



ON INVARIANTS OF FLUID MECHANICS TENSORS

KRZYSZTOF TESCH

*Gdańsk University of Technology, Fluid Mechanics Department
G. Narutowicza 11/12, 80-233 Gdansk, Poland
krzyte@pg.gda.pl*

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Abstract: This paper presents physical interpretations of the first and second invariants of tensors of fluid mechanics. Some examples of elementary applications and meanings are also given.

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Notation

- D – strain rate tensor
- D^D – deviatoric part of strain rate tensor
- D^* – normalised strain rate tensor
- e – internal energy
- $\mathcal{E}, \mathcal{E}^*$ – enstrophy
- k – kinetic energy of velocity fluctuations
- N_d – dissipation power
- p, p_e – hydrodynamic, effective pressure
- \mathbf{R} – Reynolds stress tensor
- T – temperature
- \mathbf{T} – tensor
- \vec{U} – velocity vector
- V – volume
- \vec{w} – vector
- W – vorticity measure
- γ – strain rate
- δ – Kronecker delta
- ϵ, ϵ^* – specific enstrophy
- ε^* – dissipation of kinetic energy fluctuation
- λ – heat conductivity
- μ, μ_e, μ_t, μ_v – dynamic, effective, eddy (turbulent), bulk viscosity
- ν – kinematic viscosity



ρ	– density
$\boldsymbol{\sigma}$	– stress tensor
τ_{xy}	– shear stress
ϕ_v	– dissipation function related to bulk viscosity
ϕ_μ	– dissipation function
$\vec{\Omega}$	– vorticity vector
$\boldsymbol{\Omega}$	– spin tensor
$\langle \cdot \rangle$	– average
$\ \cdot \ $	– norm
$\frac{\partial \vec{U}}{\partial \vec{r}}$	– transposed velocity gradient tensor
$\nabla \vec{U}$	– velocity gradient tensor

1. Introduction

The classical fluid mechanics provides physical interpretations to the first tensor invariants only. It is difficult to find information about the physical meaning of successive invariants. For instance in [1] one can find the following note about the second and third invariant of strain rate tensor: ‘The remaining two invariants are given by . . . , but they do not have clear physical interpretations.’

Invariants of velocity gradient tensors are often used in turbulence modelling [2]. This is because they contain all the necessary information involving the rates of rotation, stretching, and angular deformation, those being responsible for kinetic energy dissipation and vortex stretching.

Invariants of the strain rate tensor are commonly used to model the behaviour of non-Newtonian fluids [3]. This concerns the rate type fluid and generalised Newtonian fluids in particular [4].

In this paper the physical interpretations of second invariants of the basic fluid mechanics tensors are given. Additionally, discussion about third invariants of certain tensors is included.

First invariants are connected with such hydrodynamic concepts as velocity divergence or hydrodynamic pressure. Physical interpretation of second invariants requires additional concepts connected with thermodynamics. The fundamental idea that is a key concept in the case of second invariants is the dissipation function ϕ_μ . It appears in the internal energy e equation

$$\rho \frac{de}{dt} = \phi_\mu + \lambda \nabla^2 T - p \nabla \cdot \vec{U} \quad (1)$$

The above equation is known as the Fourier-Kirchhoff equation in heat transfer problems. Density is denoted here as ρ , pressure as p and temperature as T . The dissipation function ϕ_μ in the most general case, involving bulk viscosity μ_v , may be calculated from the equation

$$\phi_\mu = \mu_v (\nabla \cdot \vec{U})^2 + 2\mu \mathbf{D}^{D2} \geq 0 \quad (2)$$

