ON REFLECTION AND REFRACTION OF PLANE ELECTROMAGNETIC WAVES AT A CONDUCTING MATTER SURFACE

SERGEY LEBLE[1,2] AND SERGEY VERESHCHAGIN[1]

[1] *Immanuele Kant Baltic Federal University*
A. Nevskogo 14, 236041 Kaliningrad, Russia

[2] *Gdansk University of Technology*
Narutowicza 11//12, 80-233 Gdansk, Poland

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Abstract: A general problem of monochrome plane electromagnetic wave reflection and refraction at the interface between the conducting medium and the dielectric is formulated and solved by symbolic computation for given incident wave polarization. The conductivity account via the Ohm law directly in the Maxwell equation leads to a complex wavenumber and hence complex amplitudes of the reflected and refracted waves. Atomic absorption is taken into account via the imaginary part of permittivity. The general formula for the time-averaged Pointing vector in the conducting media as a function of the medium parameters and the incident angle is derived and used for the refraction angle definition. The result is compared with textbooks and recent publications. The dependence of intensity as a function of the angle to the interface is determined also via the Pointing vector as a function of the incident wave and medium parameters.

Keywords: electromagnetic waves, conducting medium, reflection and refraction, Fresnelformula, X-rays refraction, PACS

1. Introduction

There is a long history of problems of electromagnetic field interaction with dielectric and conducting bodies, see *e.g.* [1]. There are different aspects of such phenomenon from diffraction (see *e.g.* [2]) to focusing in different frequency ranges up to recent X-ray [3–5] problems. Modern electrodynamics involves a whole complex plane for conductivity, permeability and permittivity coefficients, *e.g.*, studying metamaterials [6].

Textbooks on electrodynamics contain the results on reflection and refraction of monochrome electromagnetic waves in the form of generalized Snell and Fresnel formulas. (In this paper $\vec{k}$ is the wave vector of the initial light, $\vec{k}'$ is the wave vector of the reflected and $\vec{k}''$ of the refracted light) The famous Stratton book [7], basing on [8], starts with complex Snell's law equalizing $k_x = k''_x$ in adjacent media one of which is supposed to be conducting. The Snell relation follows directly from the boundary condition $k \sin\theta = k'' \sin\theta''$ and trigonometry in the case of zero conductivity and absorption, but the conductivity account automatically implies complex k'', while the zero conductance yields real $\vec{k}$. It means that such generalization of Snell's law leads to complex angles as well as the necessity of its interpretation. The next question relates to the direction of propagation of the wave which, in [7], is chosen as orthogonal to the constant phase plane. In the textbooks [7, 9] the relations between complex $\vec{k}''$ and $\vec{k}$ are plugged directly into Fresnel formulas, derived in conditions of real $\vec{k}''$ that seems to be examined. Similar definitions are used by the authors of the papers [10].

The main aim of our work is to create a symbolic computing base of the reflection and refraction theory of a plane monochrome electromagnetic wave in a general case of complex permittivity and permeability with a conductivity account. In this manuscript we base on a direct solution of the complex dispersion equation (similar to *e.g.* [9]) but we do not use the real unit vector of propagation direction determined via the phase front. We rely upon the definition of propagation direction by means of the Pointing vector (as mentioned in the book [6]), with the time averaging afterwards and systematically solve the boundary conditions for field components equalizing real and imaginary parts. We restrict ourselves to the case of the incident wave propagating in a dielectric medium. We consider a plane boundary posed in the plane $z = 0$ and a conductor at $z < 0$ with conductivity and dielectric constant similar to the cited [11]. The peculiarities of the case (wave damping) imply a modification of the problem formulation. The conductivity is introduced via a direct account of the Ohm law in the Maxwell-Ampere equation, while the dielectric permittivity is supposed to be a complex function of frequency. The magnetic properties are trivial: the magnetic permeability is considered as a constant.

2. Electromagnetic waves in a conducting medium

2.1. Electromagnetic wave equations with fixed frequency

The Maxwell equations for a medium without a space charge in the Lorentz-Heaviside unit system (c – the velocity of light in vacuum) have the form

$$\begin{aligned}
&\nabla \cdot \mathfrak{D} = 0 \\
&\nabla \times \mathfrak{E} = -\frac{1}{c}\frac{\partial \mathfrak{B}}{\partial t} \\
&\nabla \cdot \mathfrak{B} = 0 \\
&\nabla \times \mathfrak{H} = \frac{1}{c}\frac{\partial \mathfrak{D}}{\partial t} + \frac{4\pi}{c}\mathfrak{J}
\end{aligned} \tag{1}$$

