WEIGHTING FUNCTION APPROXIMATION IN TRANSIENT PIPE FLOW

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Abstract: A very important problem in the transient liquid pipe flow analysis is accurate and effective modeling of hydraulic resistance. The so called integral convolution of the mean local acceleration of liquid and a weighting function need to be solved in a numerical way in order to simulate unsteady resistance. A necessary condition in effective numerical calculations is that the weighting function needs to be defined as a finite sum of exponential terms. The function keeps a constant shape in a laminar flow, in a turbulent flow, its shape changes and it is dependent on the instantaneous Reynolds number. In this article an easy method is presented to determine a proper weighting function in a very straightforward manner in a quick time. A comparison of the determined functions and function prototypes in a frequency domain will be presented as well.

Keywords: numerical fluid mechanics, transient flow, hydraulic resistance, weighting function

1. Introduction

It is important to model the hydraulic resistance occurring during a transient flow of liquids through pressure lines. Failing to consider the maximum or minimum possible pressure in a hydraulic system at the design phase can lead to major system damage or even injuries in case of long transmission pipelines. The Joukowski relationship is a simple dependence for ordinary water hammer phenomena without cavitation which is helpful in determining the maximum pressures. On the other hand, if cavitation is present, the pressure fluctuation can be significantly larger, which calls for numerical modeling of such systems.

It has been known for some time that the wall shear stress exerted on the pipe wall is a sum of the quasi-steady component τ_q and the component τ_u related to the flow unsteadiness:

$$\tau = \tau_q + \tau_u \tag{1}$$

This approach was pioneered by Zielke [1] who has demonstrated that, component τ_u for a laminar flow can be correctly described analytically in the form of a convolution integral of the product of the momentary liquid velocity variation and a weighting function (that has a fixed shape in the case of a laminar flow):

$$\tau_u = \frac{2\mu}{R} \int\limits_0^t w(t-u)\frac{\partial v}{\partial t}(u)du \tag{2}$$

where: μ – dynamic viscosity; R – inner radius of pipe; v – instantaneous mean flow velocity; t – time; u – time, used in convolution integral; $w(t)$ – weighting function.

Similarly, formula (2) can be also used for determining the component τ_u for the turbulent flow, which has been demonstrated in the works by Zarzycki [2–4] and Vardy and Brown [5–10], but with one difference: the weighting function dedicated to the turbulent flow should be used, where the shape of the function depends on the momentary value of the Reynolds number.

The literature offers two methods for resolving the convolution integral (2): the classic (inefficient) method presented by Zielke [1] (slightly improved by Vardy-Brown [11] in 2010) and the efficient method presented by Trikha [12] (later improved by Kagawa *et al.* [13] and Schohl [14]). The efficient solutions require that the weighting function is written in the form of a finite sum of exponential expressions:

$$\sum_{i=1}^k m_i e^{-n_i \cdot \hat{t}} \tag{3}$$

where: $\hat{t} = \nu \cdot t/R^2$ – dimensionless time, m_i and n_i – coefficients of weighting function

Approximating the classical weighting functions (Zielke [1] for the laminar flow and Vardy and Brown [7] or Zarzycki [3] for the turbulent flow) is not easy to accomplish. The last four decades have brought many works the authors of which have dealt with estimating coefficients of efficient weighting functions both for laminar and turbulent flows. In time, as computerization has progressed, the number of exponential expressions making an efficient function has been increasing in cycles (with the exception of the weighting function by Kagawa *et al.* that was recalled after many years, as the Japanese original article remained unknown to the world for long) which can be seen in Table 1.

In 2004, Vardy and Brown [9] presented a certain numerical method for estimating coefficients of the efficient weighting functions. However, since this method is used to determine the sought values of coefficients based on a system of equations, the number of which depends on the number of the exponential

Table 1. Matrix of works concerning efficient weighting functions for laminar and turbulent flows

			LAMINAR FLOW			
Author:	Trikha [12] **1975**	Schohl [14] **1993**	Vardy and Brown [9] **2004**	Kagawa *et al.* [13] **1983**	Vitkovsky *et al.* [15] **2004**	Urbano-wicz [16] **2009**
Number of exponential terms:	**3**	**5**	**9**	**10**	**10**	**26**
			TURBULENT FLOW			
Author:	Vitkovsky *et al.* [15] **2004**	Urbano-wicz [16] **2009**	Vardy and Brown [10] **2007**	Zarzycki and Kudźma [4] **2004**	Kudźma [17] **2005**	Urbano-wicz [16] **2009**
	basing on classic Vardy and Brown [7] weighting function			basing on classic Zarzycki [3] weighting function		
Number of exponential terms:	**10**	**16**	**17**	**6**	**8**	**22**

expressions sought, the method is inefficient and complex. The following work will present another method providing a possibility of much simpler estimation of coefficients m_i and n_i representing efficient weighting functions.