The dissipation function is of an invariant nature as a consequence of the fact that it is related to the second invariant of the strain rate tensor \mathbf{D} or its deviatoric part $\mathbf{D}^D = \mathbf{D} - 3^{-1} \delta \text{tr} \mathbf{D}$. This relation is given by means of Equation (4). The

dissipation function appears also in equations describing the second invariant of the stress tensor $\boldsymbol{\sigma}$, being a linear function of the strain rate tensor (in agreement with Newton's hypothesis)

$$\boldsymbol{\sigma} = -p\boldsymbol{\delta} + 2\mu\mathbf{D}^D \quad (3)$$

One should pay attention to the symbol \mathbf{D}^2 . It has two meanings. It may either represent the dot product of two tensors $\mathbf{D} \cdot \mathbf{D}$ (giving a tensor as a result) or the double dot product $\mathbf{D} : \mathbf{D}$ (giving scalar). The ambiguity may be decided by means of an equation rank. For instance \mathbf{D}^{D^2} in Equation (2) means $\mathbf{D}^D : \mathbf{D}^D$ because it is a scalar equation.

2. Invariants

We shall consider invariants of tensor \mathbf{T} in the form of traces $\text{tr}\mathbf{T}$, $\text{tr}\mathbf{T}^2$, $\text{tr}\mathbf{T}^3$, where $\mathbf{T}^2 \equiv \mathbf{T} \cdot \mathbf{T}$ and $\mathbf{T}^3 \equiv \mathbf{T} \cdot \mathbf{T} \cdot \mathbf{T}$. These are all invariants of symmetric tensors. It is also convenient to utilise the definition of tensor norm $\|\mathbf{T}\|^2 := \mathbf{T} : \mathbf{T}$, as described in Section 3.4.

2.1. Kronecker delta

It is hard to call the Kronecker delta $\boldsymbol{\delta}$ (identity tensor) a basic fluid mechanics tensor. However, it appears in many tensor relationships. It can be readily verified that all invariants of this tensor have the same form $\text{tr}\boldsymbol{\delta} = 3$, $\text{tr}\boldsymbol{\delta}^2 = 3$, $\text{tr}\boldsymbol{\delta}^3 = 3$.

2.2. Strain rate tensor

The first invariant of the strain rate tensor \mathbf{D} is connected with compressibility through the relation $\text{tr}\mathbf{D} = \nabla \cdot \vec{U}$. It equals zero for the incompressible case.

The second invariant is connected with the energy dissipation concept. By means of the following identity $\|\mathbf{D}\|^2 \equiv \text{tr}\mathbf{D}^2$, being true also for deviators, it can be easily shown that

$$\text{tr}\mathbf{D}^2 = \frac{\phi_\mu - \phi_v}{2\mu} + \frac{1}{3}(\nabla \cdot \vec{U})^2 \geq 0 \quad (4)$$

This is true owing to the Equation (3) and the relationship between the double dot product (norm) of a tensor and its deviatoric part in the following form $\|\mathbf{D}^D\|^2 = \|\mathbf{D}\|^2 - 3^{-1}(\text{tr}\mathbf{D})^2$. The symbol ϕ_v describes the part of the dissipation function which is related to bulk viscosity $\phi_v := \mu_v(\nabla \cdot \vec{U})^2$. If the Stokes hypothesis is valid, this function equals zero. For the incompressible case, the Equation (4) takes a simpler form

$$\text{tr}\mathbf{D}^2 = \frac{\phi_\mu}{2\mu} \quad (5)$$

2.3. Spin tensor

From the spin tensor definition $\boldsymbol{\Omega}$ it arises that its first invariant equals zero $\text{tr}\boldsymbol{\Omega} = 0$. Similarly, from the physical interpretation of its components it arises that the double dot product (norm) equals $\|\boldsymbol{\Omega}\|^2 = 2^{-1}\vec{\Omega} \cdot \vec{\Omega}$. The vorticity

vector is denoted here as $\vec{\Omega}$. Introducing the norm of a vector $\|\cdot\|$ one may find the following form of the second invariant of the spin tensor

$$\text{tr } \boldsymbol{\Omega}^2 = -\frac{1}{2} \|\vec{\Omega}\|^2 = -\epsilon^* \leq 0 \quad (6)$$

where ϵ^* stands for enstrophy (Equation (21)). This means that the second invariant in this case equals one half of the vorticity vector norm. Evidently, the norm of a vector as well as the enstrophy is of an invariant nature.