where the standard set of electric $\mathfrak{E}$, $\mathfrak{D}$ and magnetic $\mathfrak{B}$, $\mathfrak{H}$ fields is used. The material integral relations (of state) are assumed as for an isotropic medium (see *e.g.* [3]). We, following the same article and physical circumstances of, *e.g.*, synchrotron radiation, also do restrict ourselves by the fixed frequency of an incident and, hence, scattered wave:

$$\mathfrak{H} = \frac{1}{\mu}\mathfrak{B} \tag{2}$$

where the dielectric permittivity ϵ, and the magnetic permeability μ are supposed to be constant. The current density for this isotropic case is given by the simple version of Ohm's law:

$$\mathfrak{J} = \sigma\mathfrak{E} \tag{3}$$

We restrict ourselves by a real σ case, implying consumption of the imaginary part in the real part of ϵ, if it exist.

Introduce the complex field amplitudes:

$$\begin{aligned}
\mathfrak{E} &= \frac{1}{2}\left[\mathbf{E}(\vec{r})\exp(i\omega t) + \mathbf{E}^*(\vec{r})\exp(-i\omega t)\right] \\
\mathfrak{D} &= \frac{1}{2}\left[\mathbf{D}(\vec{r})\exp(i\omega t) + \mathbf{D}^*(\vec{r})\exp(-i\omega t)\right] \\
\mathfrak{B} &= \frac{1}{2}\left[\mathbf{B}(\vec{r})\exp(i\omega t) + \mathbf{B}^*(\vec{r})\exp(-i\omega t)\right] \\
\mathfrak{H} &= \frac{1}{2}\left[\mathbf{H}(\vec{r})\exp(i\omega t) + \mathbf{H}^*(\vec{r})\exp(-i\omega t)\right] \\
\mathfrak{J} &= \frac{1}{2}\left[\mathbf{J}(\vec{r})\exp(i\omega t) + \mathbf{J}^*(\vec{r})\exp(-i\omega t)\right]
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
\mathbf{D}(\vec{r}) &= \epsilon\mathbf{E}(\vec{r}) \\
\mathbf{H}(\vec{r}) &= \frac{1}{\mu}\mathbf{B}(\vec{r}) \\
\mathbf{J}(\vec{r}) &= \sigma\mathbf{E}(\vec{r})
\end{aligned} \tag{5}$$

We would mark the field and medium parameters by primes for reflected waves (in the first medium $\epsilon' = \epsilon$, $\mu' = \mu$ are real) by double primes for the second medium and refracted waves. We also suppose that for the second medium $\epsilon'' = \epsilon_1'' + i\cdot\epsilon_2''$ is complex and μ'' is real. Plugging (4), (3) and (2) into the Maxwell's equations (1), the following equations for amplitudes are obtained:

$$\nabla\cdot\epsilon''\mathbf{E}'' = 0 \tag{6}$$

$$\nabla\times\mathbf{E}'' = -\frac{i\omega}{c}\mathbf{B}'' \tag{7}$$

$$\nabla\cdot\mathbf{B}'' = 0 \tag{8}$$

$$\nabla\times\mathbf{B}'' = \left(\frac{i\omega\mu''(\epsilon_1'' - i\cdot\epsilon_2'')}{c} + \frac{4\pi\mu''}{c}\sigma\right)\mathbf{E}'' \tag{9}$$

as well as the correspondent conjugate ones. It yields the Helmholtz equation (primes omitted)

$$-\Delta\mathbf{E} = k^2\mathbf{E} \tag{10}$$

where $\left(\frac{\omega^2\mu\epsilon}{c^2} - i\frac{4\pi\omega\mu}{c^2}\sigma\right)$ is denoted as k^2. We base on the solutions of (10) and links (6) within a statement of a standard scattering problem.

Next, for the generalized harmonic wave, we write

$$\begin{aligned} \mathbf{E}'' &= \mathbf{E}_0'' e^{i\vec{k}''\vec{r}} \\ \mathbf{B}'' &= \mathbf{B}_0'' e^{i\vec{k}''\vec{r}} \end{aligned} \tag{11}$$

as for both the incoming and reflected waves. Substituting (11) into (6) yields links for complex amplitudes

$$\mathbf{E}_0'' \cdot \vec{k}'' = 0 \tag{12}$$

$$\vec{k}'' \times \mathbf{E}_0'' = -\frac{\omega}{c}\mathbf{B}_0'' \tag{13}$$

$$\mathbf{B}_0'' \cdot \vec{k}'' = 0 \tag{14}$$

$$\nabla \times \mathbf{B}_0'' e^{i\vec{k}''\vec{r}} = \left(\frac{i\omega\mu''\epsilon''}{c} + \frac{4\pi\mu''}{c}\sigma \right) \mathbf{E}_0'' e^{i\vec{k}''\vec{r}} \tag{15}$$

