KEPLER'S ORBITAL MOTION EXAMINED IN COMPARISON WITH THE METHODS OF MECHANICS DEVELOPED BY STANESCU AND TABACU

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Abstract: A new method developed by Stanescu and Tabacu to study the mechanics of a particle moving along a definite curve is compared with the well-known approach applied in calculating the Kepler's orbital motion. A particular effect of the rate of change of the kinetic energy of a body with respect to the change of a parameter defining the body position on the orbit is examined as an example.

A comparison of the two methods shows only a slight numerical difference between the results of the Stanescu-Tabacu method and the traditional formalism. The components of the body acceleration are calculated along the orbit as an application of the new approach.

Keywords: Kepler's orbital motion, mechanics by Stanescu and Tabacu, conventional classical mechanics

1. Introduction

Recently Stanescu and Tabacu have proposed a new method to study the mechanics of a particle moving along a curve without friction [1]. The main idea is to consider the force acting on such a particle together with the constraints imposed by the particle track. If the track is known, its shape allows us to decompose the force $\overrightarrow{F}$ acting on the moving body into tangential (F_t), normal (F_n) and bi-normal (F_b) components (see *e.g.* [2]); we have:

$$\overrightarrow{F} = F_t \overrightarrow{t} + F_n \overrightarrow{n} + F_b \overrightarrow{b} \tag{1}$$

Other useful parameters are the curvature radius ϱ on the track, as well as the change of the track length s traveled by a particle in effect of the change of some

parameter ϑ characterizing the track. For the Cartesian system of coordinates taken as an example, the dependence of the track on ϑ implies that:

$$x = x(\vartheta), \quad y = y(\vartheta), \quad z = z(\vartheta) \tag{2}$$

are the known functions of ϑ. The knowledge of the track allows us to compare the results of the formalism of [1] with the results of some well-established mechanical methods in a direct way.

In the present paper we study a well-known motion of a body along the Kepler orbit upon the action of the gravitational force considered as an example. A planar character of the orbit reduces the force problem to two components, F_t and F_n. Contrary to [1], where only very special cases of the orbits were chosen for consideration, our aim is to examine the motion along a general Kepler orbit. In terms of the Cartesian coordinates the orbit is defined by the components:

$$x = a\cos\vartheta, \qquad y = b\sin\vartheta \tag{3}$$

where a is a major and b is a minor semiaxis of the Kepler ellipse. In further calculations the geometry of the track given in (3) is applied in calculating the force acting on the moving body as well as the directional properties of the motion, see Section 2.

In a conventional approach (see *e.g.* [3, 4]), chosen for comparison, the coordinates x and y are defined in a similar way to (3), namely:

$$x = a\cos u, \qquad y = b\sin u \tag{4}$$

However, a special meaning of the angle between the major semiaxis and the radius of a circle on which the point having coordinates on x and y on the ellipse is projected in direction normal to that semiaxis is attributed for u here, see [3]. These geometrical properties of u, together with the radius of the circle equal to a, are applied in further calculations providing us directly with the kinetic energy of the moving body.

In effect, a comparison done in the present paper between the well-known formalism and the formalism of Stanescu and Tabacu concerns the examination of the kinetic energy changes of the body obtained in the course of its motion along the orbit.

2. Kinetic energy of Kepler's orbital motion calculated according to [1]

Let us assume that the beginning of the motion is in the perihelion point of the orbit.

In this case the force of [1] dependent on ϑ is:

$$\overrightarrow{F} = \frac{Gm_p M}{a^2} \frac{(\sqrt{1-\lambda^2} - \cos\vartheta)\,\overrightarrow{i} - \lambda\sin\vartheta\,\overrightarrow{j}}{[(\sqrt{1-\lambda^2} - \cos\vartheta)^2 + \lambda^2\sin^2\vartheta]^{3/2}} \tag{5}$$

G is the gravitational constant, M is the central mass,

$$\lambda = \frac{b}{a} = \sqrt{1-e^2} \tag{6}$$

where e is the orbit eccentricity. The mass of the moving body is assumed:

$$m_p = 1 \tag{7}$$

for the sake of simplicity.