2. Unsteady wall shear stress component

2.1. Convolution integral

The major shortcoming in the classic numerical solution of the convolution integral presented by Zielke (2) is the fact that the sought component τ_u is computed from a longer sum in each time step (since the sum takes account all the velocity fluctuations from the beginning of the transient state) [1]:

$$\tau_u = \frac{2\mu}{R} \sum_{j=1}^{k-1} (v_{i,k-j+1} - v_{i,k-j}) \cdot w \left(j\Delta\hat{t} - \frac{\Delta\hat{t}}{2} \right) \qquad (4)$$

where: k – current numerical time step, $\Delta\hat{t} = \Delta t \frac{\nu}{R^2} = \frac{L\cdot\nu}{f\cdot c\cdot R^2}$ – dimensionless time increase, L – length of the pressure line, f – number of analyzed cross pipe section, c – pressure wave velocity.

According to the foregoing equation (4), the last velocity change, that is multiplied by the value of the weighting function determined for the smallest dimensionless time $w(\Delta\hat{t}/2)$, has the strongest effect on the component τ_u. As is known, the weighting function takes large values for small dimensionless times and small values for relatively larger times (Figure 1).

Trikha [12] was the first to present a certain efficient numerical solution of the convolution integral in 1975, however, Kagawa *et al.* [13] published the
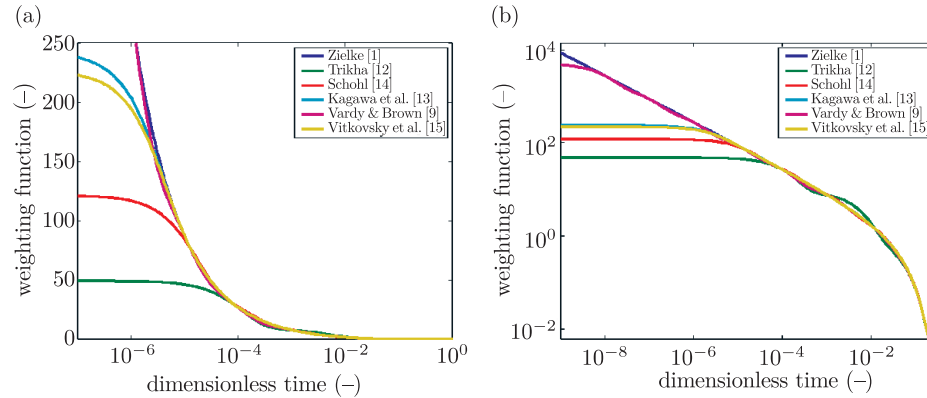
(a)

(b)

Figure 1. Pattern of laminar weighting functions

improved solution in 1983 as it had been based on too many simplifying assumptions:

$$\tau_u = \frac{2\mu}{R} \sum_{i=1}^{k} \underbrace{\left(y_i(t) \cdot e^{-n_i \cdot \Delta \hat{t}} + m_i \cdot e^{-n_i \cdot \frac{\Delta \hat{t}}{2}} \cdot [v(t+\Delta t) - v(t)] \right)}_{y_i(t+\Delta t)} \tag{5}$$

where: $y_i(t)$ – parameter computed for the previous time step (during the occurrence of the transient state, *i.e.*, for the first time step of numerical analysis $y_i(0) = 0$).

Still another efficient form of a convolution integral solution was proposed by Schohl [14] in 1993:

$$\tau_u = \frac{2\mu}{R} \sum_{i=1}^{k} \underbrace{\left(y_i(t) \cdot e^{-n_i \cdot \Delta \hat{t}} + \frac{m_i \left(1 - e^{-n_i \cdot \Delta \hat{t}}\right)}{n_i \cdot \Delta \hat{t}} \cdot [v(t+\Delta t) - v(t)] \right)}_{y_i(t+\Delta t)} \tag{6}$$

The foregoing efficient solutions (5) and (6) require that the weighting function is written in the form of a finite sum of exponential expressions.

2.2. Classic forms of the weighting function

In addition to presenting the classic convolution integral solutions (2) and (4), Zielke proposed a correct form of the weighting function for a laminar flow in his work of 1968 [1]:

$$w(\hat{t}) = \sum_{i=1}^{6} m_i \hat{t}^{(i-2)/2}, \text{ for } \hat{t} \le 0.02 \tag{7}$$

$$w(\hat{t}) = \sum_{i=1}^{5} e^{-n_i \cdot \hat{t}}, \text{ for } \hat{t} > 0.02 \tag{8}$$

where: $m_1 = 0.282095$; $m_2 = -1.25$; $m_3 = 1.057855$; $m_4 = 0.9375$; $m_5 = 0.396696$; $m_6 = -0.351563$; $n_1 = 26.3744$; $n_2 = 70.8493$; $n_3 = 135.0198$; $n_4 = 218.9216$; $n_5 = 322.5544$.

The literature offers two weighting function models for a turbulent flow:

- Vardy and Brown [7]

$$w(\hat{t}, \text{Re}) = \frac{A^* e^{-B^* \hat{t}}}{\sqrt{\hat{t}}} \tag{9}$$

where: $A^* = \sqrt{1/4\pi}$ and $B^* = \text{Re}^\kappa / 12.86$; $\kappa = \log_{10}(15.29/\text{Re}^{0.0567})$;

- Zarzycki [3]

$$w(\hat{t}, \text{Re}) = \frac{C}{\sqrt{\hat{t}}} \cdot \text{Re}^n \tag{10}$$

where: $C = 0.299635$; $n = -0.005535$.

