ON FREE CONVECTION HEAT TRANSFER FROM VERTICAL PLATE – PROPOSITION OF A NEW MODEL

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Abstract: The free convection heat transfer from an isothermal vertical plate in open space is investigated theoretically. In contrast to conventional approaches we use neither boundary layer nor self-similarity concepts. We base on expansion of the fields of velocity and temperature in a Taylor Series in x coordinate with coefficients being functions of the vertical coordinate (y). In the minimal version of the theory we restrict ourselves by cubic approximation for both functions. The Navier-Stokes and Fourier-Kirchhoff equations that describe the phenomenon give links between coefficient functions of y that after exclusion leads to the ordinary differential equation of forth order (of the Mittag-Leffleur type). Such construction implies four boundary conditions for a solution of this equation while the links between the coefficients need two extra conditions. All the conditions are chosen on the basis of the experience usual for free convection. The choice allows us to express all the theory parameters as functions of the Rayleigh number and the temperature difference. To support the conformity of the theory we derive the Nusselt-Rayleigh numbers relation that has the traditional form. The solution in the form of velocity and temperature profiles is evaluated and illustrated for air by examples of plots of data.

Keywords: Navier-Stokes equations, Fourier-Kirchhoff equation, free convective heat transfer, analytical solution, sothermal surface, boundary layer, vertical plate

1. Introduction

Free convection heat transfer is a scientific problem that still has not been explored to the end. An analytical solution of the coupled system of Navier-Stokes, Fourier-Kirchhoff and continuity equations, even after the introduction of a number of simplifying assumptions, and using the conditions of the boundary layer theory [1, 2] is still complicated and difficult. The aim of this work is to carry out theoretical considerations of free convection heat transfer from isothermal flat surfaces that will confirm or deny the validity of the use of the boundary layer theory in an analytical calculation of this problem. Using the concept of this boundary layer with thickness δ was beneficial, from a mathematical point of view, as it allowed receipt of two additional boundary conditions ($x = \delta$, $T = T_\infty$ and $W = 0$), which greatly facilitated the calculations. However, the boundary layer proposed by Prandtl and Schlichting [1, 2] should fulfill the physical condition ($\delta \ll L$), which is not met in the case of natural convection. The preliminary studies of flow patterns visualization indicate [3] that the convective structure thickness is of the same order as the height of the heating surface ($\delta \sim L$).

Another discrepancy between the boundary layer and the free convection flux is due to the fact that in the boundary layer hypothesis the fluid movement starts at the bottom (leading) edge of the hotplate [4, 5]. This is contrary to the principle of mass conservation, as well as with the results of visualization of natural convection flow structures below, and the leading edge of the vertical plate [6, 7]. This shows that it is not only the shape of free convection flows on a vertical isothermal plate that does not meet the conditions of the boundary layer theory, but also that the boundary condition on the leading edge of the plate does not correspond to reality.

In this article we want to prove in the frame of our solution that the free convective heat transfer from the vertical plate is described by a well known Nusselt-Rayleigh relation with an exponent of $n = 1/4$ and a factor $A(\tau)$, depending on the relationship between the temperature of plate T_w and the surrounding T_∞.

$$\mathrm{Nu} = A(\tau) \cdot \mathrm{Ra}^{1/4} \quad \text{where } \tau = \frac{T_w - T_\infty}{T_\infty} \tag{1}$$

This relation is supported by numerous experiments [8–10]. Such relation is derived from the energy conservation law that contains all the elements of our model: expressions for temperature and vertical velocity as a function of parameters that fix the choice of the boundary conditions. The velocity and temperature are considered as a solution of a basic system of Navier-Stokes and Fourier-Kirchhoff equations in two space variables. This system is written in the first section of this article and its origin and model simplification are described in our recent publication [11]. The continuity equation in a differential form implies the use of at least two components of velocity. The simplified model that we build in accordance with classical works [1, 2] uses the only component of velocity and therefore the differential continuity equation cannot be applied. We however apply the continuity equation in integral form (mass conservation law).

Let us consider a two dimensional stationary flow of an incompressible fluid in a gravity field. The flow is generated by convective heat transfer from a solid plate to the fluid. The plate is isothermal and vertical. In the Cartesian coordinates x (horizontal and orthogonal to the plate), y (vertical and tangent to the plate) the Navier-Stokes (NS) system of equations takes the form [8].

It follows from the second Navier-Stokes equation that the horizontal derivative of pressure in our approximation is zero. Hence, the pressure in the first Navier-Stokes equation is eliminated by differentiating with respect to y. The consequence of this operation is growth of the equation order that we take into account in the solution construction.

The algorithm of the solution's construction is as follows. First, we expand the basic fields, velocity and temperature in the power series of horizontal variable x, its substitution into the basic system gives a system of ordinary differential equations for coefficients of the expansion as functions of the vertical variable y. As such the system is generally infinite, we should cut the expansion at some power. The form of such cutting defines the model. A minimum number of terms in the model is determined by the physical conditions on velocity and temperature profiles. The minimum number of terms is taken to be three: the parabolic part guarantees maximum velocity existence, while the third term changes the sign of the velocity derivative. The temperature behavior of the same order of approximation is defined by the basic system of equations.

The first term $(C(y)x)$ in the temperature expansion is linear in x, that accounts for the boundary condition on the plate (isothermal one). The coefficient, denoted as $C(y)$, satisfies an ordinary differential equation of the fourth order. The equation relates to the Mittag-Leffler class [12], its solution is expressed in terms of the the Mittag-Leffler special function. We, in this paper, restrict ourselves by the special case of the equation the solutions of which are expressed in terms of elementary functions. The order of the equation implies four boundary conditions at the leading $(y=0)$ and separating edge $(y=L)$ (end of the plate).The differential links of other expansion coefficients with C add two integration constants, hence the necessity for two extra conditions. The formulation of the six boundary conditions in such a model is most essential and difficult.

In the second section we present the basic system in its dimensional and dimensionless forms. By means of cross-differentiation we eliminate the pressure terms and then neglect the horizontal velocity that results in two partial differential equations for the vertical component of velocity and temperature.