In the next step, a unit vector $\overrightarrow{t}$ tangential to the orbit can be calculated [1]:

$$\overrightarrow{t} = \frac{-\sin\vartheta\,\overrightarrow{i} + \lambda\cos\vartheta\,\overrightarrow{j}}{\sqrt{\sin^2\vartheta + \lambda^2\cos^2\vartheta}} \tag{8}$$

Together with the unit vector normal to $\overrightarrow{t}$:

$$\overrightarrow{n} = \frac{-\lambda\cos\vartheta\,\overrightarrow{i} - \sin\vartheta\,\overrightarrow{j}}{\sqrt{\sin^2\vartheta + \lambda^2\cos^2\vartheta}} \tag{9}$$

the curvature radius on the orbit is [1]:

$$\varrho = \frac{a}{\lambda}\left(\sin^2\vartheta + \lambda^2\cos^2\vartheta\right)^{3/2} \tag{10}$$

The next step is to calculate the tangential component of the attractive force. This gives:

$$F_t(\vartheta) = \overrightarrow{F} \cdot \overrightarrow{t} = \frac{Gm_p M}{a^2\sqrt{\sin^2\vartheta + \lambda^2\cos^2\vartheta}} \cdot$$
$$\cdot(-1)\frac{\sin\vartheta\sqrt{1-\lambda^2}(1-\cos\vartheta\sqrt{1-\lambda^2})}{[(\sqrt{1-\lambda^2}-\cos\vartheta)^2 + \lambda^2\sin^2\vartheta]^{3/2}} \tag{11}$$

In this formula the printing errors of the corresponding expression in [1] are removed.

The velocity square of the moving body at some orbit point labeled by ϑ is given by:

$$v^2 = v_0^2 + \frac{2}{m_p}\int_0^\vartheta F_t(\vartheta)L(\vartheta)\,d\vartheta \tag{12}$$

The function $L(\vartheta)$ entering the integral in Equation (12) is:

$$L(\vartheta) = a\sqrt{\sin^2\vartheta + \lambda^2\cos^2\vartheta} = \frac{ds}{d\vartheta} \tag{13}$$

where s is the length of the arc traveled by the body along the orbit. In effect, due to (7), the kinetic energy at some ϑ is given by the formula (12), on condition that the factor of $1/2$ is introduced:

$$E_{\text{kin}} = \frac{1}{2}v^2 = \frac{1}{2}v_0^2 + \int_0^\vartheta F_t(\vartheta)L(\vartheta)\,d\vartheta \tag{14}$$

3. Kinetic energy of the Kepler motion presented as a function of the variable u [3]

According to the energy balance at any point of the orbit we have:

$$E_{\text{kin}} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) \equiv \frac{1}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right] \tag{15}$$

where the potential energy is:

$$E_{\text{pot}} = -\frac{GM}{r} \tag{16}$$

and the total energy is:

$$E_{\text{tot}} = -\frac{GM}{2a} \tag{17}$$

r in Equation (16) is the distance between the moving body and the gravitational center.

A reference of u to r can be easily calculated [3, 4]. For the motion beginning in the perihelion point this relation is represented by the formula:

$$r = a(1 - e\cos u) \tag{18}$$

therefore the kinetic energy in Equation (15) becomes:

$$E_{\text{kin}} = E_{\text{tot}} - E_{\text{pot}} = -\frac{GM}{2a} + GM\frac{1}{a(1 - e\cos u)} \tag{19}$$

4. Comparison of calculations presented in Section 2 and Section 3

Since in Equation (19):

$$E_{\text{kin}} = f(u) \tag{20}$$

with the accuracy to a constant term, a comparison of the results of [1] and [3] can be done by differentiating Equation (14) with respect to ϑ and Equation (20) with respect to u. We obtain:

$$\frac{dE_{\text{kin}}}{d\vartheta} = F_t(\vartheta)L(\vartheta) = -\frac{GM}{a}F_1(\vartheta) \tag{21}$$

from Equation (14), and

$$\frac{dE_{\text{kin}}}{du} = -GM\frac{e}{a}\frac{\sin u}{(1 - \cos u)^2} = -\frac{GM}{a}F_2(u) \tag{22}$$

from Equations (19) and (20). The results of both approaches are tabulated in Table 1–3 for different eccentricity values e. Evidently, a considerable agreement between the data of the methods of [1] and [3] is attained.