Let us put (13) into (15)

$$i\vec{k}'' \times \mathbf{B}'' = i\vec{k}'' \times \mathbf{B}_0'' e^{i\vec{k}''\vec{r}} = -i\vec{k}'' \times \frac{c}{\omega}(\vec{k}'' \times \mathbf{E}_0'') e^{i\vec{k}''\vec{r}} = \left(\frac{i\omega\mu''\epsilon''}{c} + \frac{4\pi\mu''}{c}\sigma \right) \mathbf{E}_0'' e^{i\vec{k}\vec{r}} \tag{16}$$

or

$$-i\vec{k}'' \times \frac{c}{\omega}(\vec{k}'' \times \mathbf{E}_0'') = \left(\frac{i\omega\mu''\epsilon''}{c} + \frac{4\pi\mu''}{c}\sigma \right) \mathbf{E}_0'' \tag{17}$$

Using the $bac - cab$ formula and taking (12) into account we get the dispersion equation

$$-\frac{ic}{\omega}\left(\vec{k}''(\vec{k}'', \mathbf{E}_0'') - \mathbf{E}_0''(\vec{k}'', \vec{k}'') \right) = \left(\frac{i\omega\mu''\epsilon''}{c} + \frac{4\pi\mu''}{c}\sigma \right) \mathbf{E}_0'' \tag{18}$$

Due to (12) we have

$$\frac{ic}{\omega}\mathbf{E}_0'' \vec{k}''^2 = \left(\frac{i\omega\mu''\epsilon''}{c} + \frac{4\pi\mu''}{c}\sigma \right) \mathbf{E}_0'' \tag{19}$$

Hence,

$$k''^2 = \frac{\omega^2\mu''\epsilon''}{c^2} - \frac{4i\pi\omega\mu''}{c^2}\sigma \tag{20}$$

that we would name a dispersion relation, the explicit form of which for complex $\vec{k}''$ will be given in the next section after the account for the boundary conditions.

3. Energy density flux

To define the energy transfer direction, let us evaluate the time-averaged Pointing vector. The instant flux density in terms of the introduced complex amplitudes (4) is proportional to:

$$\mathfrak{E} \times \mathfrak{B} = \frac{1}{4}\left\{ \mathbf{E} \times \mathbf{B} \exp(2i\omega t) + \mathbf{E}^* \times \mathbf{B}^* \exp(-2i\omega t) + [\mathbf{E} \times \mathbf{B}^* + \mathbf{E}^* \times \mathbf{B}] \right\} \tag{21}$$

The averaged flux density inside the conducting medium is expressed via the Pointing vector:

$$\mathfrak{S}'' = \frac{c}{4\pi}\frac{1}{\tau}\int_0^\tau dt \mathfrak{E}'' \times \mathfrak{B}'' \tag{22}$$

Integrals of the exponential function over the time period give 0. Hence:

$$\mathfrak{S} = \frac{c}{16\pi}[\mathbf{E}\times\mathbf{B}^* + \mathbf{E}^*\times\mathbf{B}] \tag{23}$$

Plugging for the refracted wave

$$\mathbf{E}'' = \mathbf{E}_0'' e^{i\vec{k''}\vec{r}} \tag{24}$$

$$\mathbf{B}'' = \mathbf{B}_0'' e^{i\vec{k''}\vec{r}} \tag{25}$$

into (23) yields

$$\mathfrak{S}'' = \frac{c}{16\pi}\left[\mathbf{E}_0'' e^{i\vec{k''}\vec{r}} \times \mathbf{B}_0''^* e^{-i\vec{k''^*}\vec{r}} + \mathbf{E}_0''^* e^{-i\vec{k''^*}\vec{r}} \times \mathbf{B}_0'' e^{i\vec{k''}\vec{r}}\right] \tag{26}$$

Expressing B_0'' from (13)

$$\mathbf{B}_0'' = -\frac{c}{\omega}\vec{k''}\times\mathbf{E}_0'' \tag{27}$$

and extracting scalars from the vector product, we have

$$\mathfrak{S}'' = \frac{c}{16\pi}e^{i[\vec{k''}-\vec{k''}^*]\vec{r}}\left[\mathbf{E}_0''\times\mathbf{B}_0''^* + \mathbf{E}_0''^*\times\mathbf{B}_0''\right] \tag{28}$$

