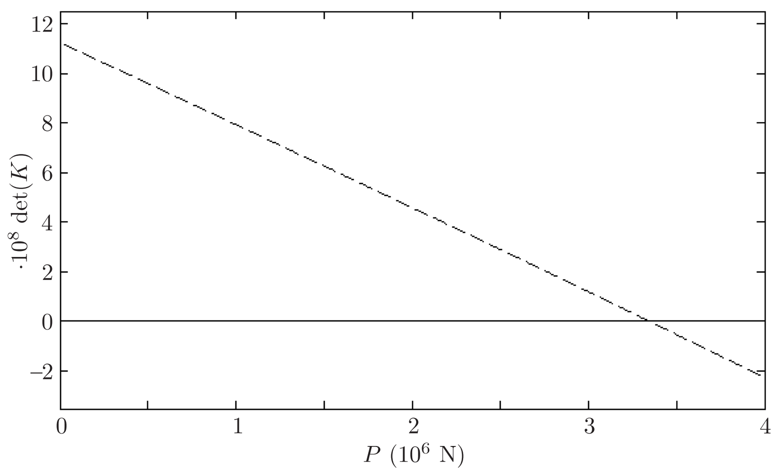

Figure 2. Graphic solution – bar model

Table 1. Critical force – bar model, analytical solution

Iteration	1	2	3	4
The value of critical force (kN)	3414.261	3342.614	3342.565	3342.565
Error (%)	–	2.1434	0.0014	0

4. Summary

The analytical solution presented in this article turns out to be effective only in the case of systems with a small number of the degrees of freedom. The analysis requires the use of large size matrices, and thus significantly increases the computation time. In addition, the computer implementation seems to be not general enough as the use of standard software is limited.

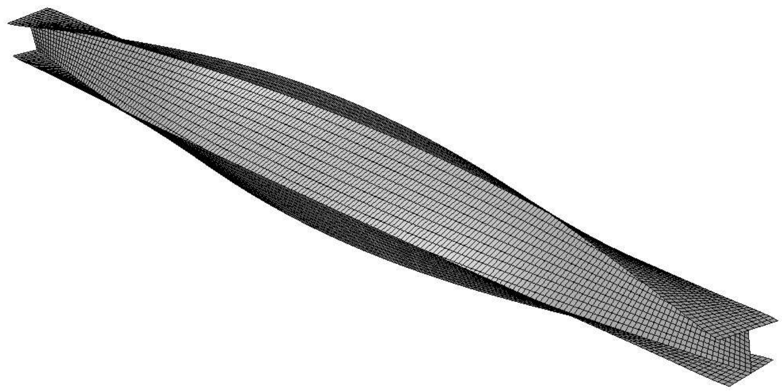

Figure 3. Torsional buckling form (second form): $P = 3354.6\,\mathrm{kN}$ – shell model, numerical solution – ABAQUS

Acknowledgements

The calculations presented in this paper were carried out at the TASK Academic Computer Center in Gdansk, Poland.

Appendix

$$\boldsymbol{C} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{bmatrix} \tag{11}$$

where:

$$[c_{11}] = \frac{\overline{\kappa}\cos(\overline{\kappa}l) - \overline{\kappa}}{\overline{\kappa}(2\cos(\overline{\kappa}l) + \overline{\kappa}l\sin(\overline{\kappa}l) - 2)}, \quad [c_{12}] = \frac{\overline{\kappa}l\cos(\overline{\kappa}l) - \sin(\overline{\kappa}l)}{\overline{\kappa}(2\cos(\overline{\kappa}l) + \overline{\kappa}l\sin(\overline{\kappa}l) - 2)}$$

$$[c_{13}] = \frac{\overline{\kappa} - \overline{\kappa}\cos(\overline{\kappa}l)}{\overline{\kappa}(2\cos(\overline{\kappa}l) + \overline{\kappa}l\sin(\overline{\kappa}l) - 2)}, \quad [c_{14}] = \frac{\sin(\overline{\kappa}l) - \overline{\kappa}l}{\overline{\kappa}(2\cos(\overline{\kappa}l) + \overline{\kappa}l\sin(\overline{\kappa}l) - 2)}$$