2.3. Simple method of approximating the weighting function

A single exponential expression will not provide for correct mapping of the classic weighting function onto the required range of dimensionless time. This means that a function approximating the classic weighting function should be a finite sum of such expressions:

$$w_{\text{apr.}}(\hat{t}) = \sum_{i=1}^{k} m_i e^{-n_i \cdot \hat{t}} \tag{11}$$

The process of computing the coefficients m_i and n_i describing subsequent exponential expressions is not as easy as it could seem. This is demonstrated, last but not least, by the first efficient weighting functions proposed in the literature, which feature great simplicity (few exponential expressions) but poor mapping of the approximated function (consider, for instance, the weighting function proposed by Trikha [12] and Schohl [14] for the laminar flow or by Zarzycki-Kudźma [4] for the turbulent flow).

The authors of many functions have made ancillary use of complex statistical and fine-tuning procedures [4, 13–15] while computing their coefficients:

- **Schohl** [14] has applied a fine-tuning procedure based on the least squares method (so he has managed to match 5 exponential expressions to 136 points describing the pattern of the classic weighting function by Zielke). Vitkovsky *et al.* [15], Kudźma and Zarzycki [4] and others have estimated their functions similarly.
- **Kagawa** *et al.* [13] have followed the classic function by Zielke (from the smallest values on the right) and fitted in new exponential expressions in real time. Vardy [9] has appreciated the potential of this approach by concluding that this method could be used for determining the smallest number of expressions required for the approximation while preserving a predefined level of accuracy. Urbanowicz [16] has used a similar method in his work.
- **Vardy and Brown** [9] have proposed a numerical procedure in which parameter n_i values are adopted for known dimensionless times and then an appropriate system of equations is construed and resolved to find individual coefficients m_i.

The following article presents a simple alternative method based on the determination of subsequent exponential expressions in steps (as in Kagawa *et al.*) and adjusting the weighting function so that the trace should cross certain points selected using the classic weighting function (as in Vardy-Brown).

The range of applicability of efficient weighting functions should be sufficient enough to ensure correct simulation of actual turbulent flows. Vardy and Brown suggest [9] that the range of applicability of the new functions should depend on the time step $\langle 10^{-2}\Delta\hat{t}; 10^3\Delta\hat{t}\rangle$ adopted for the numerical analysis. And this remark seems to be right because it implies that the range of applicability of the function should indeed depend on the tested hydraulic system.

For an approximation of a turbulent classic weighting function, it is best to perform the approximation for as small the Reynolds number as possible (for instance $\mathrm{Re} = 2\cdot 10^3$). This approach serves its purpose because, for such numbers, the shape of the turbulent weighting function resembles the shape of the laminar weighting function and, what is important, provides near-zero values (smaller than 10^{-4}) for dimensionless times $\hat{t} > 5.47\cdot 10^{-2}$. On the other hand, if large Reynolds numbers are used (such as $\mathrm{Re} = 10^6$) the classic turbulent weighting function provides near-zero values much sooner: as early as for dimensionless times $\hat{t} > 1.2\cdot 10^{-3}$. Accordingly, approximating the new function will be more difficult for such small values of the weighting function.
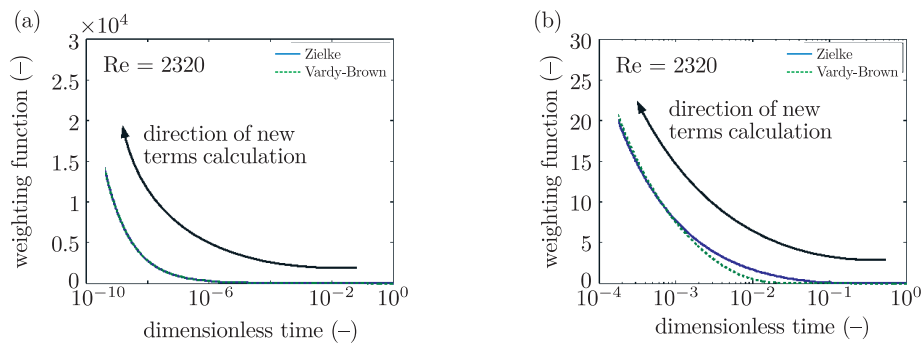

Figure 2. Direction of determination of new exponential expressions

As has been stated, the new procedure is a stepped one in which new values of coefficients m_i and n_i describing a single exponential expression $m_i\cdot\exp(-n_i\cdot\hat{t})$ will be determined in each subsequent step. The determination of exponential expressions starts from large values of dimensionless time (time $\hat{t} \approx 10^0$ can be regarded as such) for which the classic weighting function provides smallest values (near-zero) (Figure 2). The following Figure 3 presents a block diagram of subsequent steps of developing an efficient function representing a sum of five exponential expressions $w_{\mathrm{apr.}}(\hat{t}) = \sum_{i=1}^{5} m_i e^{-n_i\cdot\hat{t}}$.