As for the third invariant, through the equation $\text{tr } \boldsymbol{\Omega}^3 = \det \boldsymbol{\Omega}$, it obviously equals zero as for all asymmetrical tensors $\text{tr } \boldsymbol{\Omega}^3 = 0$.

2.4. Velocity gradient tensor and its transposition

The velocity gradient tensor $\nabla \vec{U}$ and its transposition $\frac{\partial \vec{U}}{\partial \vec{r}}$ are asymmetrical tensors. This fact results in a larger number of invariants in comparison with symmetrical tensors. Despite this, the first invariant of both tensors is the same as for the strain rate tensor \boldsymbol{D} and equals $\nabla \cdot \vec{U}$. The second invariant has the general form

$$\text{tr} \left(\frac{\partial \vec{U}}{\partial \vec{r}} \right)^2 = \text{tr} \left(\nabla \vec{U} \right)^2 = \text{tr } \boldsymbol{D}^2 - \epsilon^* \quad (7)$$

The specific form depends on the tensor \boldsymbol{D} . This is due to introduced simplifications (Stokes hypothesis or incompressibility).

For an asymmetrical tensor it true that $\boldsymbol{T} : \boldsymbol{T} \neq \text{tr}(\boldsymbol{T} \cdot \boldsymbol{T})$. However, in the case of the velocity gradient tensor and its transposition we have

$$\|\nabla \vec{U}\|^2 = \text{tr } \boldsymbol{D}^2 + \epsilon^* \quad (8)$$

where the velocity gradient tensor norm is given by $\|\nabla \vec{U}\|^2 = \nabla \vec{U} : \nabla \vec{U}$ and is discussed further towards the end.

Invariants of velocity gradient tensors and their analysis are useful because they unambiguously determine the local topology of the fluid motion [2, 5].

2.5. Stress tensor

The first invariant of the stress tensor $\boldsymbol{\sigma}$ is utilised for the definition of hydrodynamic pressure p in the form $-3p = \text{tr } \boldsymbol{\sigma}$. Thus, we have its physical interpretation.

The form of the second invariant of the stress tensor depends on the constitutive equation that describes it. For the Newton hypothesis (3) the scalar dot product of stress tensors can be evaluated as

$$\boldsymbol{\sigma}^2 = p^2 \boldsymbol{\delta} - 4p\mu \boldsymbol{D}^D + 4\mu^2 \boldsymbol{D}^{D^2} \quad (9)$$

The above equation is the most general in the sense that it concerns compressibility. If we assume incompressibility it is enough to get rid of the deviators. Utilising Equation (2) we have the following form of the second invariant of the stress tensor

$$\text{tr } \boldsymbol{\sigma}^2 = 3p^2 + 2\mu(\phi_\mu - \phi_v) \geq 0 \quad (10)$$

The total dissipation function ϕ_μ according to Equation (2) is reduced by the dissipation related to bulk viscosity ϕ_v . The above equation is of a very general

nature. It has an identical form for all cases, irrespective of the Stokes hypothesis or compressibility. Obviously, the second invariant of the stress tensor is connected with dissipation. Following the same line of reasoning it is possible to formulate the equation for the third invariant of this tensor

$$\text{tr} \boldsymbol{\sigma}^3 = -3p^3 - 6p\mu\phi_\mu + 8\mu^3 \text{tr} \mathbf{D}^3 \quad (11)$$

The only problem we encounter is the lack of interpretation of the strain rate third power trace, i.e. the third invariant of this tensor $\text{tr} \mathbf{D}^3$.

2.6. Reynolds stress tensor

We will take under consideration the Reynolds stress tensor described by means of the Boussinesq hypothesis only. This tensor is expressed by means of the averaged strain rate tensor $\langle \mathbf{D} \rangle$, the kinetic energy of turbulence k and the eddy viscosity μ_t as

$$\mathbf{R} = -\frac{2}{3}\rho k \boldsymbol{\delta} + 2\mu_t \langle \mathbf{D} \rangle \quad (12)$$

Its first invariant is connected with the turbulence energy k and it is given by $\text{tr} \mathbf{R} = -2\rho k$. We restrict ourselves to the incompressible case.

