CONVOLUTION INTEGRAL IN TRANSIENT PIPE FLOW

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Abstract: This paper is devoted to the modeling of hydraulic losses during transient flow of liquids in pressure lines. Unsteady pipe wall shear stress is presented in the form of a convolution integral of liquid acceleration and a weighting function. The weighting function depends on the dimensionless time and the Reynolds number. In its first revision (Zielke W 1968 *J. ASME* **90** 109) it had a complex and inefficient mathematical structure (featured power growth of computational time). Therefore, further work aimed at developing the so-called efficient models for correct estimation of hydraulic resistance with simultaneous linear loading of the computer's operating memory was needed. The work compared the methods of numerical solving of the convolution integral known from the literature (classic by Zielke W 1968 *J. ASME* **90** 109 and Vardy A E and Brown J M B 2010 *J. Hydraul. Eng.* **136** (7) 453 and efficient by Trikha A K 1975 *J. Fluids Eng.* p. 97, Kagawa T *et al.* 1983 *Trans. Jpn. Soc. Mech. Eng.* **49** (447) 2638 and Schohl G A 1993 *J. Fluids Eng.* **115** 420). The comparison highlighted the level of usefulness of the analyzed models in simulating the water hammer and revealed the demand for further research for the improvement of efficiency of the solutions.

Keywords: numerical fluid mechanics, transient flow, hydraulic resistance, convolution integral

1. Introduction

Many studies of unsteady flows of liquids through pressure lines assume that hydraulic losses are quasi-steady. The models provide correct results only for low frequencies or the slow velocity variation, *i.e.*, for quasi-steady flows. The approach seen in many contemporary works is that the instantaneous pipe wall shear stress τ can be presented in the form of a sum of quasi-steady quantities τ_q and quantity τ_u [1–20] variable in time:

$$\tau = \tau_q + \tau_u \tag{1}$$

Quantity τ_q is determined based on the transformed Darcy-Weisbach formula:

$$\tau_q = \frac{1}{8}\lambda\rho v|v| \tag{2}$$

where: λ – friction factor, ρ – liquid density, v – instantaneous flow velocity.

It is known that during a laminar flow liquid, molecules fill porous pipe cavities, creating a smooth "sliding surface". Many experiments have confirmed this liquid behavior. In this scenario it is assumed that hydraulic resistance is independent of the pipeline wall porosity and depends on the value of the Reynolds number only. The flow remains laminar until the critical Reynolds number value (approx. 2320) is exceeded. The friction factor in laminar flow is calculated using the Hagen-Poiseuille law:

$$\lambda = \frac{64}{\mathrm{Re}} \tag{3}$$

Once the critical value of the Reynolds number is exceeded, the flow becomes turbulent and the friction loss coefficient for coarse pipes can be computed from the Colebrook-White dependence:

$$\frac{1}{\sqrt{\lambda}} = -2\lg\left(\frac{2.51}{\mathrm{Re}\sqrt{\lambda}} + \frac{\varepsilon}{3.7D}\right) \tag{4}$$

where: ε/D – relative roughness of internal pipe walls.

For hydraulically smooth pipes the friction loss coefficient can be computed from the Prandtl-Karman equation:

$$\frac{1}{\sqrt{\lambda}} = 2\lg\left(\mathrm{Re}\sqrt{\lambda}\right) - 0.8 \tag{5}$$

Experimental results have shown that the foregoing equation (5) features fit very well for a single-phase flow for any large Reynolds number.

2. Modeling pipe wall shear stress

Zielke [20] has presented an analytical solution enabling determination of unsteady fiction losses (instantaneous pipe wall shear stress in the form of the convolution integral from local acceleration of liquid and a weighting function) for laminar flow. Zielke's model can be easily used in equations describing a 1D unsteady flow, including specifically the popular method of characteristics (MOC).

In his deliberations, Zielke has referred to the dependence presented in the paper by Brown [21], describing the impedance of a hydraulic line as a function of frequency:

$$Z_0(s) = \frac{\frac{\rho s}{\pi R^2}}{1 - \frac{2J_1\left(jR\sqrt{\frac{s}{\nu}}\right)}{jR\sqrt{\frac{s}{\nu}}\cdot J_0\left(jR\sqrt{\frac{s}{\nu}}\right)}} \tag{6}$$

where: s – Laplace transformation operator; ν – kinematic viscosity coefficient; J_0 and J_1 – Bessel functions of the first type of orders 0 and 1; j – imaginary unit; R – internal pipe radius.

By reversing the Laplace transformation, Zielke has obtained the following dependence for the instantaneous pipe wall shear stress for laminar flow [20]:

$$\tau(t) = \underbrace{\frac{4\mu}{R}v}_{\tau_q} + \underbrace{\frac{2\mu}{R}\int_0^t w(t-u)\cdot\frac{\partial v}{\partial t}(u)du}_{\tau_u} \tag{7}$$

where: $w(t)$ – weighting function, μ – dynamic viscosity coefficient.

The first expression, τ_q, of the foregoing equation (7) represents a quasi-steady quantity (being a result of inserting the expression for linear resistance rate (3) in Equation (2) for laminar flow).