The determination of new exponential expressions will be a result of the assumption that the new weighting function is supposed to cross evenly spaced points of the logarithmic scale. The points are the values of the classic weighting
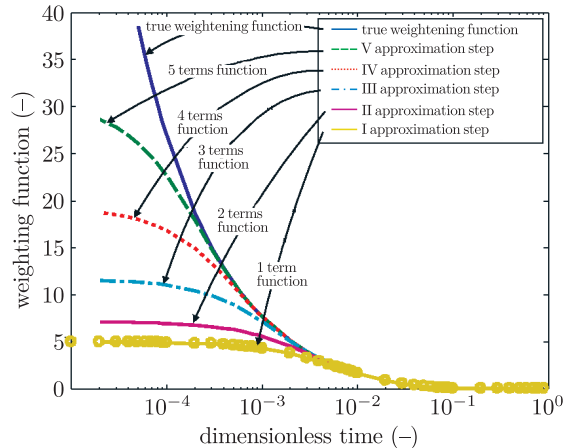
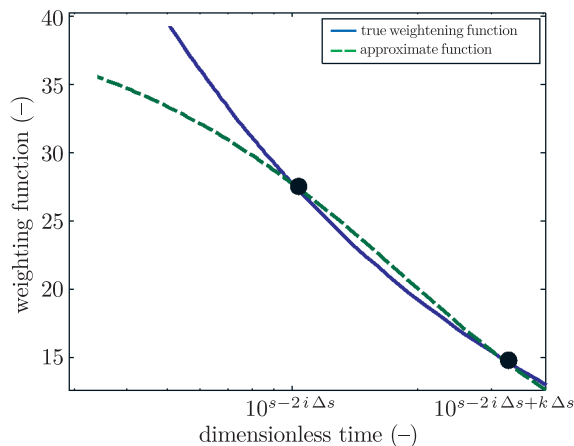**Figure 3.** Determination of subsequent exponential expressions for the new efficient weighting function

Figure 4. Transition of the new estimated efficient weighting function through two points

function $w_{\text{cl.}}$ (estimated using Zielke weighting function $w_{\text{cl., Z}}(\hat{t})$ for a laminar flow or the Vardy-Brown weighting function $w_{\text{cl., V-B}}(\hat{t},\text{Re})$ for a turbulent flow). The crossing of the new weighting function by the two points (Figure 4) relates to the meeting of the following system of equations (used each time to compute one exponential expression):

$$
\begin{cases}
w_{\text{cl.}}(\hat{t}_1) = m_{i+1} \cdot \exp(-n_{i+1} \cdot \hat{t}_1) + \sum_{r=1}^{i} m_r \cdot \exp(-n_r \cdot \hat{t}_1) \\[2mm]
w_{\text{cl.}}(\hat{t}_2) = m_{i+1} \cdot \exp(-n_{i+1} \cdot \hat{t}_2) + \sum_{r=1}^{i} m_r \cdot \exp(-n_r \cdot \hat{t}_2) \\[2mm]
\hat{t}_1 = E_1(i) = 10^{(s-2i\Delta s)}; \quad \hat{t}_2 = E_2(i) = 10^{(s-2i\Delta s + k\Delta s)}
\end{cases}
\tag{12}
$$

where: $w_{\mathrm{cl.}}$ – Zielke weighting function value $w_{\mathrm{cl.,Z}}(\hat{t})$ for a laminar flow or Vardy-Brown weighting function value $w_{\mathrm{cl.,V-B}}(\hat{t},\mathrm{Re})$ for a turbulent flow; s – starting exponent; i – step ($i = 1,2,\ldots,h$ for a laminar flow leaving the first original exponential expressions ($m_1 = 1$, $n_1 = 26.3744$); in a turbulent flow: $i = 0,1,2,\ldots,h$); Δs – exponent increment; k – increment multiplier (this work tested the values of the parameter within the $\langle 0.0001;1 \rangle$ range).

Using the following notation:

$$
\begin{cases}
w_{\mathrm{cl.}}(\hat{t}_1) = C_1 \\
\sum_{r=1}^{i} m_r \cdot \exp(-n_r \cdot \hat{t}_1) = C_2 \\
w_{\mathrm{cl.}}(\hat{t}_2) = C_3 \\
\sum_{r=1}^{i} m_r \cdot \exp(-n_r \cdot \hat{t}_2) = C_4
\end{cases}
\tag{13}
$$

will provide the following system of equations:

$$
\begin{cases}
C_1 = m_{i+1} \cdot \exp(-n_{i+1} \cdot E_1) + C_2 \\
C_3 = m_{i+1} \cdot \exp(-n_{i+1} \cdot E_2) + C_4
\end{cases}
\tag{14}
$$

where: in case of a turbulent flow if $i = 0$: $C_2 = 0$ and $C_4 = 0$; while for a laminar flow (leaving the first exponential expression of the classic Zielke weighting function) if $i = 1$: $C_2 = \exp(-26.3744 \cdot \hat{t}_1)$ and $C_4 = \exp(-26.3744 \cdot \hat{t}_2)$.