The second invariant is connected with the dissipation of the mean flow $2\mu \langle \mathbf{D} \rangle^2$ which together with the fluctuation dissipation ε^* forms the averaged dissipation function $\langle \phi_\mu \rangle$

$$\langle \phi_\mu \rangle = 2\mu \langle \mathbf{D}^2 \rangle = 2\mu \langle \mathbf{D} \rangle^2 + \rho \varepsilon^* \geq 0 \quad (13)$$

We do not introduce the homogeneity assumption here $\varepsilon^* := \langle \mathbf{D}'^2 \rangle$. Evaluating the product \mathbf{R}^2 and taking advantage of the identity $\|\mathbf{R}\|^2 \equiv \text{tr} \mathbf{R}^2$ together with relation (13) we obtain the following interpretation of the second invariant of the Reynolds stress tensor

$$\text{tr} \mathbf{R}^2 = \frac{4}{3}k^2 \rho^2 + 2\frac{\mu_t^2}{\mu} (\langle \phi_\mu \rangle - \rho \varepsilon^*) \quad (14)$$

The subsequent procedure is discussed in the next subsection. This is because the Reynolds stress tensor is a component of the total stress tensor.

2.7. Total stress tensor

The total stress tensor is composed of an averaged stress tensor and the Reynolds stress tensor $\langle \boldsymbol{\sigma} \rangle := -\langle p \rangle \boldsymbol{\delta} + 2\mu \langle \mathbf{D} \rangle + \mathbf{R}$. Utilising the Boussinesq hypothesis (12) and the following definitions of the effective pressure $p_e := \langle p \rangle + \frac{2}{3}\rho k$ and the effective viscosity $\mu_e := \mu + \mu_t$ we obtain an analogous form as for the stress tensor $\boldsymbol{\sigma}$

$$\langle \boldsymbol{\sigma} \rangle = -p_e \boldsymbol{\delta} + 2\mu_e \langle \mathbf{D} \rangle \quad (15)$$

The first invariant has an identical shape as for the stress tensor $\boldsymbol{\sigma}$ as is given by equation $\text{tr} \langle \boldsymbol{\sigma} \rangle = -3p_e$. We can find the second invariant following the same logic as in the previous subsection. The result is analogous to Equation (10)

$$\text{tr} \langle \boldsymbol{\sigma} \rangle^2 := 3p_e^2 + 2\frac{\mu_e^2}{\mu} (\langle \phi_\mu \rangle - \rho \varepsilon^*) \quad (16)$$

Again, this shows the generality of the discussed interpretation. We deal with mean flow dissipation only.

3. Examples of applications

3.1. Vorticity measure

Serrin [6] has given the following definition of the vorticity measure W , which comes originally from Truesdell: $W^2 := \|\boldsymbol{\Omega}\|^2 / \|\mathbf{D}\|^2$. The norm of vorticity vector $\|\vec{\Omega}\|$ depends on units but the vorticity measure W does not. Moreover, this measure is combined from two invariants. This means that it is itself of an invariant character. The measure equals zero for the irrotational flow $\|\vec{\Omega}\| = 0$ and $\mathbf{D} \neq \mathbf{0}$ (i.e. we deal with angular and linear deformation). The measure equals infinity for rotational flow $\|\vec{\Omega}\| \neq 0$ and $\mathbf{D} = \mathbf{0}$ (i.e. we deal with pure rotational motion without angular and linear deformation). The second case (rotational) has the largest possible vorticity measure. Employing the equations for appropriate tensor invariants we obtain

$$W^2 = \frac{\frac{1}{2}\|\vec{\Omega}\|^2}{\text{tr } \mathbf{D}^2} = \frac{\epsilon^*}{\text{tr } \mathbf{D}^2} \in [0, \infty] \quad (17)$$