The second expression, τ_u, describes the effect of the flow unsteadiness on the wall shear stress. It is the convolution integral from instantaneous liquid acceleration and a weighting function:

$$w(\hat{t}) = \sum_{i=1}^{6} m_i \hat{t}^{(i-2)/2}, \text{ for } \hat{t} \leq 0.2 \tag{8}$$

$$w(\hat{t}) = \sum_{i=1}^{5} e^{-n_i\cdot\hat{t}}, \text{ for } \hat{t} > 0.02 \tag{9}$$

where: $\hat{t} = (\nu/R^2)\cdot t$ – the dimensionless time and $m_1 = 0.282095$; $m_2 = -1.25$; $m_3 = 1.057855$; $m_4 = 0.9375$; $m_5 = 0.396696$; $m_6 = -0.351563$; $n_1 = 26.3744$; $n_2 = 70.8493$; $n_3 = 135.0198$; $n_4 = 218.9216$; $n_5 = 322.5544$.

The component of the instantaneous pipe wall shear stress, τ_u, which is variable in time, can be computed numerically using the differential approximation of the first order [20]:

$$\begin{aligned}\tau_u &= \frac{2\mu}{R}\sum_{j=1,3,\ldots}^{n-2}(v_{i,j+2} - v_{i,j})\cdot w\big((n-1-j)\Delta\hat{t}\big) \\ &= \frac{2\mu}{R}\sum_{j=1,3,\ldots}^{n-2}(v_{i,n-j+1} - v_{i,n-j-1})\cdot w(j\Delta\hat{t})\end{aligned} \tag{10}$$

where: i – the number of the subsequent computational pressure pipe cross-section changing from 1 to h; j – the number of the computational time step changing with the increment of 2 from 1 to n for $n \geq 3$; $\Delta\hat{t} = \big(v/R^2\big)\cdot\Delta t$; Δt – the time step in numerical analysis.

In the method of characteristics based on a rectangular grid the foregoing equation (10) can be written as follows [3]:

$$\begin{aligned}\tau_u &= \frac{2\mu}{R}\sum_{j=1}^{n-1}(v_{i,j+1}-v_{i,j})\cdot w\left((n-j)\Delta\hat{t}-\frac{\Delta\hat{t}}{2}\right)\\ &= \frac{2\mu}{R}\sum_{j=1}^{n-1}(v_{i,n-j+1}-v_{i,n-j})\cdot w\left(j\Delta\hat{t}-\frac{\Delta\hat{t}}{2}\right)\end{aligned} \tag{11}$$

where: j – the number of the computational time step changing with the increment of 1 from 1 to n for $n \geq 2$.

An analysis of the two last equations explains why the solution of the convolution integral by Zielke is inefficient. This is because the number of expressions representing the instantaneous value of the wall shear stress increases as part of the numerical process with each successive time step "j".

In time, it has been demonstrated [10–14, 18, 19] that the dependence (7) can be also used for transient turbulent flows. However, the weighting function in turbulent flow has no fixed pattern, as for the laminar flow. Its shape varies depending on the conditions: namely the value of the Reynolds number.

Based on the 2D (axial-symmetric) Reynolds equation, the Boussinesq hypothesis and experimental data (concerning the turbulent viscosity coefficient in the pipe cross-section), Vardy-Brown and Zarzycki have proposed their own weighting functions for turbulent flow:

- **Vardy-Brown model** [12]

$$w(\hat{t},\mathrm{Re}) = \frac{A^* e^{-B^*\hat{t}}}{\sqrt{\hat{t}}} \tag{12}$$

where: $A^* = \sqrt{1/4\pi}$ and $B^* = \mathrm{Re}^{\kappa}/12.86$; $\kappa = \log_{10}(15.29/\mathrm{Re}^{0.0567})$;

- **Zarzycki model** [19]

$$w(\hat{t},\mathrm{Re}) = C\frac{1}{\sqrt{\hat{t}}}\mathrm{Re}^n \tag{13}$$

where: $C = 0.299635$; $n = -0.005535$.

The foregoing dependences (12) and (13) for the weighting function can be used within the $2000 \leq \mathrm{Re} \leq 10^8$ range of the Reynolds number.

Vardy and Brown [9] have proposed an adjustment to the classic solution by Zielke (11) consisting in computing the integral from the weighting function. This approach having been adopted, numerical simulations begin to reflect the actual change of the wall shear stress more accurately (*inter alia*, they avoid the error in determining hydraulic resistance for the accelerated flow; see [9]).

The integral derived from Equation (8) for dimensionless time $\hat{t}$ is:

$$I_1=\left(\frac{R^2}{v}\right)\left[2m_1\cdot\hat{t}^{0.5}+m_2\cdot\hat{t}^1+\frac{2}{3}m_3\cdot\hat{t}^{1.5}+\frac{1}{2}m_4\cdot\hat{t}^2+\frac{2}{5}m_5\cdot\hat{t}^{2.5}+\frac{1}{3}m_6\cdot\hat{t}^3\right] \quad (14)$$

Whereas, for Equation (9), the integral is:

$$I_2=\left(\frac{R^2}{v}\right)\sum_{i=1}^{5}\left(-\frac{1}{n_i}\right)e^{n_i\cdot\hat{t}} \quad (15)$$

A modified solution proposed by Vardy-Brown:

$$\tau_u=\frac{2\mu}{R}\sum_{j=1}^{n-1}\frac{(v_{i,j+1}-v_{i,j})}{\Delta t}\cdot I\Big|_{(n-j-1)\cdot\Delta\hat{t}}^{(n-j)\cdot\Delta\hat{t}} \quad (16)$$

where: $I=I_1$ when $(n-j)\Delta\hat{t}\leq 0.02$ and $I=I_2$ when $(n-j)\Delta\hat{t}>0.02$.