Transforming the foregoing system of equations (14) the following equation can be produced:

$$
\frac{C_1 - C_2}{\exp(-n_{i+1} \cdot E_1)} - \frac{C_3 - C_4}{\exp(-n_{i+1} \cdot E_2)} = 0
\tag{15}
$$

Using the foregoing equation it is possible to determine the unknown "n_{i+1}" numerically, using the FZERO function representing a module of MATLAB or, for instance, using the BISECTION method. Once "n_{i+1}" is found, "m_{i+1}" is computed using one of the following equations:

$$
\frac{C_1 - C_2}{\exp(-n_{i+1} \cdot E_1)} = m_{i+1} \quad \text{or} \quad \frac{C_3 - C_4}{\exp(-n_{i+1} \cdot E_2)} = m_{i+1}
\tag{16}
$$

A complete numerical procedure used to determine new expressions of the weight function for laminar and turbulent flows is presented schematically in Appendix A.

More accurate functions (featuring better matching, or representing a smaller relative percentage error) are obtained by applying smaller values of the parameter k (increment multiplier). This is because a reduction of this parameter can be followed by a reduction of the parameter Δs. Numerous simulation tests using the foregoing method demonstrated that the parameter Δs had certain limits. The lower limit for the laminar flow was $\Delta s = 0.24$ (at $k = 0.0001$). The approximating weighting function for this value features the best match to the Zielke function.

After adjusting the values of the estimated parameters "m_i" by multiplying the values by the correction factor $z_l = 0.999615$ ($m_{icl} = z_l \cdot m_i$ – the role of the correction factor is to spread evenly the distribution of the relative percentage error to minimize the absolute percentage error) the relative error in the domain of time was within the $\pm 0.04\%$ range.

For the turbulent flow, on the other hand, the approximating function was most accurate for $\Delta s = 0.235$ (at $k = 0.0001$). The relative percentage error for the function remained within the $\pm 0.032\%$ range (with the correction factor $z_t = 0.99964$).

It is not possible to use smaller values of the parameter Δs as such values would make the proposed procedure unstable and produce estimation errors.

In addition to good matching in the time domain, the estimated functions feature good matching in the frequency domain which is confirmed in Figure 5 below showing the matching of the new estimated laminar function.
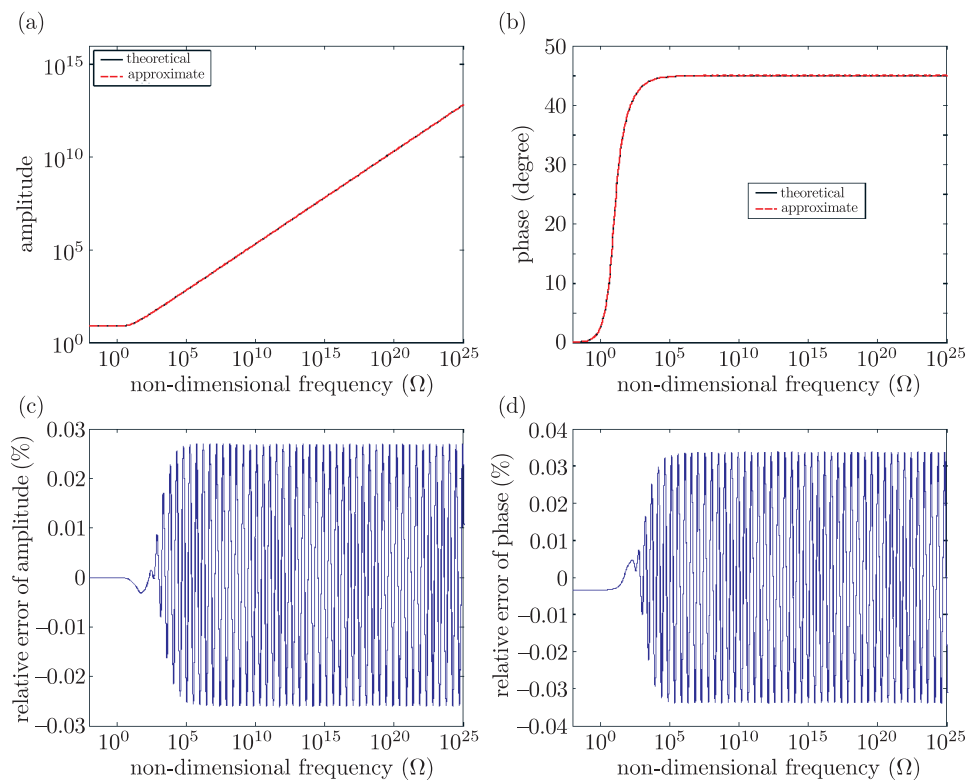

Figure 5. Analytical laminar weighting function and estimated approximation

It should be noted that selecting correct starting parameters is important for the numerical resolution of non-linear equations, such as Equation (15). See Appendix B for broader coverage of the starting value of parameter "n_{i+1}".

3. Conclusion

The following paper presents a simple method for rapid determination of new weighting functions written in the form of a finite sum of exponential expressions. This notation in the form of the finite sum will allow efficient determination of unsteady friction losses (using the efficient solution of the convolution integral presented by Kagawa *et al.* [12] or Schohl [13]). The proposed method will make it much simpler to simulate transient states in complex hydraulic, water supply or heating networks.

Main Points:

1. Given that $k = 0.0001$, the new weighting function is much better matched (with a smaller relative percentage error) than for $k = 1$ as the reducing parameter k allows reduction of the lower increment limit of the exponent Δs.