In the third section we expand both the velocity and temperature fields into a Taylor series in x and derive ordinary differential equations for the coefficients by direct substitution into the basic system. The minimal (cubic) version is obtained by restricting the infinite system of equations using a special constraint. The fourth section deals with formulation of the boundary conditions and their explicit form in terms of the coefficient functions of basic fields. It is important to stress that the set of boundary conditions and conservation laws determines all the necessary parameters as the function of Grashof and Rayleigh numbers in the stationary regime under consideration.

The fifth section contains solution $C(y)$ in explicit form via elementary functions and expressions of constants of integrations in terms of boundary conditions. The solution parameters are denoted as l, p. The parameter l is the width in the x direction of the stream at the leading edge.

In the sixth section the parameters l and p are expressed in terms of Gr and Pr numbers solving the conservation laws of mass and energy equations in integral form. The equations link the field functions of temperature and velocity between leading and separating edges of the plate. As the main result of our study we have expressed the total velocity and temperature fields as a function of Rayleigh Ra and Prandtl Pr numbers.

In the last section we illustrate our results by the plot of $C(y)$ and velocity and temperature profiles for hypothetical conditions of air.

The solution of the problem allows expressing the Nusselt number $\mathrm{Nu} = \alpha \cdot L/\lambda$ via the heat transfer coefficient α in terms of Rayleigh Ra in the well known form $\alpha \sim A(\tau) \cdot \mathrm{Ra}^{1/4}$. The numerical value of $A(\tau)$ differs insignificantly from the experimental one due to the simplified version of the Mittag-Leffler equation that we use.

2. The basic equations and the solution

The basic equations of the model in dimensionless form [11] are

$$\frac{\partial}{\partial x}\left[W\frac{\partial W}{\partial y} - \mathrm{Gr}(T+1) - \left(\frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial x^2}\right)\right] = 0 \tag{2}$$

$$\mathrm{Pr}W\frac{\partial T}{\partial y} = \left(\frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial x^2}\right) \tag{3}$$

where $W(y,x)$ is the vertical component of velocity and $T(y,x)$ is the temperature in two space dimensions: horizontal x and vertical y, Gr and Pr are well known Grashof and Prandtl numbers.

The transition to the dimensional variables, marked by primes, is given by:

$$x' = xL \qquad y' = yL \qquad T' = (T - T_w)/(T_w - T_\infty) \qquad W' = WW_o \tag{4}$$

where L is the height of the vertical plate, T_w, T_∞ are the temperatures of the plate and the surrounding fluid at large distance and W_0 is a characteristic velocity proportional to viscosity ν

$$W_o = \frac{\nu}{L} \tag{5}$$

In such notations the Grashof number is expressed as:

$$\mathrm{Gr} = \frac{g\beta(T_w - T_\infty)L^3}{\nu^2} \tag{6}$$

where: g is the acceleration of gravity and $\beta = -\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_p = \frac{1}{T_\infty}$ is the volumetric expansion coefficient.

We use the standard boundary conditions on the plate:

$$T(0,y)=0 \qquad W(0,y)=0 \tag{7}$$

The first condition corresponds to a special choice of the temperature scale $T_w=0$, it means that the dimensionless temperature of the surrounding fluid is -1. The rest of boundary conditions are chosen on the basis of physical reasons and the mathematical structure of equations for the functional parameter $C(y)$ [11].

The main idea of the model is a result of polynomial approximation for both basic variables

$$\begin{aligned} W(x,y) &= \gamma x+(C_1 y+C_2)x^2-\frac{\mathrm{Gr}}{6}C(y)x^3 \\ T(x,y) &= C(y)x-\frac{1}{6}\frac{d^2C(y)}{dy^2}x^3 \end{aligned} \tag{8}$$

where the principal functional parameter $C(y)$ is a solution of the Mittag-Leffler equation [12]:

$$\frac{1}{6}\frac{d^4C(y)}{dy^4}+\mathrm{Pr}(yC_1+C_2)\frac{dC(y)}{dy}=0 \tag{9}$$

while C_1, C_2 are constant parameters of the theory that appears in the integration procedure and approximations of basic Navier-Stokes and Fourier-Kirchhoff equations.

Equation (9) is an ordinary differential equation of the fourth order and requires four boundary conditions. Apart from this the equation coefficients contain three parameters: γ, C_1, C_2 and the well known physical parameters Gr and Pr. Hence, for the explicit determination of $W(x,y)$ and $T(x,y)$ we need seven conditions that will be considered in the next section.

3. Problem formulation and conditions for temperature and velocity

Let us first fix the model by the choice of $C_1=0$. It simplifies the Equation (9): it is now the ordinary differential equation with constant coefficients the solutions of which are expressed in elementary functions. Such a model worsens the vertical profiles of temperature and velocity gradients (via the functional parameter $C(y)$) in the vicinity of the plate, but it simplifies the theory parameters calculation. Next, we introduce a boundary condition for the temperature and velocity. We will also determine here a conditional boundary of the domain under consideration in which the polynomial approximation (8) is valid. The domain is restricted by the straight vertical line of the plate, a straight line segment $y=0, x\in[0,l]$ (leading edge), the adiabatic curve that links the leading and trailing edges $y=1$, $x\in t[0,X]$ (see Figure 1).