5. Application of the method of [1] in calculating the dependence of the body acceleration on the orbit

The method of [1] provides us immediately with the tangential component of acceleration of the body motion on the orbit:

$$a_t = F_t(\vartheta) = -\frac{GM}{a}\frac{F_1(\vartheta)}{L(\vartheta)} = -\frac{GM}{a}F_3(\vartheta) \tag{23}$$

due to (7).

Table 1. Comparison of $F_1(\vartheta)$ from Equation (21) with $F_2(u)$ from Equation (22) for $e = 1/\sqrt{2}$; angles $\vartheta = u$ are given in degrees

angles	F_1	F_2
0	0.0000 0	0.00000
15	1.82174	1.82093
30	2.35323	2.35271
45	2.00000	1.99988
60	1.46542	1.46531
75	1.02328	1.02321
90	0.70710	0.70710
105	0.48804	0.48803
120	0.33424	0.33423
135	0.22222	0.22222
150	0.13600	0.13600
165	0.06461	0.06461
180	0.00000	0.00000

Table 2. Comparison of $F_1(\vartheta)$ from Equation (21) with $F_2(u)$ from Equation (22) for $e = 1/\sqrt{3}$; angles $\vartheta = u$ are given in degrees

angles	F_1	F_2
0	0.0000 0	0.00000
15	0.76389	0.76365
30	1.15483	1.15463
45	1.16589	1.16583
60	0.98823	0.98815
75	0.77086	0.77080
90	0.57735	0.57735
105	0.42212	0.42210
120	0.30108	0.30107
135	0.20586	0.20586
150	0.12831	0.12830
165	0.06158	0.06158
180	0.00000	0.00000

Table 3. Comparison of $F_1(\vartheta)$ from Equation (21) with $F_2(u)$ from Equation (22) for $e = 1/\sqrt{4} = 1/2$; angles $\vartheta = u$ are given in degrees

angles	F_1	F_2
0	0.0000 0	0.00000
15	0.48415	0.48403
30	0.77775	0.77763
45	0.84606	0.84602
60	0.76985	0.76978
75	0.63724	0.63718
90	0.50000	0.50000
105	0.37864	0.37862
120	0.27713	0.27712
135	0.19298	0.19298
150	0.12175	0.12117
165	0.05884	0.05884
180	0.00000	0.00000

On the other hand, the method of [3] combined with that of [1] gives the normal component of the acceleration of the moving body on the orbit:

$$a_n = \frac{\dot{x}^2 + \dot{y}^2}{\varrho} = \frac{\frac{GM}{a}\left(\frac{1}{1-e\cos\vartheta} - \frac{1}{2}\right)}{\frac{a}{\lambda}(\sin^2\vartheta + \lambda^2\cos^2\vartheta)^{3/2}} = \frac{GM}{a^2}\lambda F_4(\vartheta) \qquad (24)$$

Here the formula for the curvature radius of [1] (see Equation (10)) is combined with that for the kinetic energy in Equation (19); $\vartheta = u$.