2. There are certain bottom and upper limits of the parameter Δs (exponent increment) between which new exponential expressions can be determined. Once the limits are exceeded, the approximation of new expressions can produce errors (*e.g.*, estimating subsequent parameters with values smaller than the previous ones: $m_{i+1} < m_i$ or $n_{i+1} < n_i$) or even can be impossible to complete.

3. The lower (minimum) limit of parameter Δs (exponent increment) required for correct estimation of coefficients describing the weighting function is different for laminar and turbulent flows. It could be said that it depends on the pattern of the classic weighting function.

4. The number of the estimated exponential expressions "h" should depend on the actually simulated transient state. We can follow the recommendation of Vardy and Brown formulated in their paper of 2004 [9] that proposes to adopt the value of the time step Δt from the $\langle 10^{-2}\Delta t; 10^3\Delta t \rangle$ range as the determinant for identifying the number of expressions. For $t > 10^3\Delta t$ the values of the weight function should be assumed as null.

5. Note that each change of the form of the classic turbulent weighting function reflects on the minimum value of Δs that can be used for the foregoing procedure. This is because the turbulent function is partly based on experimental data and – considering the inputs from the ongoing, increasingly more accurate, experimental research – the form of the function will evolve by slightly changing its pattern.

The future work will be oriented towards developing a similar simple method (for determination of coefficients of efficient weight functions) directly in the domain of frequency.

Appendix A

The proposed algorithms are presented in Figures 6–7 on following pages.

Appendix B

The known efficient laminar weighting functions (Figure 8) were analyzed in detail to ensure correct selection of the starting values of coefficients "n_{i+1}". Apart
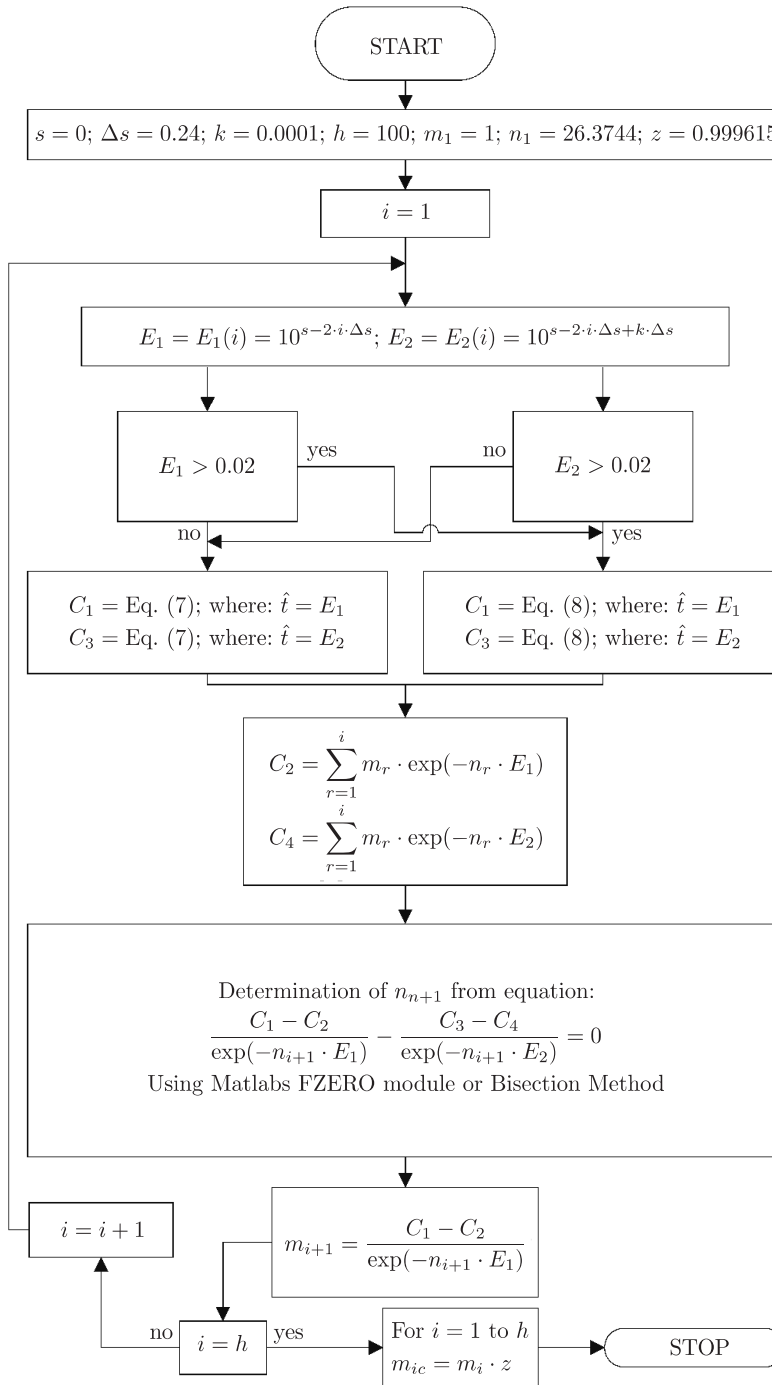
START

$s = 0$; $\Delta s = 0.24$; $k = 0.0001$; $h = 100$; $m_1 = 1$; $n_1 = 26.3744$; $z = 0.999615$