In the dimensional form the segment under consideration has the characteristic length of the incoming flow l' which we identify with a parameter of our model. In the conventional Prandtl theory this length is taken as zero. According

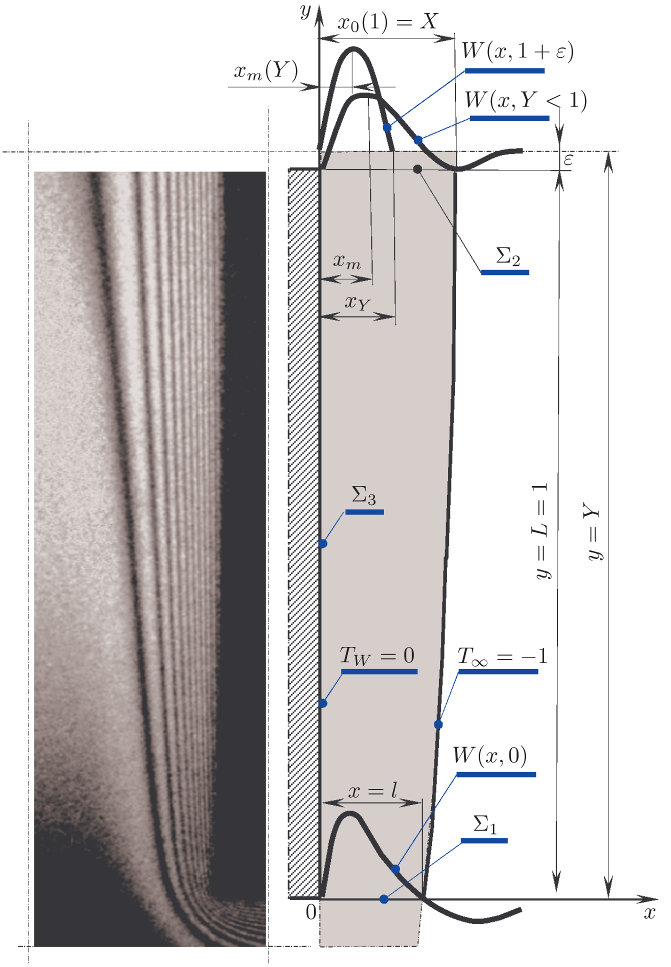

Figure 1. Result of interferometric study [13] and exemplary curves of conventional boundary layer theory (left side); coordinate system and notations for our model of two-dimensional convective fluid flow from isothenmal vertical plate (right side)

to these assumptions we choose the following boundary conditions for velocity and temperature at the level $y=0$.

$$\begin{aligned} W(l,0) &= \gamma l + C_2 l^2 - \frac{\mathrm{Gr}}{6} C(0) l^3 = 0 \\ T(l,0) &= C(0) l - \frac{1}{6} \left. \frac{d^2 C(y)}{dy^2} \right|_{y=0} l^3 = -1 \end{aligned} \tag{10}$$

That gives the first two boundary conditions.

The first boundary condition

$$C(0) = \frac{1}{l^2 \mathrm{Gr}} (6\gamma + 6lC_2) \tag{11}$$

The second boundary condition

$$\left.\frac{d^2C(y)}{dy^2}\right|_{y=0} = \frac{6(6\gamma+6lC_2+l\mathrm{Gr})}{l^4\mathrm{Gr}} \tag{12}$$

Having a polynomial approximation we choose the coefficients on the basis of the boundary conditions above the upper trailing edge $y=1$, we assume that when the fluid loses contact with the heated plate, the temperature gradient values quickly tend to zero at the level $y=1+\varepsilon=Y$, $\varepsilon \ll 1$. So, we impose the condition $\partial T/\partial x|_{x=0,y=Y}=0$ (see for example isotherms in the numerical modeling for a finite vertical plate [14]).

The third boundary condition

This yields the third boundary condition at the trailing edge:

$$C(1+\varepsilon)=0 \tag{13}$$

The velocity profile has to have two extrema: maximum at the distance $x_m(y)$ and minimum at $x_0(y)$. The extrema for the curve are defined by a derivative of $W(x,y)$ as a function of x. Hence, the relation $\frac{dW}{dx}=\gamma+2C_2x-3\frac{\mathrm{Gr}}{6}C(y)x^2=0$ indicates that for $\alpha<0$, $\beta>0$ and $\gamma>0$ we have two extrema points

$$\begin{aligned} x_m &= \frac{2C_2}{\mathrm{Gr}C(y)} - \sqrt{\frac{4C_2^2}{\left(\mathrm{Gr}C(y)\right)^2}+\frac{2\gamma}{\mathrm{Gr}C(y)}} \\ x_0 &= \frac{2C_2}{\mathrm{Gr}C(y)} + \sqrt{\frac{4C_2^2}{\left(\mathrm{Gr}C(y)\right)^2}+\frac{2\gamma}{\mathrm{Gr}C(y)}} \end{aligned} \tag{14}$$

if $2C_2^2+\gamma\mathrm{Gr}C(y)>0$.

In the exceptional case of $2C_2^2+\gamma\mathrm{Gr}C(y)=0$, the expression for the position is simplified to

$$x_m(Y)=-\frac{\gamma}{2C_2} \tag{15}$$

which is positive for $C_2<0$. The second extremum of the velocity at this level $y=Y$ does not exist now (see Figure 1).

Hence, we define $x_Y=2x_m(Y)=-\frac{\gamma}{C_2}$ as a border of the domain under consideration.

The fourth boundary condition

At the level $y=1$ at which

$$W(x_0(1),1)=0 \tag{16}$$

where $x_0(1)\equiv X$ denotes the boundary layer thickness analogue. Equation (16) is solved with respect to $C(1)$, which gives:

$$C(1)=-\frac{3}{2\gamma}\frac{C_2^2}{\mathrm{Gr}} \tag{17}$$

as a function of the problem parameters. Then, plugging (17) for the expression for the X yields

$$X = -2\frac{\gamma}{C_2} \tag{18}$$

The fifth boundary condition

The expression for the temperature (8) contains two terms, the second one is small compared to the first as will be proved after calculation of the second derivative of $C(y)$ at the level of $y = 1$. So we neglect the second term and substitute (17) and (18) into (8) equalizing it to the temperature of the surrounding fluid ($T = -1$).

$$T(x = X, 1) = C(1) \qquad X = -1(19)$$

we have:

$$C_2 = -\frac{\mathrm{Gr}}{3} \qquad X = 6\frac{\gamma}{\mathrm{Gr}} = 6a \qquad x_Y = \frac{3\gamma}{\mathrm{Gr}} = 3a \qquad C(1) = -\frac{1}{6a} \tag{20}$$

where:

$$a = \frac{\gamma}{\mathrm{Gr}} \tag{21}$$

4. Solution of the equation for C(y)

From Equation (9) after plugging C_2 (20) and taking into account $C_1 = 0$ we arrive at an equation with constant coefficients

$$\frac{1}{2}\frac{d^4C(y)}{dy^4} - \mathrm{PrGr}\frac{dC(y)}{dy} = 0 \tag{22}$$

Its solution is expressed via elementary functions (see also [15]).

$$C(y) = A_0 + A_1 \exp[sy] + \exp\left[-\frac{sy}{2}\right]\left(B_1 \cos\left[\frac{\sqrt{3}}{2}sy\right] + B_2 \sin\left[\frac{\sqrt{3}}{2}sy\right]\right) \tag{23}$$

where the main parameter of the solution

$$s = \sqrt[3]{2\mathrm{Ra}} \tag{24}$$

is expressed via Rayleigh number $\mathrm{Ra} = \mathrm{GrPr}$.

