WHEN THE SUPERSYMMETRY IS NOT ENOUGH: THE PARASUPERSYMMETRIC ALGEBRAS OF THE BOUSSINESQ EQUATIONS

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Abstract: In this article we look at a conundrum that the Boussinesq-type equations pose for mathematicians allowing a Miura-type transformation while at the same time exhibiting no trace of a supersymmetric structure. We demonstrate that this riddle should be unraveled by dropping the standard supersymmetric approach in favor of its generalization: the "*parasupersymmetry*".

Keywords: supersymmetry, Boussinesq equation

1. Introduction

Among the multitude of interesting results produced by more then a century of studies on the Korteweg-de Vries (KdV) equation there is one that has looked particularly intriguing ever since its discovery by Miura in 1968 [1]: the existence of a direct relationship between the solution of the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0 \tag{1}$$

and its counterpart, the modified KdV (mKdV)

$$v_t - 6v^2 v_x + v_{xxx} = 0 \tag{2}$$

What Miura has shown is that the functions

$$u_{\pm}(x,t) = v(x,t)^2 \pm v_x(x,t), \qquad (x,t) \in \mathbb{R}^2 \tag{3}$$

will always be the solutions of (1) provided that the function v in the KdV equation (3) is itself a solution of the mKdV equation (2). This relationship,

called the *Miura transformation* can be utilized in a number of ways: to construct new solutions of KdV (mKdV), to study the properties of these solutions, etc. Interestingly enough, despite its obvious simplicity, the exact nature of this unusual relationship remained in the dark for more than twenty years, until 1990 when Andreev and Burova revealed an underling supersymmetric structure of the KdV and mKdV systems (including the lower KdV equations) [2], with the Miura transformations being its direct offspring.

This simple and elegant reasoning had but one little flaw: it was not applicable for the other, non-KdV types of equations. This became evident in 1993, when Gesztesy, Race and Weikard found the Miura-type transformations for the Boussinesq (Bq) and the modified Boussinesq (mBq) equations [3]. It had been well established by that time that a supersymmetric (SUSY) algebra could only be realized via the even-order linear operators [4]. However, both Bq and mBq, while permitting the Lax pairs (and, hence, the corresponding differential operators), require the differential equations to be of the third-order. Such a system cannot produce a SUSY algebra, and we are once again left in an ambiguous situation of possessing a valid transformation with no reasonable explanation as to why it works.

At about the same time these developments were taking place, a number of prominent mathematical physicists came up with a new approach for the extension of a SUSY quantum formalism to the systems with a triple degeneracy of the energy spectrum [5–7]. This new approach was based on the transformations obeying not superalgebra, but a *parasuperalgebra*. From a physical point of view, unlike the SUSY which binds one bosonic and one fermionic levels of the same energy, the parasupersymmetry (PSUSY) instead binds one bosonic level with *two* parafermionic ones [8]. The reader could compare the case with the orthosymplectic superalgebra, the generators of which are combined to build the Hamiltonian in [9].

What we are planning to do in this article then is to demonstrate how these different ideas come together for the Boussinesq systems by showing that the algebraic structure of Bq and mBq systems of equations is actually the PSUSY and, furthermore, that Miura-type transformations between these systems [3] is a direct consequence of the PSUSY structure.

2. From supersymmetry to parasupersymmetry

Let us begin by reminding the reader what a supersymmetric algebra is and how it can be constructed for the particular case of a KdV equation (1). The first step here would be a construction of a corresponding Lax pair:

$$
\begin{aligned}
\frac{dL_1}{dt} &= [A_1, L_1] = A_1 L_1 - L_1 A_1 \\
L_1 &= \partial^2 + u_1(x,t) \\
A_1 &= -4\partial^3 - 6u_1(x,t)\partial - 3\partial u_1(x,t)
\end{aligned}
\tag{4}
$$

where $u(x,t) = -u_1(x,t)$ is a solution of the KdV equation, [...] denotes the Poisson bracket and we have used the notation $\partial = d/dx$.