3.2. Shear stress

It is possible to find another interpretation of the second invariant (10) for an incompressible case. This second interpretation is possible thanks to the strain rate γ which appears in the simple shear flow $\tau_{xy} = \mu \frac{\partial U_x}{\partial y} = \mu\gamma$. It may be generalised to the three-dimensional case

$$\gamma = \sqrt{2\mathbf{D}:\mathbf{D}} = \sqrt{2}\|\mathbf{D}\| \quad (18)$$

Obviously, the above relation comes to its one-dimensional equivalent. By means of the identity $\|\mathbf{D}\|^2 \equiv \text{tr } \mathbf{D}^2$ and Equation (5) one may find a very simple relationship between the shear strain rate and the dissipation function $\phi_\mu = \mu\gamma^2$. This equation gives the relationship between the shear stress (for instance on the wall) and the dissipation function. From this equation another physical interpretation of the stress tensor second invariant for an incompressible case arises

$$\text{tr } \boldsymbol{\sigma}^2 = 3p^2 + 2\mu^2\gamma^2 \quad (19)$$

which has no compressible equivalent.

3.3. Enstrophy

The enstrophy \mathcal{E} is defined as the integral of the velocity gradient tensor norm over certain volume V

$$\mathcal{E} = \iiint_V \|\nabla \vec{U}\|^2 dV = \iiint_V \epsilon dV \quad (20)$$

where $\epsilon := \|\nabla \vec{U}\|^2$ may be treated as a specific enstrophy. As shown previously, the norm of the velocity gradient tensor is invariant. So is the enstrophy. The less general definition of enstrophy takes under consideration only the asymmetrical part of the velocity gradient tensor, i.e. the spin tensor $\boldsymbol{\Omega}$.

This results in

$$\mathcal{E}^* = \iiint_V \|\boldsymbol{\Omega}\|^2 dV = \iiint_V \frac{1}{2} \|\vec{\Omega}\|^2 dV = \iiint_V \epsilon^* dV \quad (21)$$

where $\epsilon^* := 2^{-1} \|\vec{\Omega}\|^2$ may be again treated as specific enstrophy and it is also the second invariant of the spin tensor $\boldsymbol{\Omega}$. Basically, the specific enstrophy is defined as one half the square of the vorticity. The definition (21) arises from Equation (20) through identity $\|\nabla \vec{U}\|^2 \equiv \|\vec{\Omega}\|^2 + \nabla \cdot (\vec{U} \cdot \nabla \vec{U})$ and two further conditions. The former being incompressibility $\nabla \cdot \vec{U} = 0$ and the latter assumes velocity \vec{U} to decay rapidly at infinity. Finally, integration of the definition (20) by means of the Gauss theorem results in Equation (21).

The definition of enstrophy plays important role in the theory of turbulence [7]. It determines the rate of dissipation of kinetic energy being a global measure of the dissipation rate. To see this, it is necessary to recall the dissipation power definition [8] $N_d := \iiint_V \phi_\mu dV$. The dissipation function ϕ_μ for an incompressible case is given by Equation (5). Keeping in mind Equation (8) in the form of $\|\nabla \vec{U}\|^2 = \|\mathbf{D}\|^2 + \|\boldsymbol{\Omega}\|^2$ as well as the earlier identity $\|\nabla \vec{U}\|^2 \equiv \|\vec{\Omega}\|^2 + \nabla \cdot (\vec{U} \cdot \nabla \vec{U})$, it can be easily shown that

$$2\mu \iiint_V \|\mathbf{D}\|^2 dV = 2\mu \iiint_V \|\boldsymbol{\Omega}\|^2 dV \quad (22)$$

This means that for the incompressible case where velocity \vec{U} decays rapidly at infinity $N_d = 2\mu \mathcal{E}^*$, i.e. dissipation power is proportional to enstrophy. However, locally it is not true $\|\mathbf{D}\|^2 \neq \|\boldsymbol{\Omega}\|^2$.