3. Efficient solution of convolution integral

Trikha [5] was the first to present an efficient numerical solution of the convolution integral (second expression in Equation (7)) in 1975:

$$\tau_u(t+\Delta t)\approx\frac{2\mu}{R}\sum_{i=1}^{j}\underbrace{\left[y_i(t)\cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot\Delta t}+m_i\cdot[v_{(t+\Delta t)}-v_t]\right]}_{y_i(t+\Delta t)} \quad (17)$$

To obtain the foregoing solution, it was necessary to write the weighting function in the form of a finite sum of exponential expressions:

$$w(\hat{t})=\sum_{i=1}^{j}m_ie^{-n_i\cdot\hat{t}} \qquad \text{where: } i=1,2,\ldots,j \quad (18)$$

this is because only this form of the function allows an efficient solution of the convolution integral to be achieved.

As Trikha has made too many simplifications while deriving his equations for the efficient solution of convolution integral (17) and (18), Kagawa *et al.* [2], and then Schohl [4], have proposed more accurate solutions.

Schohl's solution is slightly different from the solution proposed by Kagawa *et al.* See the following for derivation of the solutions:

$$\begin{aligned}\tau_u(t)&=\frac{2\mu}{R}\int_0^t w_{\text{apr}}(t-u)\cdot\frac{\partial v}{\partial u}(u)du=\frac{2\mu}{R}\int_0^t\sum_{i=1}^{j}w_i(t-u)\cdot\frac{\partial v}{\partial u}(u)du\\&=\frac{2\mu}{R}\sum_{i=1}^{j}\int_0^t w_i(t-u)\cdot\frac{\partial v}{\partial u}(u)du\end{aligned} \quad (19)$$

$$\tau_u(t)=\frac{2\mu}{R}\sum_{i=1}^{j}y_i(t) \quad (20)$$

$$
\begin{aligned}
y_i(t) &= \int_0^t w_i(t-u)\cdot\frac{\partial v}{\partial u}(u)du = \int_0^t m_i\cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot(t-u)}\cdot\frac{\partial v}{\partial u}(u)du \\
y_i(t) &= m_i\cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot t}\cdot\int_0^t e^{n_i\cdot\frac{\nu}{R^2}\cdot u}\cdot\frac{\partial v}{\partial u}(u)du
\end{aligned}
\tag{21}
$$

Using the method of characteristics to solve the system of partial differential equations describing a transient flow requires that the computation is performed for certain predefined time steps Δt. The notation for the subsequent time step can be as follows:

$$
\begin{aligned}
y_i(t+\Delta t) &= \int_0^{t+\Delta t} w_i(t+\Delta t-u)\cdot\frac{\partial v}{\partial u}(u)du = \\
&= \int_0^t w_i(t+\Delta t-u)\cdot\frac{\partial v}{\partial u}(u)du + \int_t^{t+\Delta t} w_i(t+\Delta t-u)\cdot\frac{\partial v}{\partial u}(u)du = \\
&= \int_0^t m_i\cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot(t+\Delta t-u)}\cdot\frac{\partial v}{\partial u}(u)du + \int_t^{t+\Delta t} m_i\cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot(t+\Delta t-u)}\cdot\frac{\partial v}{\partial u}(u)du = \\
&= \underbrace{e^{-n_i\cdot\frac{\nu}{R^2}\cdot\Delta t}\cdot\int_0^t m_i\cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot(t-u)}\cdot\frac{\partial v}{\partial u}(u)du}_{y_i(t)} + \\
&\quad \underbrace{+m_i\cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot(t+\Delta t)}\cdot\int_t^{t+\Delta t} e^{n_i\cdot\frac{\nu}{R^2}\cdot u}\cdot\frac{\partial v}{\partial u}(u)du}_{\Delta y_i(t)}
\end{aligned}
\tag{22}
$$

Ultimately:

$$
y_i(t+\Delta t) = y_i(t)\cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot\Delta t} + \Delta y_i(t) \tag{23}
$$

where:

$$
\Delta y_i(t) = m_i\cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot(t+\Delta t)}\cdot\int_t^{t+\Delta t} e^{n_i\cdot\frac{\nu}{R^2}\cdot u}\cdot\frac{\partial v}{\partial u}(u)du \tag{24}
$$

Assuming for the foregoing expression that function $v(u)$ is linear function $[v(u)=au+b]$ within the range $\langle t,t+\Delta t\rangle$, its derivative after time $\partial v(u)/\partial u$ can be considered as a constant, the value of which is computed as follows:

$$
\frac{[v_{(t+\Delta t)}-v_t]}{\Delta t} \tag{25}
$$

Given this assumption, $\Delta y_i(t)$ can be written as follows, as in Schohl [4]:

$$\begin{aligned}\Delta y_i(t) &\approx m_i \cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot(t+\Delta t)} \cdot \frac{\left[v_{(t+\Delta t)} - v_t\right]}{\Delta t} \cdot \int\limits_t^{t+\Delta t} e^{n_i\cdot\frac{\nu}{R^2}\cdot u} du = \\ &= m_i \cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot(t+\Delta t)} \cdot \frac{\left[v_{(t+\Delta t)} - v_t\right]}{\Delta t} \cdot \frac{R^2}{n_i \nu} \cdot \left[e^{n_i\cdot\frac{\nu}{R^2}\cdot(t+\Delta t)} - e^{n_i\cdot\frac{\nu}{R^2}\cdot t}\right] \\ &= \frac{m_i R^2}{\Delta t n_i \nu} \cdot \left[1 - e^{-n_i\cdot\frac{\nu}{R^2}\cdot\Delta t}\right] \cdot \left[v_{(t+\Delta t)} - v_t\right]\end{aligned} \tag{26}$$