The boundary conditions: (11), (12), (13) that after substitution of C_2 and $\gamma = a\mathrm{Gr}$ from (20), (21) give:

$$\begin{aligned}
C(0) &= A_0 + A_1 + B_1 = \frac{1}{l^2\mathrm{Gr}}\left(6a\mathrm{Gr} + 6l\left(-\frac{\mathrm{Gr}}{3}\right)\right) = \frac{1}{l^2}(6a - 2l) \\
C''(0) &= \frac{1}{2}s^2\left(2A_1 - B_1 - \sqrt{3}B_2\right) = -6\left(\frac{1}{l^3} - 6\frac{a}{l^4}\right) \\
C(1+\varepsilon) &= A_0 + e^{-\frac{1}{2}s}\left(B_1 \cos\frac{1}{2}\sqrt{3}s + B_2 \sin\frac{1}{2}\sqrt{3}s\right) + A_1 e^s = 0 \\
C(1) &= A_0 + A_1 \exp[s] + \exp\left[-\frac{s}{2}\right]\left(B_1 \cos\left[\frac{s\sqrt{3}}{2}\right] + B_2 \sin\left[\frac{s\sqrt{3}}{2}\right]\right) = -\frac{1}{6a}
\end{aligned} \tag{25}$$

Solving the system approximately [15, 11] with respect to: A_0, A_1, B_1, B_2 we list the results in the table introducing the new parameter

$$p = a/l \tag{26}$$

$$\begin{aligned}
A_0 &\approx -\frac{1}{6pl} \\
A_1 &\approx \frac{1}{6ple^s} \\
B_1 &\approx \frac{1}{6pl} + \frac{1}{l}(6p-2) \\
B_2 &\approx -\frac{1}{\sqrt{3}}\left(p-\frac{1}{6}\right)\frac{72p - l^2s^2 + 6l^2ps^2}{l^3ps^2}
\end{aligned} \tag{27}$$

Plugging the results to (23) yields the explicit expression for $C(y)$.

$$\begin{aligned}
C(y) = \frac{e^{s(y-1)}-1}{6lp} + \frac{(6p-1)^2}{6lpe^{\frac{1}{2}sy}}\cos\frac{s\sqrt{3}}{2}y \\
-\frac{\sqrt{3}(6p-1)(72p - l^2s^2 + 6l^2ps^2)}{18l^3ps^2e^{\frac{1}{2}sy}}\sin\frac{s\sqrt{3}}{2}y
\end{aligned} \tag{28}$$

This expression reproduces the solution from [11] after the plugging (26) and the first approximation in ε that means a shift of the boundary condition position to point $1+\varepsilon(\mathrm{Ra})$.

5. Application of the conservation laws

As has been explained in the introduction we use the continuity equation in integral form, hence the mass conservation law in a steady state is written as the surface integral over the boundary Σ of the domain we described in the previous sections:

$$\int_{\Sigma} \rho \overrightarrow{W} \cdot \overrightarrow{n}\, dS = 0 \tag{29}$$

where: Σ is the sum of lateral surfaces Σ_i: $\Sigma = \sum_{i=1}^{4} \Sigma_i$. Σ_1 is the boundary related to the leading edge ($y=0$). The trailing edge corresponds to Σ_2 ($y=1$), while the area Σ_3 is the heating surface ($x=0$) and Σ_4 is the rest of the boundary through which the mass flux is neglected.

Along the heating surface of the vertical plate $y \in (Y_0, Y_L)$ the solution of the problem having the form $C(y, \mathrm{Ra}, \tau)$ (28) shows stability $C(y) \approx \mathrm{const}$, which conforms to the shape of isotherms also shown in Figure 3. The left side of this figure represents interferometric study results from [13] identical with [6, 7] and two examples of curves ($\delta \sim y^{1/4}$) (red lines), starting at $x=0$, $y=0$, according to the conventional boundary layer theory.

Let us bear in mind that the integral form of the law of conservation of mass (29) is formulated by the division of surface Σ into just two lower Σ_1 and upper Σ_2 boundaries. The horizontal mass flux is neglected, due to the assumption that $W_x = 0$. According to our main assumption about the two-dimensionality of the stream, we neglect the dependence of variables on z coordinate.

We also introduce the parameter l (see introduction) as the width of the incoming flow ($y = 0$, $W(l,0) = 0$) and hence we restrict our theory by $x \in [0,l]$ at the leading edge $y = 0$.

Hence, the condition of total mass conservation is as follows:

$$\int_{\Sigma_1} \rho \overrightarrow{W} \cdot \overrightarrow{n} dS = \int_{\Sigma_2} \rho \overrightarrow{W} \cdot \overrightarrow{n} dS \tag{30}$$

where the flow from below Σ_1 is approximately the product of density at temperature $T = -1$ and the velocity of the incoming flow in the interval $x \in [0,l]$.

The next boundary condition is connected with the conservation of energy in a control volume V (area S with the unit width see Figure 1), which arises from the FK Equation (3) by integration over the volume.

$$\Pr \int_V \left(W \frac{\partial T}{\partial y} \right) dV = \int_V \left(\frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial x^2} \right) dV = \int_S (\mathrm{grad} T) \overrightarrow{n} dS \tag{31}$$

The left-hand side of the energy conservation Equation (31) is transformed in a similar way by applying the identity $\mathrm{div}(T\overrightarrow{W}) = T \,\mathrm{div} \overrightarrow{W} + \overrightarrow{W} \cdot \nabla T$ and $\partial W_x / \partial x + \partial W_y / \partial y = 0$.