Plugging here (27) results in

$$\mathfrak{S}'' = -\frac{c^2}{16\pi\omega}e^{i[\vec{k''}-\vec{k''}^*]\vec{r}}\left[\mathbf{E}_0''\times\left[\vec{k''}^*\times\mathbf{E}_0''^*\right] + \mathbf{E}_0''^*\times\left[\vec{k''}\times\mathbf{E}_0''\right]\right] \tag{29}$$

Choosing such polarization that the field vector $\mathbf{E}$ lies inside the plane of the vector $\vec{k}$, hence in the two dimensional case via $bac-cab$, it writes

$$\mathfrak{S}'' = -\frac{c^2}{16\pi\omega}e^{i[\vec{k''}-\vec{k''}^*]\vec{r}}\left[\vec{k''}^*(\mathbf{E}_0''\mathbf{E}_0''^*) - \mathbf{E}_0''^*(\mathbf{E}_0''\vec{k''}^*) + \vec{k''}(\mathbf{E}_0''^*\mathbf{E}_0'') - \mathbf{E}_0''(\mathbf{E}_0''^*\vec{k''})\right] \tag{30}$$

After some algebra (see Appendix) we arrive at a compact formula for the Pointing vector for the refracted wave:

$$\mathfrak{S}'' = -\frac{c^2}{8\pi\omega}e^{2i(\Im\vec{k''},\vec{r})}\left(\Re\vec{k''}(\mathbf{E}_0'',\mathbf{E}_0''^*) + 2\left[\Im\vec{k''}\times[\Im\mathbf{E}_0''\times\Re\mathbf{E}_0'']\right]\right) \tag{31}$$

The generalized Snell law may be written as follows. For the incident wave we put

$$\left(\frac{\vec{S}}{S},\vec{i}\right) = \sin\alpha \tag{32}$$

and, for the refracted wave

$$\left(\frac{\vec{S''}}{S''},\vec{i}\right) = \sin\beta \tag{33}$$

where $\vec{S''}$ is proportional to (31). So, the ratio of $\frac{\sin\beta}{\sin\alpha}$ (relative refraction index) is defined completely by the Pointing vectors of the waves. At $z=0$ the expression (33) is written as

$$\sin\beta = \frac{\Re k_x''(\mathbf{E}_0'',\mathbf{E}_0''^*) - E_{0x}''^*(\mathbf{E}_0'',\vec{k''}^*) - E_{0x}''(\mathbf{E}_0''^*,\vec{k''})}{\left|\Re\vec{k''}(\mathbf{E}_0''\mathbf{E}_0''^*) - \mathbf{E}_0''^*(\mathbf{E}_0''\vec{k''}^*) - \mathbf{E}_0''\right.}(\mathbf{E}_0''^*\vec{k''})\Big| \tag{34}$$

4. Boundary conditions

Let us choose the boundary plane at $x=0$. A normal projection at the boundary gives us

$$\epsilon\mathbf{E}_{0n}e^{i\vec{k}\vec{r}} + \epsilon\mathbf{E}_{0n}'e^{i\vec{k'}\vec{r}} = \epsilon''\mathbf{E}_{0n}''e^{i\vec{k''}\vec{r}} \tag{35}$$

and the tangential one is equal to

$$\mathbf{E}_{0t}e^{i\vec{k}\vec{r}}+\mathbf{E}'_{0t}e^{i\vec{k}'\vec{r}}=\mathbf{E}''_{0t}e^{i\vec{k}''\vec{r}} \tag{36}$$

with similar relations for the magnetic field. As the vector $\vec{r}$ lies in the plane $x=0$, the scalar products in the exponents do not contain z-components of the wavevectors, hence

$$k''_x=k'_x=k_x. \tag{37}$$

Hence the wave vector $\vec{k''}$ should have the real component k''_x and, eventually, the complex component $k''_z=k''_{z1}+i\cdot k''_{z2}$. Therefore the sum of its squares in (20) yields

$$k''^2_x+k''^2_z=k''^2_x+k''_{z1}{}^2+2ik''_{z1}k''_{z2}-k''_{z2}{}^2=\frac{\omega^2\mu''(\epsilon''_1-i\cdot\epsilon''_2)}{c^2}-\frac{4i\pi\omega\mu''}{c^2}\sigma \tag{38}$$