$$[c_{21}] = \frac{\sin(\overline{\kappa}l)}{2\cos(\overline{\kappa}l) + \overline{\kappa}l\sin(\overline{\kappa}l) - 2}, \quad [c_{22}] = \frac{\cos(\overline{\kappa}l) + \overline{\kappa}l\sin(\overline{\kappa}l) - 1}{\overline{\kappa}(2\cos(\overline{\kappa}l) + \overline{\kappa}l\sin(\overline{\kappa}l) - 2)}$$

$$[c_{23}] = \frac{-\sin(\overline{\kappa}l)}{2\cos(\overline{\kappa}l) + \overline{\kappa}l\sin(\overline{\kappa}l) - 2}, \quad [c_{24}] = \frac{1 - \cos(\overline{\kappa}l)}{\overline{\kappa}(2\cos(\overline{\kappa}l) + \overline{\kappa}l\sin(\overline{\kappa}l) - 2)}$$

$$[c_{31}] = \frac{-\overline{\kappa}\cos\left(\frac{\overline{\kappa}l}{2}\right)}{\overline{\kappa}l\cos\left(\frac{\overline{\kappa}l}{2}\right) - 2\sin\left(\frac{\overline{\kappa}l}{2}\right)}, \quad [c_{32}] = \frac{-\sin\left(\frac{\overline{\kappa}l}{2}\right)}{\overline{\kappa}l\cos\left(\frac{\overline{\kappa}l}{2}\right) - 2\sin\left(\frac{\overline{\kappa}l}{2}\right)}$$

$$[c_{33}] = \frac{\overline{\kappa}\cos\left(\frac{\overline{\kappa}l}{2}\right)}{\overline{\kappa}l\cos\left(\frac{\overline{\kappa}l}{2}\right) - 2\sin\left(\frac{\overline{\kappa}l}{2}\right)}, \quad [c_{34}] = \frac{-\sin\left(\frac{\overline{\kappa}l}{2}\right)}{\overline{\kappa}l\cos\left(\frac{\overline{\kappa}l}{2}\right) - 2\sin\left(\frac{\overline{\kappa}l}{2}\right)}$$

$$[c_{41}] = \frac{l\sin(\overline{\kappa}l)\overline{\kappa}^2 + \cos(\overline{\kappa}l)\overline{\kappa} - \overline{\kappa}}{\overline{\kappa}(2\cos(\overline{\kappa}l) + \overline{\kappa}l\sin(\overline{\kappa}l) - 2)}, \quad [c_{42}] = \frac{\sin(\overline{\kappa}l) - \overline{\kappa}l\cos(\overline{\kappa}l)}{\overline{\kappa}(2\cos(\overline{\kappa}l) + \overline{\kappa}l\sin(\overline{\kappa}l) - 2)}$$

$$[c_{43}] = \frac{\overline{\kappa}\cos(\overline{\kappa}l) - \overline{\kappa}}{\overline{\kappa}(2\cos(\overline{\kappa}l) + \overline{\kappa}l\sin(\overline{\kappa}l) - 2)}, \quad [c_{44}] = \frac{\overline{\kappa}l - \sin(\overline{\kappa}l)}{\overline{\kappa}(2\cos(\overline{\kappa}l) + \overline{\kappa}l\sin(\overline{\kappa}l) - 2)}$$

References

[1] Kujawa M 2012 *TASK Quart.* **16** (1/2) 5

[2] Rheinboldt W C 1981 *Computer and Structures* **13** 103

[3] Chapra S C and Canale R P 1998 *Numerical Methods for Engineers*, McGraw-Hill Book Company

[4] Hutchinson J W and Koiter W T 1970 *Appl. Mech. Rev.* **23** 1353

[5] Thompson J M T and Hunt G W 1973 *A General Theory of Elastic Stability*, Wiley and Sons

[6] Huseyin K 1975 *Nonlinear Theory of Elastic Stability*, Noordhoff International Publishing

[7] Wolfram S 1991 *Mathematica: a System for Doing Mathematics by Computer*, Second Edition, Addison-Wesley, USA

[8] Kwon Y W and Bang H 2000 *The Finite Element Method Using MATLAB*, Second Edition, CRC Press, New York

[9] ABAQUS, Inc. *ABAQUS Theory Manual*, http://www.abaqus.com

[10] Vlasov V Z 1961 *Thin-walled Elastic Beams*, Second Edition, Israel Program for Scientific Translations, Jerusalem

[11] Szymczak C 1978 *Engineering Transaction* **26** (2) 323 (in Polish)