According to our assumptions, we are left with the flows across Σ_1, Σ_2, Σ_3 and on the basis of the homogeneity of the problem. The horizontal energy flux from the control volume V of the fluid to the surrounding fluid is neglected due to the small temperature difference on the boundary of the control volume.

5.1. Mass conservation law

The basic equation (30) contains the density

$$\rho = \rho_\infty (1 - \beta(T - T_\infty)) \tag{32}$$

where ρ_∞ is the density of a liquid in the non-disturbed area where the temperature is T_∞. The part of gravity force $g\beta(T - T_\infty)$ arises from the dependence of the extra density on temperature, β is the coefficient of thermal expansion of the fluid. In the case of gases $\beta = \frac{1}{T_\infty}$.

Let us transform the left-hand side of the Equation (30) taking into account the relations (8), $\rho = \rho_\infty$, (20), plugging the relations $\gamma = a\mathrm{Gr}, C_2 = -\frac{\mathrm{Gr}}{3}$ and $C(0) = \frac{6a - 2l}{l^2}$, for the incoming flow we have:

$$\int_{\Sigma_1} \rho \overrightarrow{W} \cdot \overrightarrow{n} dS = \frac{1}{36} \rho_\infty l^3 \mathrm{Gr}(9p - 1) \tag{33}$$

The outgoing flow through Σ_2 (r.h.s. of (30)) is taken at the end of the stability range $y = Y$,where the width of the stream is $x_{0Y} = 6a$ (20). In this expression the dependence of the density on temperature is taken into account:

$$\rho_\infty (1 - \beta(T_w - T_\infty)(T' + 1)) = \rho_\infty \left(1 - \tau \left(C(Y)x - \frac{1}{6} \left. \frac{d^2 C(y)}{dy^2} \right|_{y=Y} \cdot x^3 + 1 \right) \right) \tag{34}$$

where: $T' = (T - T_w)/(T_w - T_\infty)$, $\tau = \frac{T_w - T_\infty}{T_\infty}$.

Neglecting the small second derivative term it is expressed as:

$$\int_{\Sigma_2} \rho \overrightarrow{W} \cdot \overrightarrow{n}\, dS = -\frac{3}{5} l^3 p^3 \rho_\infty \mathrm{Gr}(3\tau - 5) \tag{35}$$

The mass conservation law (30) yields

$$45p + 324p^3\tau - 540p^3 - 5 = 0 \tag{36}$$

The solutions of the above equation, for the case $\tau = 0$ are $p = \frac{1}{6}$ and $p = -\frac{1}{3}$. The second solution is non-physical because a and l should be positive, see the definition of p (26).

It means that in conditions of absence of thermal expansion the width of the convective stream at the trailing edge: $x_Y = 6pl = l$ is equal to the one at the leading edge.

In the case of $\tau > 0$ we have three roots $p_i(\tau)$ which we estimate numerically for the temperature $T_\infty = 293.15\,\mathrm{K}$ and the temperature difference $T_w - T_\infty = 20\,\mathrm{K}$ ($\tau = \frac{20}{293.15} = 6.8224 \cdot 10^{-2}$). Its values are: $p = 0.1498$, -0.3396 and 0.1898. The first we exclude because $x_{Y,1} < l$ and the second is negative. Hence we choose the third one

$$p = 0.1898 \tag{37}$$

Generally, the dependence of the third root of equation (36) is shown on the plot in Figure 2.

For this root $x_Y = 6pl = 1.1388 \cdot l > l$

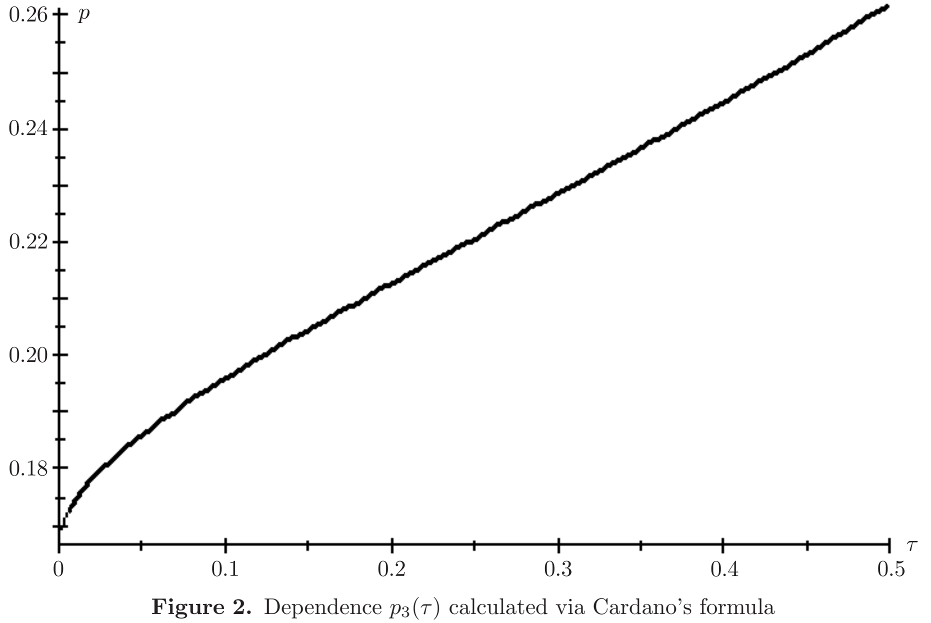

Figure 2. Dependence $p_3(\tau)$ calculated via Cardano's formula

5.2. Energy conservation law

The energy conservation equation (31) is transformed in a similar way, we are left with the heat flows across $\Sigma_1, \Sigma_2, \Sigma_3$ (Figure 1) and on the basis of the homogeneity of the problem with respect to coordinate z. To link the incoming fluid temperature $T=-1$ (from the bottom edge flow) we put $T(x,0)=-1$ at $y=0$ and the outgoing fluid (see (8)) which we take at the stability end level $y=Y$ results in:

$$\int_0^1 \frac{\partial T}{\partial x}\bigg|_{x=0} dy - \mathrm{Pr}\left(-\int_0^l (-1)\left(\gamma x + C_2 x^2 - \frac{\mathrm{Gr}}{6}C(0)x^3\right)dx \right. \\ \left. + \int_0^{x_Y} (C(Y)x)\left(\gamma x + C_2 x^2 - \frac{\mathrm{Gr}}{6}C(Y)x^3\right)dx\right) = 0 \tag{38}$$