Splitting the real and imaginary parts gives:

$$k''^2_x+k''_{z1}{}^2-k''_{z2}{}^2=\frac{\mu''\epsilon''_1\omega^2}{c^2} \tag{39}$$

$$2k''_{z1}k''_{z2}=-\frac{\mu''\epsilon''_2\omega^2}{c^2}-\frac{4\pi\mu''\omega}{c^2}\sigma=-\frac{\mu''\omega^2}{c^2}\left(\epsilon''_2+\frac{4\pi}{\omega}\sigma\right) \tag{40}$$

we restrict ourselves by real conductivity σ and $\epsilon''_2=0,\epsilon''_1=\epsilon''$. Solving this biquadratic system and denoting $k''_x=\frac{\omega\sin\alpha}{c}$, we get

$$k''_{z1}=\frac{\omega}{\sqrt{2}c}\sqrt{\left(\epsilon''-\sin^2(\alpha)\right)+\sqrt{\left(\epsilon''-\sin^2(\alpha)\right)^2+\left(\frac{4\pi\sigma}{\omega}\right)^2}} \tag{41}$$

$$k''_{z2}=\frac{\omega}{\sqrt{2}c}\sqrt{-\left(\epsilon''-\sin^2(\alpha)\right)+\sqrt{\left(\epsilon''-\sin^2(\alpha)\right)^2+\left(\frac{4\pi\sigma}{\omega}\right)^2}} \tag{42}$$

The general case gives similar expressions.

To evaluate the intensities of reflected and refracted waves we will use the Pointing vectors defined by (31) via vectors $\vec{k'}$, $\vec{k''}$ given by (39) and amplitudes $\vec{E'_0}$, $\vec{E''_0}$.

Using (35) and (36) together with those for the magnetic field and dropping one dimension by choosing the axes as follows,

$$\epsilon E_{0z}e^{i\vec{k}\vec{r}}+\epsilon E'_{0z}e^{i\vec{k'}\vec{r}}=\epsilon''E''_{0z}e^{i\vec{k''}\vec{r}} \tag{43}$$

$$E_{0x}e^{i\vec{k}\vec{r}}+E'_{0x}e^{i\vec{k'}\vec{r}}=E''_{0x}e^{i\vec{k''}\vec{r}} \tag{44}$$

$$B_{0z}e^{i\vec{k}\vec{r}}+B'_{0z}e^{i\vec{k'}\vec{r}}=B''_{0z}e^{i\vec{k''}\vec{r}} \tag{45}$$

$$\frac{1}{\mu}B_{0x}e^{i\vec{k}\vec{r}}+\frac{1}{\mu}B'_{0x}e^{i\vec{k'}\vec{r}}=\frac{1}{\mu''}B''_{0x}e^{i\vec{k''}\vec{r}} \tag{46}$$

that fix the only polarization It is possible only if $\vec{k}\vec{r}=\vec{k'}\vec{r}=\vec{k''}\vec{r}$, hence

$$\epsilon E_{0z}+\epsilon E'_{0z}=\epsilon''E''_{0z} \tag{47}$$

$$E_{0x}+E'_{0x}=E''_{0x} \tag{48}$$

$$B_{0z}+B'_{0z}=B''_{0z} \tag{49}$$

$$B_{0x} + B'_{0x} = \frac{\mu}{\mu''} B''_{0z} \tag{50}$$

Expressing the magnetic field components via (13) yields

$$\epsilon E_{0z} + \epsilon E'_{0z} = \epsilon'' E''_{0z} \tag{51}$$

$$E_{0x} + E'_{0x} = E''_{0x} \tag{52}$$

$$(k_z E_{0x} - k_x E_{0z}) - (k_z E'_{0x} + k_x E'_{0z}) = \frac{\mu}{\mu''}((k''_{z1} + i \cdot k''_{z2}) E''_{0x} - k''_x E''_{0z}) \tag{53}$$

Putting the first and second equation in the third one we get

$$\begin{gathered}(k_z E_{0x} - k_x E_{0z}) + (k_z E'_{0x} + k_x E'_{0z}) = \\ \frac{\mu}{\mu''}\Big((k''_{z1} + i \cdot k''_{z2})(E_{0x} + E'_{0x}) - k''_x \frac{\epsilon}{\epsilon''}(E_{0z} + E'_{0z})\Big)\end{gathered} \tag{54}$$

Using the link between the components from (6) for a reflected wave with

$$E'_{0z} = \frac{k_x}{k_z} E'_{0x} \tag{55}$$

because $k'_z = -k_z$.