Or as follows, as in Kagawa *et al.* [2]:

$$\begin{aligned}\Delta y_i(t) &\approx m_i \cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot(t+\Delta t)} \cdot \frac{\left[v_{(t+\Delta t)} - v_t\right]}{\Delta t} \cdot \int\limits_t^{t+\Delta t} e^{n_i\cdot\frac{\nu}{R^2}\cdot u} du = \\ &= m_i \cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot(t+\Delta t)} \cdot \frac{\left[v_{(t+\Delta t)} - v_t\right]}{\Delta t} \cdot e^{n_i\cdot\frac{\nu}{R^2}\cdot\left(t+\frac{\Delta}{t}2\right)} \cdot \int\limits_t^{t+\Delta t} du = \\ &= m_i \cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot\left(\frac{\Delta t}{2}\right)} \cdot \frac{\left[v_{(t+\Delta t)} - v_t\right]}{\Delta t} \cdot (t+\Delta t - t) = \\ &= m_i \cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot\left(\frac{\Delta t}{2}\right)} \cdot \left[v_{(t+\Delta t)} - v_t\right]\end{aligned} \tag{27}$$

The final efficient numerical solution of the convolution integral by Schohl is as follows:

$$\tau_u(t+\Delta t) \approx \frac{2\mu}{R} \cdot \sum_{i=1}^{j} \underbrace{\left[y_i(t) \cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot\Delta t} + \frac{m_i R^2}{\Delta t n_i \nu} \cdot \left[1 - e^{-n_i\cdot\frac{\nu}{R^2}\cdot\Delta t}\right] \cdot \left[v_{(t+\Delta t)} - v_t\right]\right]}_{y_i(t+\Delta t)} \tag{28}$$

While the solution by Kagawa *et al.* is the following:

$$\tau_u(t+\Delta t) \approx \frac{2\mu}{R} \cdot \sum_{i=1}^{j} \underbrace{\left[y_i(t) \cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot\Delta t} + m_i \cdot e^{-n_i\cdot\frac{\nu}{R^2}\cdot\left(\frac{\Delta t}{2}\right)} \cdot \left[v_{(t+\Delta t)} - v_t\right]\right]}_{y_i(t+\Delta t)} \tag{29}$$

As the simulation starts from the steady flow ($v = \text{const.}$), the wall shear stress parameter, τ_u, occurring during the transient flow and the values of all components $y_i(t)$ is equal to 0 in the first computational time step. In each subsequent time step the values of components change according to Equation (23).

4. Simulation results

The following section presents the results of illustrative simulations of fluctuations of parameter τ_u using the solutions of the convolution integral (3 efficient and 2 inefficient ones) discussed in the two preceding sections. The simulation results were obtained for a known experimental pattern (Figure 1) of variation of the mean liquid velocity (occurring during simple water hammering in the center of the cross-section of a pressure pipe – $\mathrm{Re} = 1111$, $v_o = 0.066\,\mathrm{m/s}$,

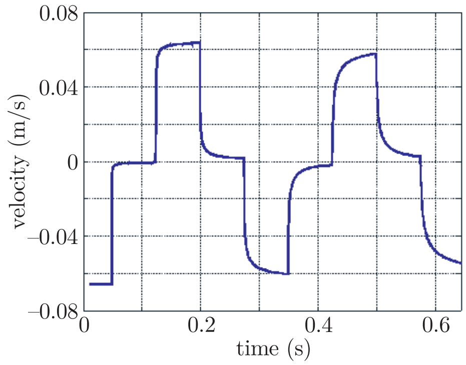

Figure 1. Mean velocity profile – pipe midpoint

$L = 98.11$ m, $R = 0.008$ m, $\nu = 9.493 \cdot 10^{-7}$ m^2/s i $c = 1305$ m/s [1]). The experiment consisted in a sudden closure of the terminal valve of a pipe transporting liquid from a constant pressure tank.

As can be seen in Figure 1, the work analyzed the effect of the velocity variation on the pattern of parameter τ_u only for the first two water hammer effect periods (within $t = 0.644$ s from the occurrence of the transient state).

The simulation tested the following:

1) The effect of the number of time steps "n" (zero-dimensional time step $\Delta\hat{t}$) on the pattern of parameter τ_u. Three cases were analyzed:
   - **CASE I** ($n_1 = 96$ time steps, $\Delta\hat{t} = 1 \cdot 10^{-4}$)
   - **CASE II** ($n_2 = 266$ time steps, $\Delta\hat{t} = 3.6 \cdot 10^{-5}$)
   - **CASE III** ($n_3 = 2561$ time steps, $\Delta\hat{t} = 3.7 \cdot 10^{-6}$)
2) The effect of the quantity of the exponential terms describing the efficient weighting function on the pattern of parameter τ_u. Also three cases were analyzed (Figure 2):
   - Function consisting of **18 exponential terms**
   - Function consisting of **22 exponential terms**
   - Function consisting of **26 exponential terms**

   See paper [7] for details of coefficients used in the weighting function.
3) The quality of matching the results obtained using efficient solutions of a convolution integral [derived] compared to the fit of results obtained using classic (or inefficient) solutions.