Evaluation of integrals and substitution of the parameters ($\gamma = a\mathrm{Gr}$, $C_2 = -\frac{\mathrm{Gr}}{3}$, $x_{0Y} = 6a$, $C(0) = \frac{6a-2l}{l^2}$, $C(Y) = -\frac{1}{6a}$ and $a = pl$) yields

$$\int_0^1 \frac{\partial T}{\partial x}\bigg|_{x=0} dy = \mathrm{Ra}\frac{1}{180}l^3(-45p + 216p^3 + 5) \tag{39}$$

This relation gives the last link between the parameters p, l and Ra. In the approximation we follow (see [11], Figure 3), $\int_0^1 \frac{\partial T}{\partial x}\big|_{x=0} dy = \int_0^1 C(y)dy \approx -\frac{1}{6pl}$, that has the evident sense of a heat transfer from the plate, hence

$$\frac{1}{6pl} + \mathrm{Ra}\frac{1}{180}l^3(-45p + 216p^3 + 5) = 0 \tag{40}$$

This equation gives the expression for the width l of the leading edge flow at $y=0$

$$l = (\mathrm{Ra})^{-1/4}\sqrt[4]{\frac{30}{p(45p - 216p^3 - 5)}} \tag{41}$$

5.3. Nusselt-Rayleigh numbers relation

The main result of our investigation (39) arises from the energy conservation law. Plugging the expression for l as the function of Ra and $p(\tau)$ we arrive at the Nusselt-Rayleigh number relation

$$\mathrm{Nu} = \int_0^1 \frac{\partial T}{\partial x}\bigg|_{x=0} dy = \frac{(\mathrm{Ra})^{1/4}}{6p\cdot\sqrt[4]{\frac{30}{p(45p-216p^3-5)}}} = A(p)(\mathrm{Ra})^{1/4} \tag{42}$$

6. Analysis of solution

We have approximate formulas that define the expression for $C(y)$ (28) as the function of parameters Ra and p via the plate height L and τ.

We plot the dependence $C(y)$ for parameter values after substitution of the expression for $s = \sqrt[3]{2\mathrm{Ra}}$ (Figure 3).

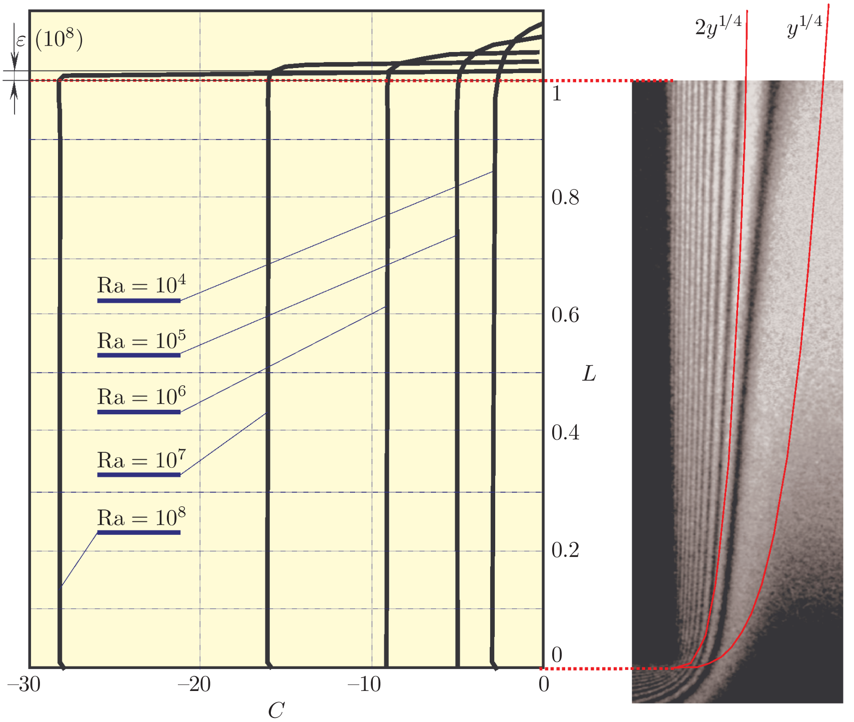

Figure 3. Dependence of C which physical interpretation connected with isothermal lines, on vertical variable y and Ra and a qualitative comparison with the real isotherm distrybution on vertical plate during free convection heat transfer [13]

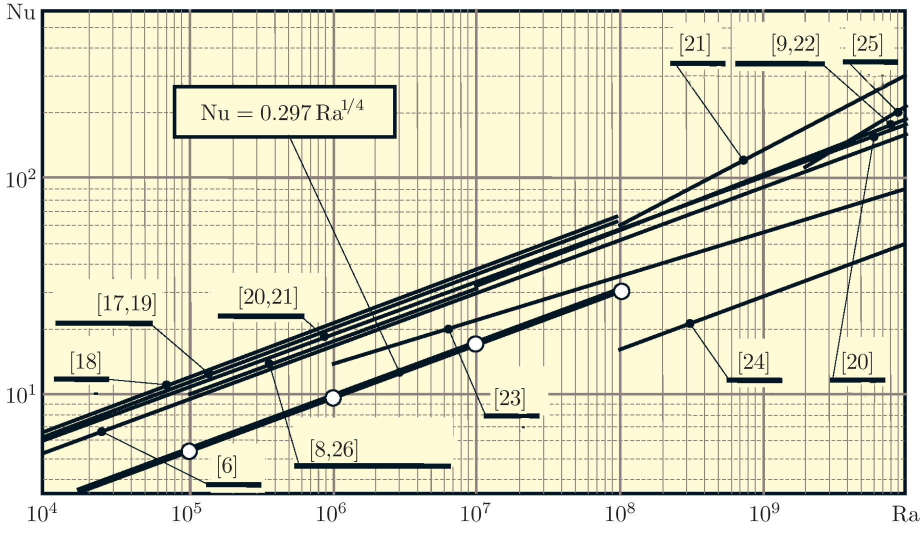

Figure 4. The comparison of the obtained Nusselt-Rayleigh relation with the literature data