The crux of the SUSY method is an observation that the linear differential operator L_1 can be rewritten as a product of *two* new operators:

$$L_1 = q^+ q, \qquad q = \partial + g, \qquad q^+ = \partial - g, \tag{5}$$

where the function $g(x,t)$ satisfies the following Riccati equation:

$$g_x - g^2 = u_1. \tag{6}$$

Remark 1

Note, that a simple transformation $g \to \psi$ by the formula

$$g = -\partial \ln|\psi| = -\frac{\psi_x}{\psi} \tag{7}$$

transforms the Riccati equation (6) into the Schrödinger equation on a zero background:

$$L_1 \psi = (\partial^2 + u_1)\psi = 0.$$

The operators q and q^+ are noncommutative, therefore in addition to L_1 we can also construct a new, different operator L_2 as:

$$L_2 = qq^+ = \partial^2 + u_2. \tag{8}$$

Fascinatingly, the transformation $L_1 \to L_2$ is identical to the famous Darboux transformation (DT) for the Schrödinger equation [10], which for our purposes amounts to the replacement of an old potential u_1 with a new potential $u_2 = u_1 - 2g_x$.

In order to define the superalgebra we shall introduce the supersymmetric Hamiltonian H and supergenerators Q, Q^+ as follows [11]:

$$H = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix}, \qquad Q^+ = \begin{pmatrix} 0 & q^+ \\ 0 & 0 \end{pmatrix} \tag{9}$$

where the supergenerators are nilpotent of order two,

$$Q^2 = \left(Q^+\right)^2 = 0$$

The resulting superalgebra has the form

$$\{Q,Q^+\} = QQ^+ + Q^+Q = H, \qquad [Q,H] = [Q^+,H] = 0 \tag{10}$$

Using this example as a template, let us now construct a PSUSY. In order to do it we must add to L_1 (an original Schrödinger operator) and L_2 (the

Schrödinger operator after one DT) an additional operator L_3, produced by two consecutive DTs:

$$L_1 = q^+ q \;\rightarrow\; L_2 = qq^+ = \tilde{q}^+ \tilde{q} + \mu \;\rightarrow\; L_3 = \tilde{q}\tilde{q}^+ + \mu \tag{11}$$

where $\mu = \text{const}$,

$$\tilde{q} = \partial + \tilde{g}, \qquad \tilde{q}^+ = \partial - \tilde{g} \tag{12}$$

the new "modified" function $\tilde{g}$ should satisfy a new Riccati equation

$$\tilde{g}_x - \tilde{g}^2 = u_2 - \mu \tag{13}$$

and the new operator L_3 can be rewritten as

$$L_3 = \partial^2 + u_1 - 2(g_x + \tilde{g}_x) + \mu \tag{14}$$

Acting by analogy with (9), we now define the parasuperhamiltonian and the parasupergenerators as

$$H = \begin{pmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & L_3 - \mu \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 0 & 0 \\ q & 0 & 0 \\ 0 & \tilde{q} & 0 \end{pmatrix}, \qquad Q^+ = \begin{pmatrix} 0 & q^+ & 0 \\ 0 & 0 & \tilde{q}^+ \\ 0 & 0 & 0 \end{pmatrix} \tag{15}$$

These parasupergenerators are nilpotent of order three,

$$Q^3 = \left(Q^+\right)^3 = 0 \tag{16}$$

and when $\mu = 0$ its direct products QQ^+ and Q^+Q produce the truncated versions of the parasuperhamiltonian H:

$$QQ^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & L_3 \end{pmatrix}, \qquad Q^+Q = \begin{pmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{17}$$

which means that for $\mu = 0$ they satisfy the following parasuperalgebra [7]:

$$Q^+QQ^+ = Q^+H, \qquad (Q^+)^2Q + Q(Q^+)^2 = Q^+H, \qquad [H,Q] = [H,Q^+] = 0 \tag{18}$$