4.1. CASE I ($n_1 = 96$ time steps, $\Delta\hat{t} = 1 \cdot 10^{-4}$)

The foregoing diagrams show clearly what errors can result from using the efficient solution of the convolution integral by Trikha (17) for modeling an unsteady flow. This is because unsteady wall shear stresses, τ_u, simulated using the solution depend mostly on the adopted weighting function. The more expressions the ultimate form of the function contains, the worse the results are. A good visual example is comparing the results shown in Figure 3a (28) with those in Figure 3g

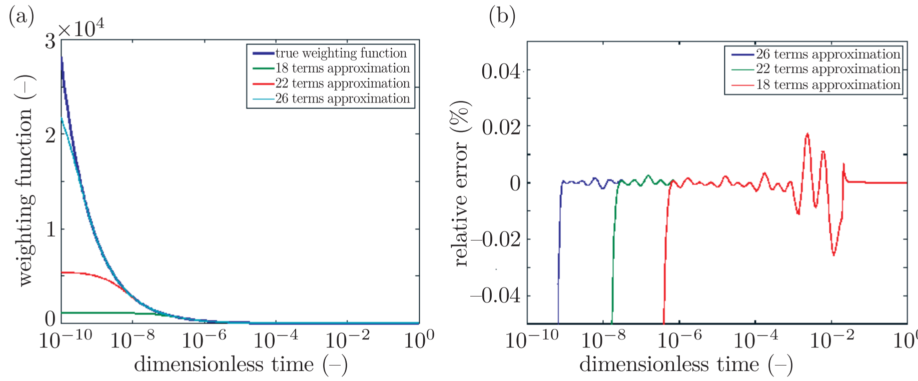

Figure 2. Weighting function

(Zielke-Vardy-Brown solution). It is clear that the results of the simulation using the solution by Trikha are approx. 500 times larger than the results provided by the exact adjusted classic solution by Zielke-Vardy-Brown (16). Therefore, using the solution of the convolution integral by Trikha should be avoided in numerical computations of unsteady hydraulic resistance. Similarly wrong results have been obtained for this solution in the next two cases (CASE II and CASE III). Due to its incompatibility with the classic solutions, the solution by Trikha will not be compared or considered in the following sections of this work.

Figure 3d (zoom) shows that the efficient solution by Schohl (28) is slightly dependent on the quantity of exponential expressions making up the weighting function. The larger the quantity of exponential expressions, the higher the consistency with the results provided by the classic adjusted solution by Zielke-Vardy-Brown (16).

On the other hand, the effect of the quantity of the exponential expressions making up the weighting function is not observed for the efficient solution by Kagawa *et al.* (Figure 3e).

Figure 3g shows that the results of simulated parameter τ_u are understated for the classic solution by Zielke (11). This means that simulated hydraulic resistance is understated if this solution is used.

4.2. CASE II ($n_2 = 266$ time steps, $\Delta\hat{t} = 3.6 \cdot 10^{-5}$)

A review of the results shown in Figure 4a has confirmed the trend noted for the results obtained for the previous case (CASE I). The more expressions the efficient weighting function contains, the more accurate the simulation for the solution by Schohl is (this is related to the fact that the efficient function containing more expressions is matched to the classic weighting function by Zielke (8)–(9) within a broader range of dimensionless time).

Again, the number of expressions did not affect the results obtained using the solution by Kagawa *et al.* (Figure 4c).

Figure 4e shows the same dependence that was observed for the previous case (CASE I): the results obtained using the classic solution by Zielke (11) were