The data for $F_3(\vartheta)$ and $F_4(\vartheta)$ obtained from Equations (23) and (24), respectively, are plotted in Figures 1–2.
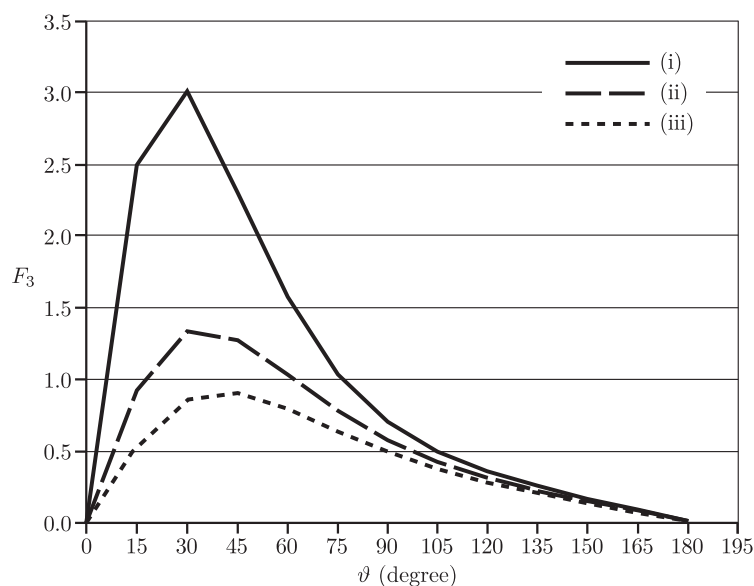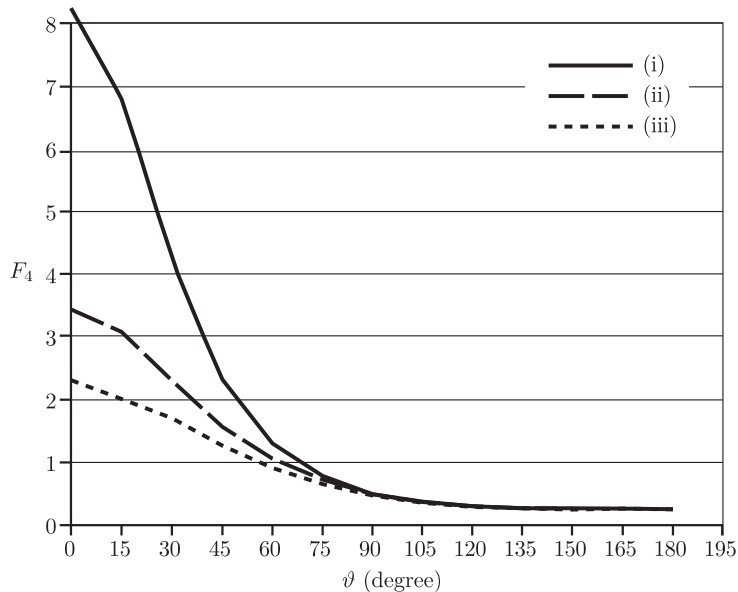

Figure 1. The plot of F_3 entering the formula (23) for the acceleration component a_t presented as a function of ϑ. The upper curve (i) is for $e = 1/\sqrt{2}$, middle curve (ii) is for $e = 1/\sqrt{3}$, lower curve (iii) is for $e = 1/\sqrt{4} = 1/2$

A characteristic point is that for $\vartheta = \frac{\pi}{2}$ the same value $F_4 = 0.5$ is obtained for any e, whereas for $\vartheta = 180°$ the values of F_4 for all e listed in Figure 2 are equal approximately to 0.25. Evidently, for $\vartheta = 180°$ and $e = 0$ the values of F_4 rise to 0.5.

6. Conclusions

The paper concerns a comparison of a new approach to the mechanics of a moving particle proposed recently by Stanescu and Tabacu [1] with the well-known mechanical methods. In [1] such a comparison is limited solely to very special points of the body motions on their tracks neglecting a general check of the motion as a whole.

The present paper bridges this gap for the case of the body motion along Kepler's orbit by extending the calculations of the kinetic energy of the moving

Figure 2. The plot of F_4 entering the formula (24) for the acceleration component a_n presented as a function of ϑ. The upper curve (i) is for $e = 1/\sqrt{2}$, middle curve (ii) is for $e = 1/\sqrt{3}$, lower curve (iii) is for $e = 1/\sqrt{4} = 1/2$

body practically to the arbitrary points on the Kepler track. An agreement between both approaches, that of [1] and the traditional one, is satisfactory.

References

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