Remark 2

The requirement $\mu = 0$ implies that only a special kind of DT – the *binary DT* – is permissable as a foundation for a parasuperalgebra. We have seen in Remark 1 that setting $g = -\psi_x/\psi$ produces the Schrödinger equation $L_1\psi = 0$. Similarly, the setting $\tilde{g} = -\tilde{\psi}_x/\tilde{\psi}$ will produce a spectral problem $L_2\tilde{\psi} = \mu\tilde{\psi}$. We can then rewrite (14) in a simplified form (again, assuming $\mu = 0$):

$$L_3 = \partial^2 + u_3 = \partial^2 + u_1 - 2\partial^2 \ln\left|\psi\tilde{\psi}\right| \tag{19}$$

This form of L_3 implies the existence of two interesting options for $\tilde{\psi}$. The first one is when $\tilde{\psi} = 1/\psi$. In this case $u_3 = L_3 - \partial^2 = u_1$ and $L_1 = L_3$, so it simply transforms the operator L_1 into itself. The more interesting possibility for $\tilde{\psi}$ would be

$$\tilde{\psi} = \frac{1}{\psi} \int dx \psi^2$$

in which case

$$u_3 = u_1 + 2\partial^2 \log \int dx \psi^2 \tag{20}$$

(20) is a so called binary DT (also called DT squared). It is a fundamental relationship in the positron theory [12]. For example, one can show that in order

to construct a one-positron (or one-negatron) solution of the KdV equation it will suffice to use the formula (20) for $u_1 = 0$. In other words, the positron potentials require PSUSY while the solitons need SUSY.

3. The PSUSY structure of Boussinesq integrable systems

In Section 1 we mentioned how Andreev and Burova have shown the connection between KdV and mKdV to have a SUSY structure [2]. The crucial point of the proof was the construction of a *supercharge* – a square root of a SUSY Hamiltonian, defined as a 2×2 matrix operator θ that satisfies the property[1]:

$$\theta^2 = H \tag{21}$$

and is identical to the L-operator of the mKdV hierarchy.

Let us now turn our attention to the Bq system

$$a_{1t} = (2b_1 - a_{1x})_x, \qquad b_{1t} = \left(b_{1x} - \frac{2}{3} a_{1xx} - \frac{1}{3} a_1^2 \right)_x \tag{22}$$

and to its "modified" version, provided by [3]

$$f_{1t} = f_{1xx} - 2f_1 f_{1x} - \frac{2}{3} \left(2f_1 + f_2 \right)_{xx} - \frac{2}{3} \left(f_1 f_2 - (f_1 + f_2)^2 \right)_x$$
$$f_{2t} = f_{2xx} - 2f_2 f_{2x} - \frac{2}{3} \left(f_2 - f_1 \right)_{xx} - \frac{2}{3} \left(f_1 f_2 - (f_1 + f_2)^2 \right)_x \tag{23}$$

One of the goals of the work [3] was to establish the connection between the solutions a_1, b_1 of (22) and f_1, f_2 of (23) that would be a Miura-type transformation. The authors indeed found such a transform, and it was of a very unusual sort, for its algebraic structure did not correspond to anything akin to the Andreev and Burova model. Therefore, it is our goal to resolve this puzzle by showing that (22) and (23) are indeed connected, only by the PSUSY. To demonstrate this we will begin with the Lax representation (4) for (22) where

$$L_1 = \partial^3 + a_1 \partial + b_1, \qquad A_1 = \partial^2 + \frac{2}{3} a_1 \tag{24}$$

It is a well known fact that [3],

$$L_1 = \left(\partial + f_3 \right) \left(\partial + f_2 \right) \left(\partial + f_1 \right) = q_3 q_2 q_1, \tag{25}$$

where

$$f_1 + f_2 + f_3 = 0$$
$$a_1 = \left(f_2 + 2f_1 \right)_x - f_1^2 - f_2^2 - f_1 f_2 \tag{26}$$
$$b_1 = f_{1xx} + f_1 \left(f_2 - f_1 \right)_x - f_1 f_2 \left(f_1 + f_2 \right)$$