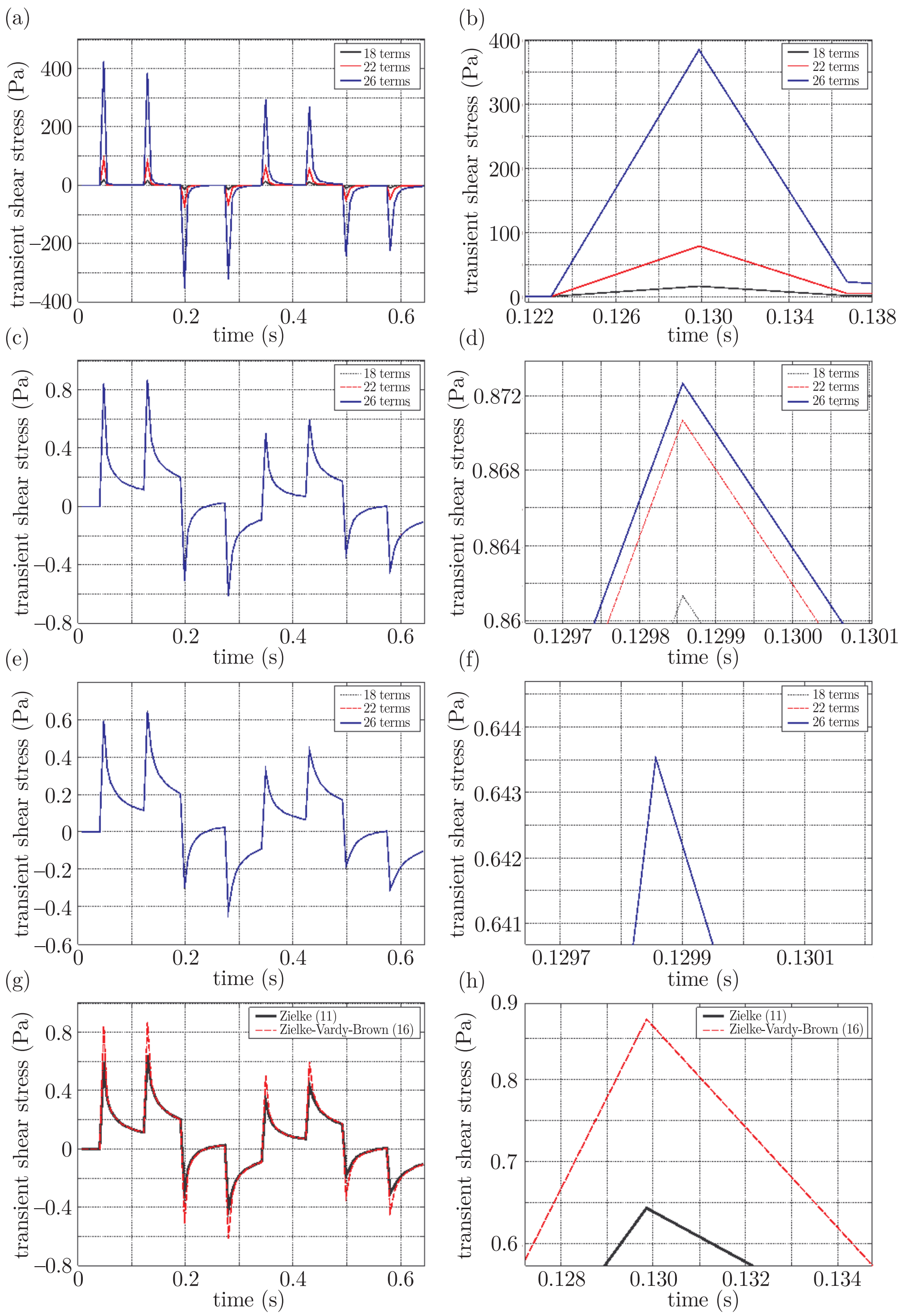

Figure 3. CASE I – results of simulated τ_u transient shear stress parameter runs using: (a), (b) Trikha efficient solution [5], (c), (d) Schohl efficient solution [4], (e), (f) Kagawa *et al.* efficient solution [2], (g), (h) Zielke [20] and Zielke-Vardy-Brown [9] inefficient solution (right panels are zoom of peak from left panels)