Next, solving with respect to E'_{0x}, we have

$$E'_{0x} = E_{0x} \frac{\mu'' \epsilon'' k_x^2 + \mu'' \epsilon'' k_z^2 - k''_z \mu \epsilon'' k_z + \mu \epsilon k_x k_z}{\mu \epsilon k_x^2 - \mu'' \epsilon'' k_x^2 + \mu'' \epsilon'' k_z^2 + k''_z \mu \epsilon'' k_z} \tag{56}$$

Similarly

$$E''_{0x} = E_{0x} \frac{\mu \epsilon k_x^2 + 2\mu'' \epsilon'' k_z^2 + \mu \epsilon k_x k_z}{\mu \epsilon k_x^2 - \mu'' \epsilon'' k_x^2 + \mu'' \epsilon'' k_z^2 + k''_z \mu \epsilon'' k_z} \tag{57}$$

with the account for link as (55)

$$E''_{0z} = -\frac{k_x}{k''_z} E''_{0x} \tag{58}$$

$$E''_{0z} = -E_{0x} k_x \frac{\mu \epsilon k_x^2 + 2\mu'' \epsilon'' k_z^2 + \mu \epsilon k_x k_z}{k''_z(\mu \epsilon k_x^2 - \mu'' \epsilon'' k_x^2 + \mu'' \epsilon'' k_z^2 + k''_z \mu \epsilon'' k_z)} \tag{59}$$

Now we can plug the results in (34) to derive the generalized Snell law and Fresnel formulas. To evaluate the intensities of reflected and refracted waves we will use the Pointing vectors defined by (31) via vectors $\vec{k}'$, $\vec{k}''$ given by (39) and amplitudes $\vec{E}'_0$, $\vec{E}''_0$. It simplifies as

$$\mathfrak{S}'' = -\frac{c^2}{16\pi\omega} e^{-2\Im k''_z z} \left[\Re \vec{k}'' (\mathbf{E}''_0, \mathbf{E}''^*_0) - \mathbf{E}''^*_0 (\mathbf{E}''_0 \vec{k}''^*) - \mathbf{E}''_0 (\mathbf{E}''^*_0 \vec{k}'')\right] \tag{60}$$

From this it follows that the parameter $\Im k''_z$ should be positive. It defines the decrement of attenuation. The first term is equal to

$$\Re \vec{k}'' (\mathbf{E}''_0, \mathbf{E}''^*_0) = \Re \vec{k}'' (E''_{0x} E''^*_{0x} + E''_{0z} E''^*_{0z}) = \Re \vec{k}'' (1 + \frac{k_x^2}{|k''_z|^2}) E''_{0x} E''^*_{0x} \tag{61}$$

while the second one is expanding as

$$-\mathbf{E}''^*_0 (\mathbf{E}''_0, \vec{k}''^*) - \mathbf{E}''_0 (\mathbf{E}''^*_0, \vec{k}'') = 2i \mathbf{E}''^*_0 (\mathbf{E}''_0, \Im \vec{k}''_z) - 2i \mathbf{E}''_0 (\mathbf{E}''^*_0, \Im \vec{k}''_z) \tag{62}$$

For the x-component we obtain

$$2i\mathbf{E}_{0x}''^{*}\mathbf{E}_{0z}''\Im k_z'' - 2i\mathbf{E}_{0x}''\mathbf{E}_{0z}''^{*}\Im k_z'' = -2ik_x\left(\frac{k_z''^{*}}{|k_z''|^2} - \frac{k_z''}{|k_z''|^2}\right)E_{0x}''^{*}E_{0x}''\Im k_z'' = \\ -4k_x\frac{(\Im k_z'')^2}{|k_z''|^2}E_{0x}''^{*}E_{0x}'' \tag{63}$$

while for the z-component we have zero $2i\mathbf{E}_{0z}''^{*}\mathbf{E}_{0z}''\Im k_z'' - 2i\mathbf{E}_{0z}''\mathbf{E}_{0z}''^{*}\Im k_z'' = 0$. We should plug it into (34). Doing the transformations of the vector module, we have

$$\left|\left(\Re\vec{k}''(\mathbf{E}_0'', \mathbf{E}_0''^{*}) - \mathbf{E}_0''^{*}(\mathbf{E}_0'', \vec{k}''^{*}) - \mathbf{E}_0''(\mathbf{E}_0''^{*}, \vec{k}'')\right)\right| = \\ \sqrt{\left(k_x\left(1+\frac{k_x^2}{|k_z''|^2}\right) - 4k_x\frac{(\Im k_z'')^2}{|k_z''|^2}\right)^2 + \left(\Re k_z''\left(1+\frac{k_x^2}{|k_z''|^2}\right)\right)^2}\,E_{0x}''^{*}E_{0x}'' \tag{64}$$

Finally, the Snell law is written as

$$\sin\beta = \frac{-4k_x(\Im k_z'')^2 + k_x\left(|k_z''|^2 + k_x^2\right)}{\sqrt{\left(k_x\left(|k_z''|^2+k_x^2\right) - 4k_x(\Im k_z'')^2\right)^2 + \left(\Re k_z''\left(|k_z''|^2+k_x^2\right)\right)^2}} \tag{65}$$

or

$$\sin\beta = \frac{1 - \frac{4(\Im k_z'')^2}{|k_z''|^2+k_x^2}}{\sqrt{\left(1 - \frac{4(\Im k_z'')^2}{|k_z''|^2+k_x^2}\right)^2 + \left(\frac{\Re k_z''}{k_x}\right)^2}} \tag{66}$$

From (41) we evaluate $\Re k_z''$ and the imaginary part $\Im k_z''$ as functions of the matter parameters and the incident angle α.