3.4. Norms and normalising functions

The concept of tensor norm \mathbf{T} can be introduced by analogy to a vector norm. The norm of vector \vec{w} is associated with its magnitude $w \equiv \|\vec{w}\| := (\vec{w} \cdot \vec{w})^{1/2}$. The norm of a tensor is defined as $\|\mathbf{T}\| := (\mathbf{T} : \mathbf{T})^{1/2}$. The relationship between the norm of a symmetrical tensor and the second invariant $\|\mathbf{T}\| := (\text{tr}(\mathbf{T} \cdot \mathbf{T}))^{1/2}$ results from this definition.

Let us consider the deviatoric part of the strain rate tensor [8]. By means of Equation (2) and Stokes hypothesis we can obtain

$$\|\mathbf{D}^D\| = \sqrt{\frac{\phi_\mu}{2\mu}} \quad (23)$$

The norm of the strain rate tensor or its deviator is connected with dissipation. Both the dissipation and norm $\|\mathbf{D}^D\|$ are of an invariant character. The normalisation of vector $\vec{w}/\|\vec{w}\|$ gives a unit vector towards \vec{w} . Analogously, we can consider normalisation of tensors. Denoting the normalised strain rate tensor by \mathbf{D}^* we have for its deviator

$$\mathbf{D}^{*D} = \frac{\mathbf{D}^D}{\|\mathbf{D}^D\|} = \mathbf{D}^D \sqrt{\frac{2\mu}{\phi_\mu}} \quad (24)$$

It arises that the dissipation function performs the role of normalising function on the strain rate tensor or its deviator. After normalisation we have $\|\mathbf{D}^{*D}\| = 1$.

4. Summary

The set of all first and second invariants of basic tensors of fluid mechanics is given in Table 1. This table extends the current physical interpretations of particular invariants up to the second order. Additionally, one can find there the third invariants of the Kronecker delta, the spin tensor and the stress tensor for inviscid fluid.

To grant a physical interpretation to third invariants one should first find the physical interpretation of the strain rate tensor third invariant D^3 .

Table 1. Invariants

T	$\text{tr}T$	$\text{tr}T^2$	$\text{tr}T^3$
δ	3	3	3
D	$\nabla \cdot \vec{U}$	$\frac{\phi_\mu - \phi_v}{2\mu} + \frac{1}{3}(\nabla \cdot \vec{U})^2 \geq 0$	
$D, \mu_v = 0$	$\nabla \cdot \vec{U}$	$\frac{\phi_\mu}{2\mu} + \frac{1}{3}(\nabla \cdot \vec{U})^2 \geq 0$	
$D, \rho = \text{const}$	0	$\frac{\phi_\mu}{2\mu} \geq 0$	
D^D	0	$\frac{\phi_\mu - \phi_v}{2\mu} \geq 0$	
$D^D, \mu_v = 0$	0	$\frac{\phi_\mu}{2\mu} \geq 0$	
Ω	0	$-\epsilon^* \leq 0$	0
$\frac{\partial \vec{U}}{\partial \vec{r}}, \nabla \vec{U}$	$\nabla \cdot \vec{U}$	$\text{tr}D^2 - \epsilon^*$	
σ	$-3p$	$3p^2 + 2\mu(\phi_\mu - \phi_v) \geq 0$	
$\sigma, \rho = \text{const}$	$-3p$	$3p^2 + 2\mu\phi_\mu \geq 0$	
$\sigma, \mu = 0$	$-3p$	$3p^2 \geq 0$	$-3p^3$
R	$-2\rho k$	$\frac{4}{3}k^2\rho^2 + 2\frac{\mu^2}{\mu}(\langle\phi_\mu\rangle - \rho\epsilon^*)$	
$\langle\sigma\rangle, \rho = \text{const}$	$-3p_e$	$3p_e^2 + 2\frac{\mu^2}{\mu}(\langle\phi_\mu\rangle - \rho\epsilon^*)$	

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