Acting by analogy with (8) and (11) we produce a chain of operators by means of two DTs:

$$L_1 \to L_2 \to L_3 \tag{27}$$

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1. In fact, there are two operators that satisfy this property: θ and $\theta' = i\sigma_3\theta$, where σ_3 is the Pauli matrix.

or

$$q_3 q_2 q_1 \rightarrow q_1 q_3 q_2 \rightarrow q_2 q_1 q_3 \tag{28}$$

where

$$L_2 = \partial^3 + a_2 \partial + b_2, \qquad L_3 = \partial^3 + a_3 \partial + b_3 \tag{29}$$

with

$$a_2 = a_1 - 3 f_{1x}, \qquad a_3 = a_2 - 3 f_{2x} \tag{30}$$

and for all further calculations we will set $b_2 = b_3 = 0$.

Just as for the usual PSUSY we construct the first parasuperhamiltonian,

$$H_I = \begin{pmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & L_3 \end{pmatrix} \tag{31}$$

where it is important to keep in mind that, in contrast to (15), L_i ($i = 1, 2, 3$) are the linear differential operators of the third order.

We now require a parasupercharge M that satisfies the condition

$$M^3 = H \tag{32}$$

It is actually easy to verify that the operator

$$M = \begin{pmatrix} 0 & 0 & q_3 \\ q_1 & 0 & 0 \\ 0 & q_2 & 0 \end{pmatrix} \tag{33}$$

satisfies our requirement. The rest of the roots of equation (32) can be obtained from M by multiplication to the matrix

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & (\lambda_1 \lambda_2)^{-1} \end{pmatrix} \tag{34}$$

where $\lambda_{1,2}$ are the arbitrary (nonvanishing) complex numbers.

The operator (33) has actually first arose in [3]: namely, the Lax equation for the (23) is

$$\frac{dM}{dt} = [H_{II}, M] \tag{35}$$

so the parasupercharge M serves as L-operator for (23). It then must become very clear why (23) can indeed be called a "modified Bq system". The A-operator H_{II} has the form

$$H_{II} = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{pmatrix} \tag{36}$$

where $A_i = \partial^2 + \frac{2}{3} a_i$ (see (30)). Hence, (36) has the exact structure of the parasuperhamiltonian (15). To show that (36) = (15) (with $\mu = 0$) we should find such functions g and $\tilde{g}$ that

$$A_1 = (\partial - g)(\partial + g), \qquad A_2 = (\partial + g)(\partial - g) = (\partial - \tilde{g})(\partial + \tilde{g}), \qquad A_3 = (\partial + \tilde{g})(\partial - \tilde{g}) \tag{37}$$

This can be done by looking at (26); it is easy to see that

$$g = f_1 + c_1, \qquad \tilde{g} = f_2 + c_2 \tag{38}$$

with some constants c_1 and c_2.[2] After the calculations we get the nonlinear equation for f_1,

$$2(2c_2 - f_1)\big(f_{1x} + 2(c_2 - c_1)f_1\big)_x + f_{1x}^2 + \tag{39}$$
$$\big((f_1 + 2c_1 - c_2)^2 - 3(c_1^2 + c_2^2)\big)\big(3(f_1 - c_2)^2 - (c_1 + c_2)^2 - 2c_1c_2\big) = 0$$

and

$$f_2 = \frac{f_{1x} + f_1^2 - 2c_1 f_1 - c_1^2 - 2c_2^2}{2(2c_2 - f_1)}, \qquad f_3 = \frac{f_{1x} - f_1^2 + 2(2c_2 - c_1)f_1 - c_1^2 - 2c_2^2}{2(f_1 - 2c_2)} \tag{40}$$

The equation (39) can be written in a more compact form,

$$2FF_{xx} - F_x^2 + 4\alpha FF_x - \big((F - 3c_2 + 2\alpha)^2 - 3(\alpha^2 + 2c_2^2 - 2c_2\alpha)\big) \times$$
$$\big(3(F - c_2)^2 - \alpha^2 - 6c_2^2 + 6\alpha c_2\big) = 0 \tag{41}$$

where $F = 2c_2 - f_1$, $\alpha = c_2 - c_1$.