$i = 1$

$E_1 = E_1(i) = 10^{s - 2 \cdot i \cdot \Delta s}$; $E_2 = E_2(i) = 10^{s - 2 \cdot i \cdot \Delta s + k \cdot \Delta s}$

$E_1 > 0.02$ — yes / no — $E_2 > 0.02$

no / yes

$C_1 = $ Eq. (7); where: $\hat{t} = E_1$
$C_3 = $ Eq. (7); where: $\hat{t} = E_2$

$C_1 = $ Eq. (8); where: $\hat{t} = E_1$
$C_3 = $ Eq. (8); where: $\hat{t} = E_2$

$$C_2 = \sum_{r=1}^{i} m_r \cdot \exp(-n_r \cdot E_1)$$
$$C_4 = \sum_{r=1}^{i} m_r \cdot \exp(-n_r \cdot E_2)$$

Determination of n_{n+1} from equation:
$$\frac{C_1 - C_2}{\exp(-n_{i+1} \cdot E_1)} - \frac{C_3 - C_4}{\exp(-n_{i+1} \cdot E_2)} = 0$$
Using Matlabs FZERO module or Bisection Method

$i = i + 1$

$$m_{i+1} = \frac{C_1 - C_2}{\exp(-n_{i+1} \cdot E_1)}$$

no — $i = h$ — yes

For $i = 1$ to h
$m_{ic} = m_i \cdot z$

STOP

Figure 6. Simplified block diagram of determination of subsequent exponential expressions for laminar flow
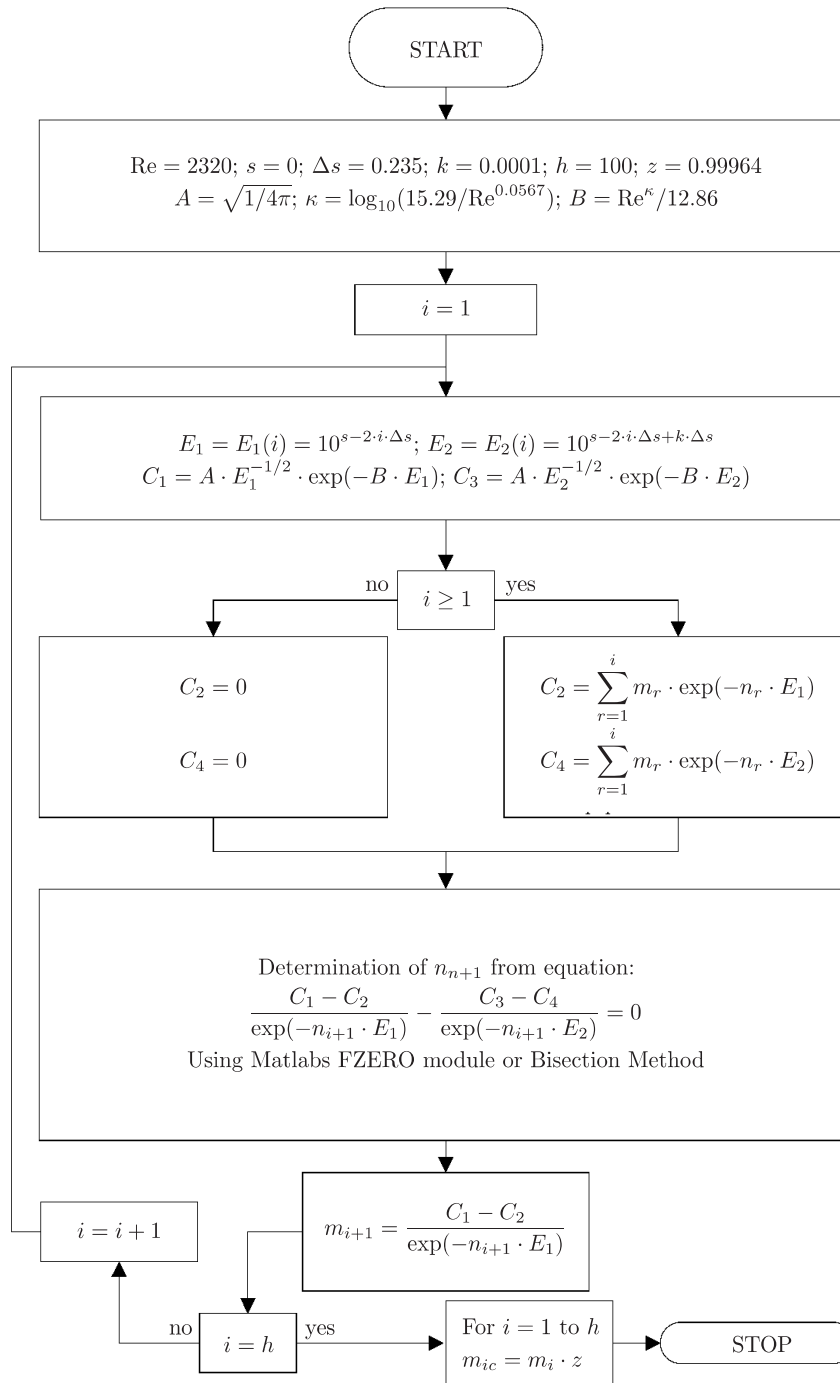
Figure 7. Simplified block diagram of determination of subsequent exponential expressions for turbulent flow
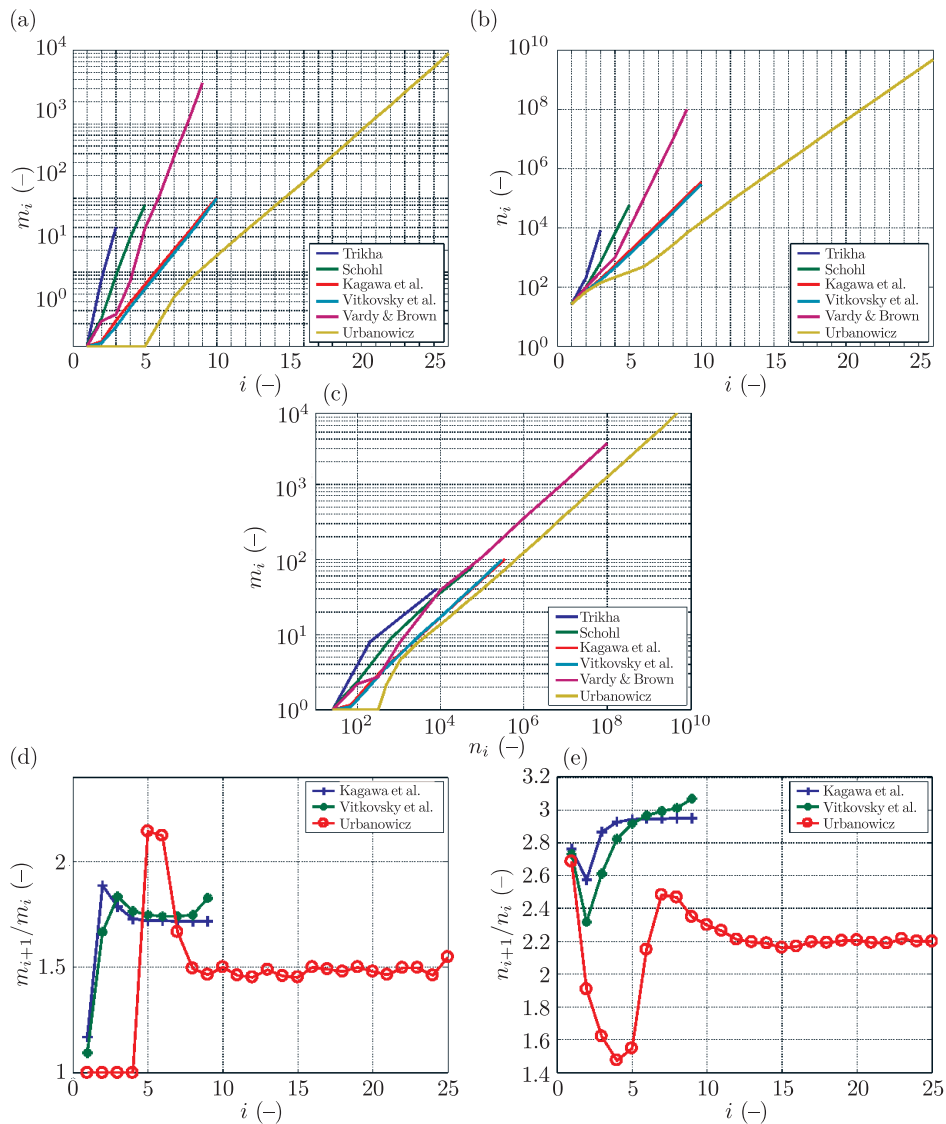
Figure 8. Review of coefficients describing laminar efficient weighting functions