In the case of $\Im k_z'' = 0$, $k_x = \frac{\sqrt{\epsilon}\omega\sin\alpha}{c}$ we arrive at the conventional Snell's law

$$\sin\beta = \frac{\frac{\omega}{c}\sin\alpha}{\sqrt{\frac{\epsilon''\omega^2}{c^2}}} = \frac{\sqrt{\epsilon}\sin\alpha}{\sqrt{\epsilon''}} \tag{67}$$

after account for

$$k_x^2 + k_{z1}^2{}'' = \frac{\epsilon''\omega^2}{c^2} \tag{68}$$

For the Fresnel formulas, based on explicit formulas (60), (56), (57), there is a program written by means symbolic computation.

5. Discussion

By Bergmann, [8] (see also [7]), the Snell law for a conductive medium is expressed in a rather ambiguous way (notations of the book [7] are used). The real angle is determined by

$$\sin\theta_1 = \sin\beta = \frac{\alpha_2\sin\theta_0}{\sqrt{q^2 + \alpha_2^2 sin^2\theta_0}} \tag{69}$$

where θ_0 is the angle of the incident wave, $\alpha_2 = \omega\sqrt{\mu_2\epsilon_2} = \omega/c_2$, and

$$q = \rho(\alpha_1\cos\gamma - \beta_1\sin\gamma) \tag{70}$$

is defined via COMPLEX $\cos\theta_1 = \rho e^{i\gamma}$. The index "1" marks the conducting medium, and

$$k_1^2 = \omega^2 \mu_1 \epsilon_1 + i\omega\sigma_1\mu_1 = \left(\vec{k''}, \vec{k''}\right) \tag{71}$$

the r.h.s. is written via the notation of this article with

$$k_1 = \alpha_1 + i\beta_1 \tag{72}$$

The expression via the medium parameters are given by the formula (60) of [7]

$$q^2 = \frac{1}{2}\left[\alpha_1^2 - \beta_1^2 - \alpha_2^2 \sin^2\theta_0 + \sqrt{4\alpha_1^2\beta_1^2 + (\alpha_1^2 - \beta_1^2 - \alpha_2^2\sin^2\theta_0)^2}\right] \tag{73}$$

The solution with respect to α_1, β_1 gives

$$\alpha_1 = \sqrt{\frac{\omega^2\mu_1\epsilon_1}{2} + \sqrt{\left(\frac{\omega^2\mu_1\epsilon_1}{2}\right)^2 + \left(\frac{\omega\sigma_1\mu_1}{2}\right)^2}} \tag{74}$$

$$\beta_1 = \frac{\omega\sigma_1\mu_1}{2\sqrt{\frac{\omega^2\mu_1\epsilon_1}{2} + \sqrt{\left(\frac{\omega^2\mu_1\epsilon_1}{2}\right)^2 + \left(\frac{\omega\sigma_1\mu_1}{2}\right)^2}}} \tag{75}$$

The relations between notations of (66) and (69) are extracted from (20) and (71)

$$k_1^2 = \alpha_1^2 + 2i\alpha_1\beta_1 - \beta_1^2 = k_x^2 + (k''_{1z})^2 + 2i\Re k''_z \Im k''_z - (\Im k''_z)^2 \tag{76}$$

Equalizing real and imaginary parts yields the system

$$\alpha_1^2 - \beta_1^2 = k_x^2 + (\Re k''_z)^2 - (\Im k''_z)^2 \tag{77}$$

$$\alpha_1\beta_1 = \Re k''_z \Im k''_z \tag{78}$$

The system is a biquadratic equation with respect to $\Re k''_z$, $\Im k''_z$.

For a test let us take the case

$$\mu_1 = \mu_2 = \epsilon_2 = 1 \qquad \epsilon_1 = 2 \qquad \sigma_1 = 3 \tag{79}$$

Then $\alpha_2 = \frac{\omega}{c}$.

The evaluation of $\sin\beta$ by the expressions (69) (conventional) and (66) is plotted as a function of frequency in Figure 1.