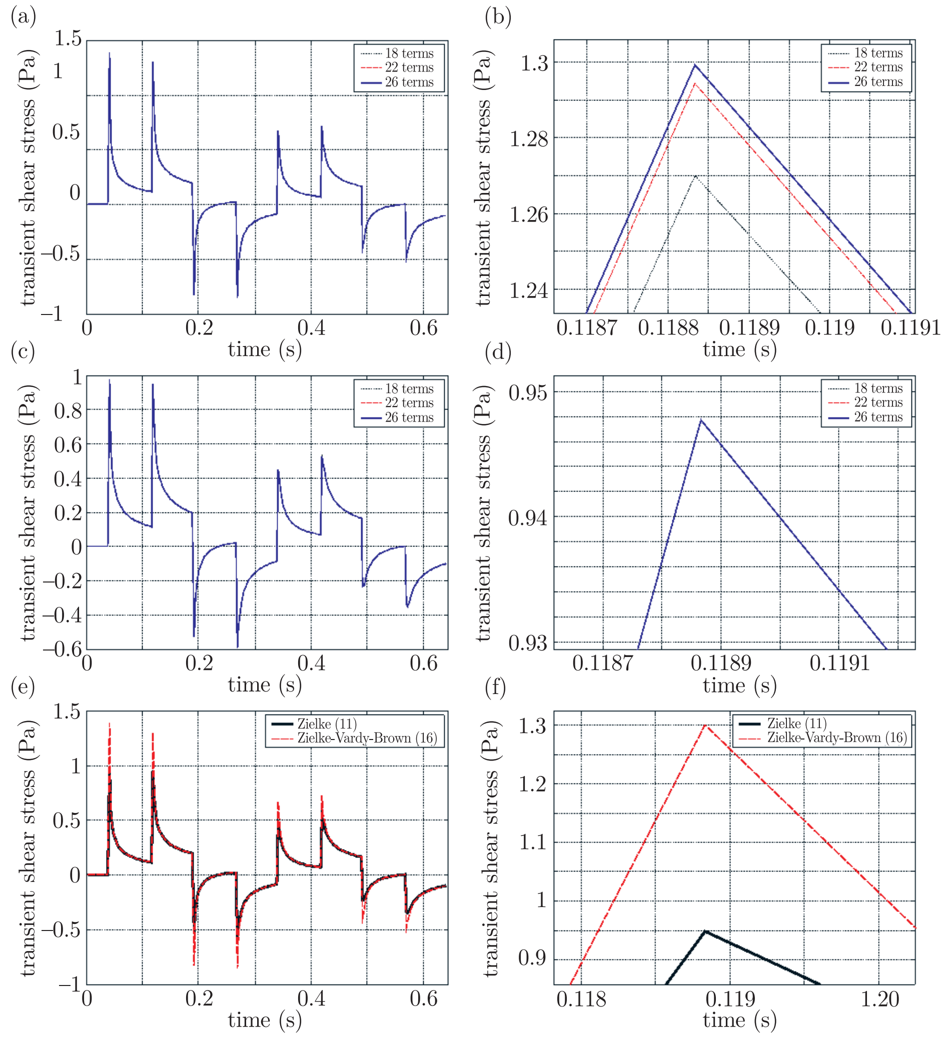

Figure 4. CASE II – results of simulated τ_u transient shear stress parameter runs using: (a), (b) Schohl efficient solution [4], (c), (d) Kagawa *et al.* efficient solution [2], (e), (f) Zielke [20] and Zielke-Vardy-Brown [9] inefficient solution (right panels are zoom of peak from left panels)

understated *vs.* the results provided by the corrected model by Zielke-Vardy-Brown (16).

4.3. *CASE III* ($n_3 = 2561$ *time steps,* $\Delta\hat{t} = 3.7 \cdot 10^{-6}$)

The foregoing illustrative comparisons (for all cases: CASE I, II and III) show clearly that the efficient solution of the convolution integral by Kagawa *et al.* (29) corresponds to the classic solution by Zielke (11). However, as Vardy and Brown [9] have correctly noted, the classic solution by Zielke is unable to provide a correct simulation (as shown by Vardy and Brown in the example of

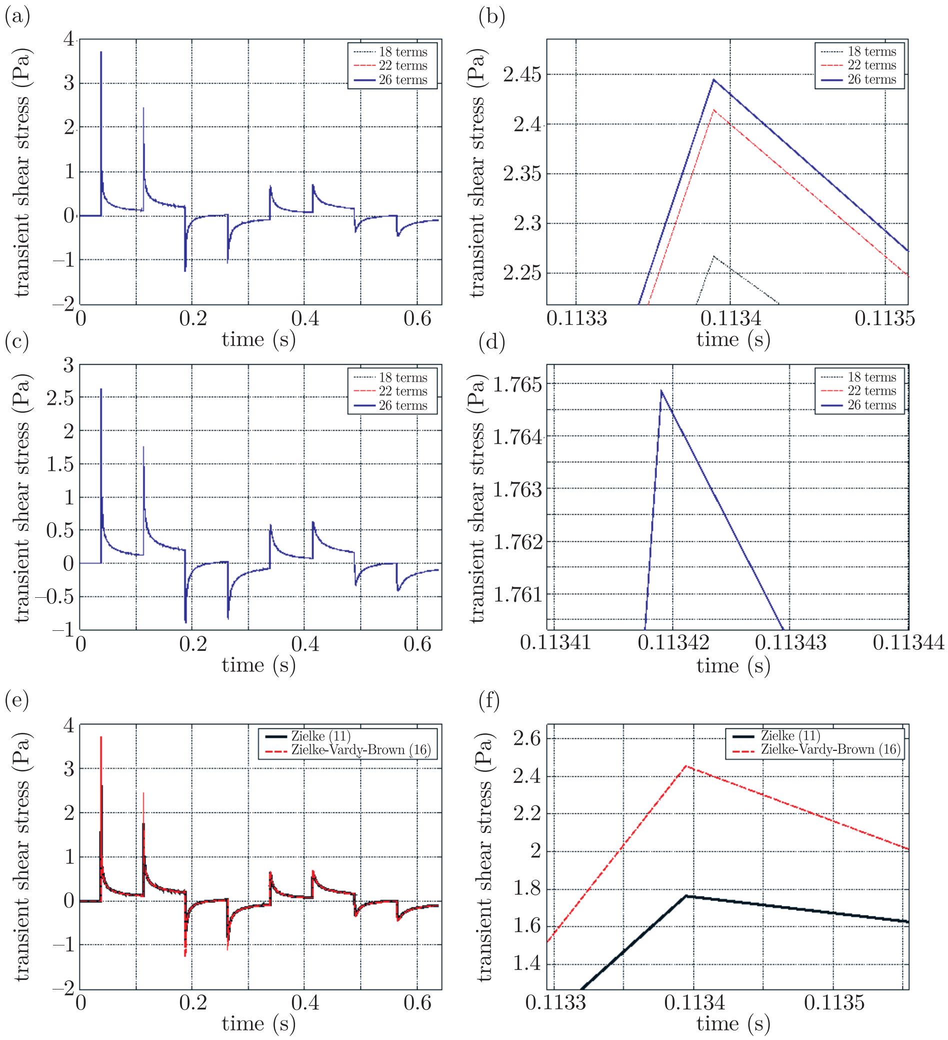

Figure 5. CASE III – results of simulated τ_u transient shear stress parameter runs using: (a), (b) Schohl efficient solution [4], (c), (d) Kagawa *et al.* efficient solution [2], (e), (f) Zielke [20] and Zielke-Vardy-Brown [9] inefficient solution (right panels are zoom of peak from left panels)

an accelerated flow) due to the simplification consisting in not computing the integral from the weighting function.

The solution by Schohl (28) is an efficient solution that computes the integral from the weighting function. And, as shown by the qualitative analysis of the foregoing results, the solution corresponds to the adjusted classic solution by Zielke-Vardy-Brown (16) with good fit.