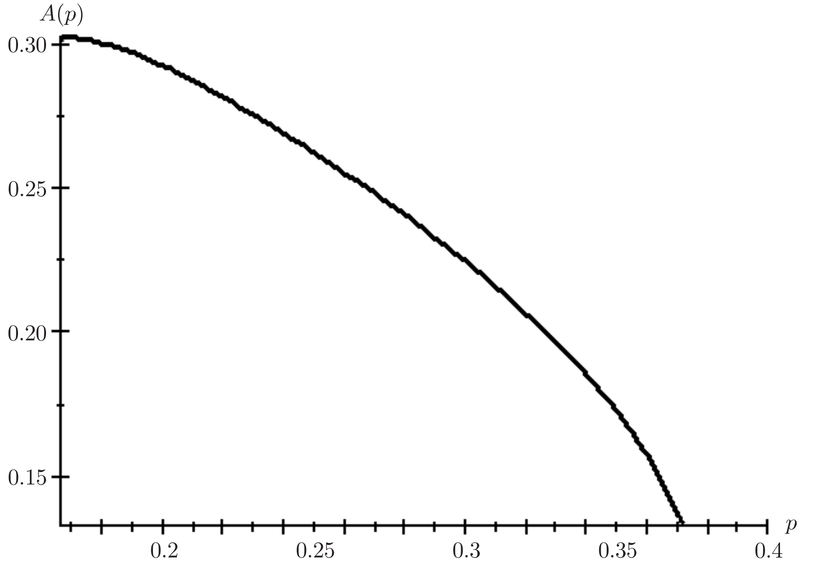

Figure 5. Dependence of $Nu-Ra$ coefficient on theory parameter $p(\tau)$ for $\tau>0$

Dependence between coefficient $A(p)$ (42) and $p(\tau)$ defined by (36) is plotted for $\tau\geq 0$ and $p\geq 0.167$ in Figure 5.

We consider the parameter l as the basic characteristic scale of the phenomenon in our theory which enters in the expressions of velocity and temperature (8).

$$W(x,y)=\frac{1}{6\mathrm{Pr}}x\mathrm{Ra}\big(-2x-x^2C(y)+6lp\big) \tag{43}$$

$$T(x,y)=C(y)x-\frac{1}{6}\frac{d^2C(y)}{dy^2}x^3 \tag{44}$$

where the functional parameter $C(y)$ is given by (28) (see Figure 3). Examples of profiles are shown in Figure 6.

The profiles are given at the height $y=0$, the profiles at nonzero y differ from Figure 6 by the terms that are small because the functional parameter $C(y)$ is almost constant at the range $(0,1)$.

We choose the following set of data for air and our setup [3]: $L=0.5\,\mathrm{m}$, $\nu=16\cdot 10^{-6}\,\mathrm{m}^2/\mathrm{s}$, $b=\frac{1}{303}\,\mathrm{K}^{-1}$, $g=10\,\mathrm{m/s}^2$, $T_w=40°\mathrm{C}$, $T_\infty=20°\mathrm{C}$, $T_w-T_\infty=20\,\mathrm{K}$, $T=T'(T_w-T_\infty)+T_w\,°\mathrm{C}$, $\mathrm{Pr}=0.71$. It gives: $\mathrm{Ra}=2.2561\cdot 10^8$, $s=767.0$, $W_o=\frac{\nu}{L}=3.2\cdot 10^{-5}\,\mathrm{m/s}$.

The width of the incoming flow at the leading edge is expressed by (41) via $p=0.1898$ as $l=2.9582\cdot\mathrm{Ra}^{-1/4}$ and the width on the level Y at the trailing edge is equal to $x_{0Y}=6pl=3.3688\cdot\mathrm{Ra}^{-1/4}$

The correspondent profiles of velocity and temperature are illustrated in Figure 7 and Figure 8.