from the starting parameters m_i and n_i, diagrams 8a, 8b and 8c show a nearly linear relationship of the growth of the parameters sought in the logarithmic scale. Diagram 8d shows that the m_{i+1}/m_i ratio varies for the most accurate of the known functions within the 1–2.15 range. Also, the diagram shows that this relationship stabilizes to some extent starting from $i = 3$ for the functions by Kagawa *et al.* and Vitkovsky *et al.* and starting from $i = 8$ for the function by Urbanowicz (at 1.72 for Kagawa *et al.*, 1.78 for Vitkovsky *et al.* and 1.48 for Urbanowicz). Diagram 8e confirms the foregoing observation for diagram 8d. Namely, a similar trend is visible for the ratio of coefficients n_{i+1}/n_i that ranges

from 1.46 to 3.1. In this case it is also clear that starting from $i = 4$ for the function by Kagawa *et al.* and starting from $i = 12$ for the function by Urbanowicz the ratio stabilizes to a certain extent (at 2.94 for Kagawa *et al.* and 2.2 for Urbanowicz). However, this stabilization was not observed for the function by Vitkovsky *et al.*, where the n_{i+1}/n_i ratio initially declined but then showed a regular growth trend starting from $i = 2$.

Also, a review of the foregoing diagrams shows clearly that subsequent values of these parameters can be estimated in practice with a small error, which enables the research algorithm to estimate the exact values of the parameters without any error.

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