Also, the analysis of all the results has answered the question concerning the effect of the time step on the simulation results. Namely, it is clear that the maximum values of peaks occurring in the patterns of parameter τ_u grow as the value of the dimensionless time step $\Delta\hat{t}$ decreases. It is a regularity justified by the fact that the values of the weighting function are the larger, the smaller the time step in numerical computations is. This means that velocity increments are multiplied by larger values (the solution by Trikha was a marked exception as it was the only solution that displayed different behavior, which is an argument for a definitive need for avoiding this solution in simulations).

4.4. Quantitative analysis

Apart from a standard qualitative analysis, the paper contains a quantitative one. The qualitative analysis demonstrates clearly that the efficient solution by Schohl conforms to the adjusted classic solution by Zielke-Vardy-Brown and that the efficient solution by Kagawa *et al.* conforms to the classic solution by Zielke (until recently considered to be the most accurate solution). Therefore, in the following sections the results of the simulation performed using the solution by Kagawa *et al.* are compared with those provided by the classic solution by Zielke and the results provided by the efficient solution by Schohl are compared with those provided by the classic solution adjusted by Zielke-Vardy-Brown.

It was only the absolute percentage errors of the maximum and minimum values occurring in the simulated patterns of parameter τ_u that were analyzed (marked with circles in the following Figure 6).

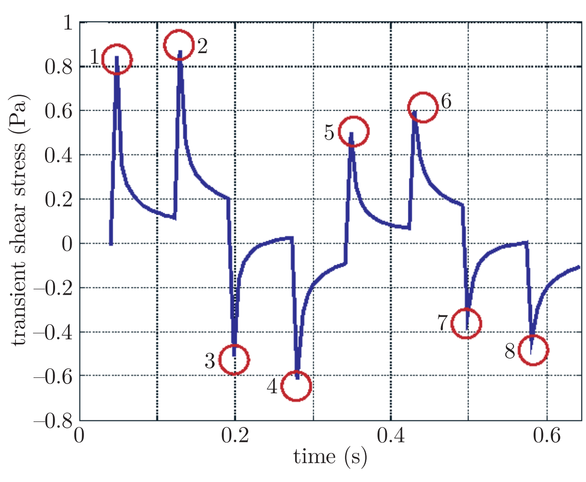

Figure 6. Analyzed shear stress peaks

Eight errors from the stimulated patterns having been calculated, the errors were used to estimate a single parameter, "E", representing the arithmetic mean of all the errors, using the following equation:

$$E = \frac{\sum_{i=1}^{8} \left| \frac{\tau i_{\mathrm{eff.(max,min)}} - \tau i_{\mathrm{ineff.(max,min)}}}{\tau i_{\mathrm{ineff.(max,min)}}} \right| \cdot 100\%}{8} \tag{30}$$

where: $\tau i_{\text{eff.(max,min)}}$ – maximum and minimum values of effective runs (Kagawa and Schohl solution); $\tau i_{\text{ineff.(max,min)}}$ – maximum and minimum values of ineffective runs (Zielke and Zielke-Vardy-Brown solution)

Table 1 shows the results of the proposed quantitative analysis.

Table 1. Mean absolute error of actual results

Case	**Error parameter E (%)**	
	Kagawa *vs.* Zielke	Schohl *vs.* Zielke-Vardy-Brown
CASE I	0.0051	0.075
CASE II	0.0015	0.118
CASE III	0.0019	0.230

It follows clearly from the foregoing table that the fit of the results obtained using the efficient solution by Kagawa is very good and this is why the solution used to be the most popular one. However, the one-way tendency to improvement of the fit as the time step in the numerical becomes smaller is missed. A reverse trend (where the time step reduction deteriorates the results) can be observed for the fit of the results obtained using the efficient solution by Schohl. Without doubt, this behavior relates to the incorrect result of integration using the weighting function for the last time step. Even the first drawing shows that the efficient weighting function approaches a certain fixed value (namely) rather than infinity for the dimensionless time approaching zero. Without doubt, the incorrect calculation of the integral using the weighting function for the last time step is the source of the error (as the last change of velocity is multiplied by the result of integration calculated using the weighting function within the 0 to range), which could possibly be eliminated by adjusting the efficient solution by Schohl.

5. Conclusion

The paper analyzes three solutions of the convolution integral known from the literature: Trikha [5], Kagawa *et al.* [2] and Schohl [4]. The results of the research show that Trikha's simplifications are responsible for significant errors and this model should be ruled out as a tool for simulating hydraulic resistance.

Also, the results show that the efficient solution by Kagawa *et al.* (often used in the past by the authors of the paper) features very good correspondence to the classic solution by Zielke. As recently demonstrated [9], the solution is not error-free as it underestimates unsteady hydraulic resistance. Further, comparisons show that the adjusted solution of the convolution integral used to calculate the exact integral using the weighting function has its efficient counterpart: the solution by Schohl.

The qualitative analysis of the results provided by the efficient solution by Schohl demonstrates also that increasing the number of expressions describing the weighting function improves slightly the fit of the simulation results compared to the results obtained using the accurate classic solution by Zielke-Vardy-Brown.

It should be also noted that the quantitative results signal a slight problem that has not been solved to date: the computation of the integral using the weighting function for the last time step in the efficient solutions generates an error that increases as the time step in the numerical analysis is smaller (Table 1). This problem should be eliminated and this will be the subject of our next paper.

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