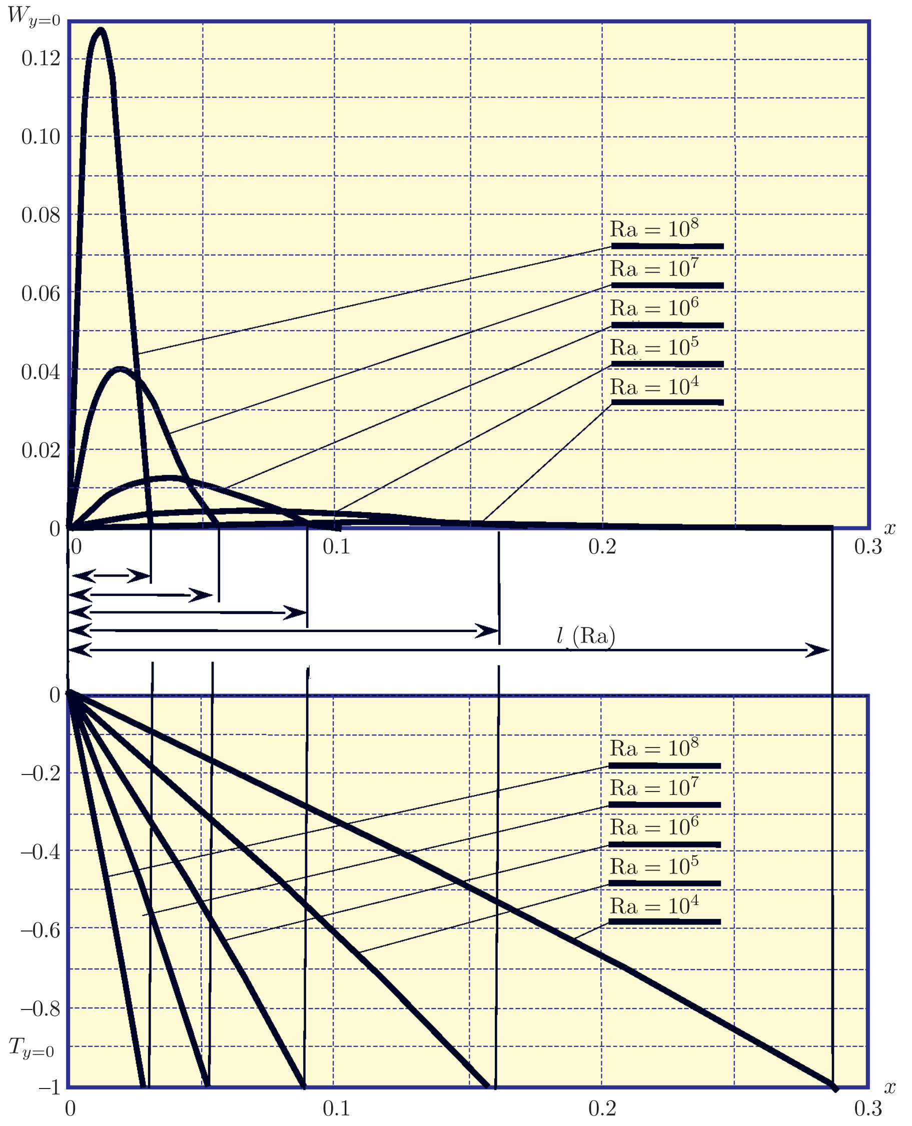

Figure 6. Distributions of velocity $W_{y=0}$ and temperature $T_{y=0}$ of natural convective fluid flow on the level $y = 0$ of the vertical plate in the function of Rayleigh number Ra

7. Conclusions

In this paper we were looking for a uniform solution of free convective heat transfer problems in terms of explicit expressions for velocity and temperature as functions of the two variables that define the heat transfer phenomena across the flow from the vertical isothermal plate. We also avoided any boundary layer or self similarity assumptions. As an important result we found the corresponding approximations and the set of the boundary conditions yielding the unique solution of the stationary boundary problem. We derived analytical formulas

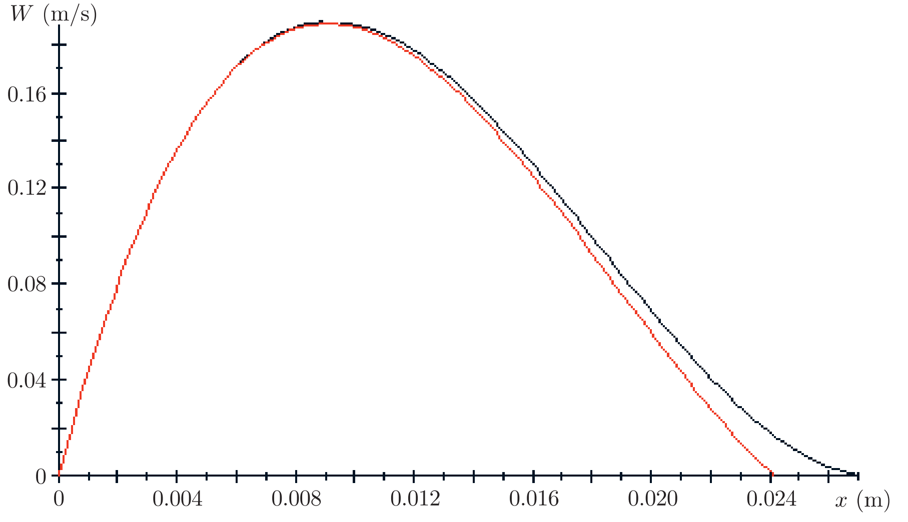

Figure 7. Profiles of velocity on levels $y=0.5$ (black) and $y=0$ (red) of vertical plate of $L=0.5\,\mathrm{m}$, for $t_\infty=293.15$, $t_w-t_\infty=20\,\mathrm{K}$ ($\tau=6.8224\cdot10^{-2}$, $\mathrm{Ra}=2.2561\cdot10^{8}$)

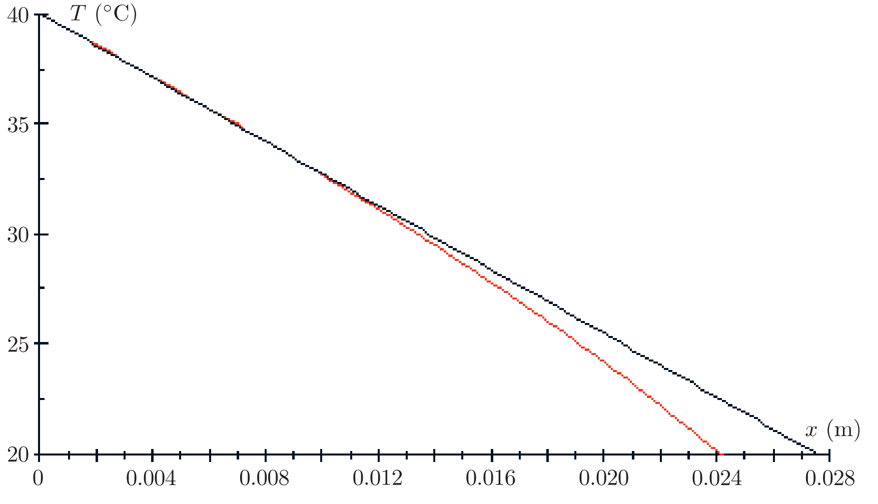

Figure 8. Profiles of temperature on level $y=0.5$ (black) and $y=0$ (red) of vertical plate of $L=0.5\,\mathrm{m}$, for $t_\infty=293.15$, $t_w-t_\infty=20\,\mathrm{K}$ ($\tau=6.8224\cdot10^{-2}$, $\mathrm{Ra}=2.2561\cdot10^{8}$)

for the main fields of velocity and temperature parameterized by the Rayleigh number Ra and the relative temperature difference τ between the plate and the surrounding flow. The results are illustrated by plots for an air example. On the basis of our rather simple solution we obtained the Nusselt-Rayleigh relation reasonably corresponding to the experimental ones (Figure 5).

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