Substituting (39) and (40) into the (23) one gets

$$f_{1t} = -2c_1\big(f_1^2 - 2c_1 f_1 + 2f_1 f_2 - 4c_2 f_2 - c_1^2 - 2c_2^2\big)$$
$$f_{2t} = 2c_2\big(f_2^2 - 2c_2 f_2 + 2f_1 f_2 - 4c_1 f_1 - c_2^2 - 2c_1^2\big) \tag{42}$$

Thus, if $c_1 = c_2 = 0$ then we get the stationary solutions of the mBq equation.

The equations for the f_{1x} (and f_{2x}) are compatible with the equations for the f_{1t} (and f_{2t}) if $c_1 = c_2$ or when

$$F_t = 2c_1 F_x \tag{43}$$

Therefore, if $c_1 \neq c_2$ then $F = F(\xi)$ with $\xi = x + 2c_1 t$ and $F(\xi)$ should be a solution of (41) with substitution $F_x \rightarrow F_\xi$.

Thus, $H_{_{II}}$ (36) is parasuperhamiltonian if

$$\frac{2}{3}a_1 = f_{1x} - (f_1 + c_1)^2$$
$$\frac{2}{3}a_2 = -f_{1x} - (f_1 + c_1)^2 \tag{44}$$
$$\frac{2}{3}a_3 = 4c_1 f_1 - 2f_1 f_2 - 2f_2^2 + 2c_1^2$$

where f_1 and f_2 are defined by (39), (40).

4. The complete PSUSY algebra

As we have seen, the usual PSUSY (18) is valid for a special kind of potentials only. On the other hand, the complete PSUSY algebra shall be constructed from the parasuperhamiltonian $H_{_I}$ (31) rather than $H_{_{II}}$ (36), since $H_{_{II}}$ is connected with the auxiliary dynamical problem whereas all the information regarding the

2. Note that in this case the functions f_1 and f_2 are not arbitrary!

mBq equation is stored in H_I. Thus, it is with the aid of this operator that one can obtain the complete PSUSY algebra. Also note that in contrast to the SUSY algebra (10) the complete PSUSY algebra is defined by the superhamiltonian H_I and *three* parasupergenerators,

$$Q_1 = \begin{pmatrix} 0 & 0 & 0 \\ q_1 & 0 & 0 \\ 0 & q_2 & 0 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} 0 & 0 & q_3 \\ 0 & 0 & 0 \\ 0 & q_2 & 0 \end{pmatrix}, \qquad Q_3 = \begin{pmatrix} 0 & 0 & q_3 \\ q_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (45)$$

whereas the SUSY (10) requires only the superhamiltonian and two supergenerators Q and Q^+.

The corresponding parasuperalgebra has the form

$$M^3 = H_I, \qquad M^2 = Q_1^2 + Q_2^2 + Q_3^2, \qquad \{Q_i, Q_k\} = M^2, \qquad i \neq k$$
$$Q_1 Q_2 Q_3 = Q_2 Q_3 Q_1 = Q_3 Q_1 Q_2 = Q_1^2 = Q_2^2 = Q_3^2 = 0 \qquad (46)$$
$$Q_1 Q_3 Q_2 + Q_2 Q_1 Q_3 + Q_3 Q_2 Q_1 = 2H_I, \qquad [Q_k, H_I] = 0$$

with $i, k = 1, 2, 3$.

(46) is the para-generalization of (10). The proof of the last three equations is based on the easily checked intertwining relations

$$q_1 L_1 = L_2 q_1, \qquad q_2 L_2 = L_3 q_2, \qquad q_3 L_3 = L_1 q_3. \qquad (47)$$

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