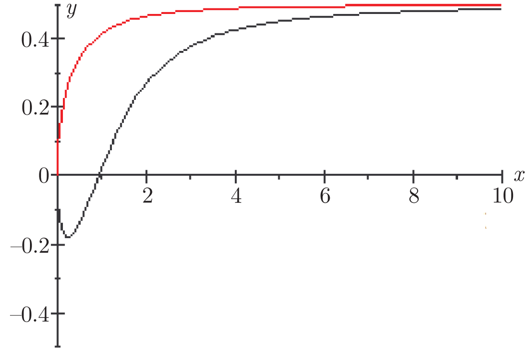

Figure 1. Evaluation of $\sin\beta$ by the expressions (69) (conventional) and (66) in function of frequency

6. Conclusion

The expression for the sine of the angle of refraction is derived via the Pointing vector as a function of the attenuation parameter $\Im k_z''$ and wave vector components of the incident wave and the conducting medium parameters ϵ'', σ, μ''. The resulting formula is compared to the conventional one obtained by means of complex angle introduction and its interpretation in terms of phase front propagation. There is a difference between the results that is plotted as a function of frequency at some hypothetical values of parameters. There are lot of possible applications of the results of this paper in X-ray optics [5] and, after slight development, in the theory of electromagnetic wave propagation in metamaterials [6].

Appendix 1

Let us rewrite additional terms of (30) into a simpler form

$$-\mathbf{E}_0''^*(\mathbf{E}_0'',\vec{k''^*})-\mathbf{E}_0''(\mathbf{E}_0''^*,\vec{k''}) \tag{80}$$

$$-\mathbf{E}_0''^*(\mathbf{E}_0'',\vec{k''}-2i\Im\vec{k''})-\mathbf{E}_0''(\mathbf{E}_0''^*,\vec{k''^*}+2i\Im\vec{k''}) \tag{81}$$

$$-\mathbf{E}_0''^*(\mathbf{E}_0'',\vec{k''})-\mathbf{E}_0''^*(\mathbf{E}_0'',-2i\Im\vec{k''})-\mathbf{E}_0''(\mathbf{E}_0''^*,\vec{k''^*})-\mathbf{E}_0''(\mathbf{E}_0''^*,2i\Im\vec{k''}) \tag{82}$$

But $(\mathbf{E}_0'',\vec{k''})=(\mathbf{E}_0''^*,\vec{k''^*})=0$, so we obtain

$$\mathbf{E}_0''^*(\mathbf{E}_0'',2i\Im\vec{k''})-\mathbf{E}_0''(\mathbf{E}_0''^*,2i\Im\vec{k''}) \tag{83}$$

$$(\mathbf{E}_0''-2i\Im\mathbf{E}_0'')(\mathbf{E}_0'',2i\Im\vec{k''})-\mathbf{E}_0''(\mathbf{E}_0''-2i\Im\mathbf{E}_0'',2i\Im\vec{k''}) \tag{84}$$

$$\mathbf{E}_0''(\mathbf{E}_0'',2i\Im\vec{k''})-2i\Im\mathbf{E}_0''(\mathbf{E}_0'',2i\Im\vec{k''})-\mathbf{E}_0''(\mathbf{E}_0'',2i\Im\vec{k''})+2\mathbf{E}_0''(i\Im\mathbf{E}_0'',2i\Im\vec{k''}) \tag{85}$$

$$4\left[\Im\mathbf{E}_0''(\mathbf{E}_0'',\Im\vec{k''})-\mathbf{E}_0''(\Im\mathbf{E}_0'',\Im\vec{k''})\right] \tag{86}$$

Using $bac-cab$ into reverse we finally get

$$4\left[\Im\vec{k''}\times[\Im\mathbf{E}_0''\times\mathbf{E}_0'']\right] \tag{87}$$

Hence the Pointing vector is:

$$\begin{aligned}\mathfrak{S}''=&-\frac{c^2}{8\pi\omega}e^{i[\vec{k''}-\vec{k''^*}]\vec{r}}\left[\Re\vec{k''^*}(\mathbf{E}_0'',\mathbf{E}_0''^*)+2\left[\Im\vec{k''}\times[\Im\mathbf{E}_0''\times\mathbf{E}_0'']\right]\right]=\\&-\frac{c^2}{8\pi\omega}e^{i[\vec{k''}-\vec{k''^*}]\vec{r}}\left[\Re\vec{k''^*}(\mathbf{E}_0'',\mathbf{E}_0''^*)+2\left[\Im\vec{k''}\times[\Im\mathbf{E}_0''\times(\Re\mathbf{E}_0''+i\Im\mathbf{E}_0'')]\right]\right]\end{aligned} \tag{88}$$

Finally

$$\mathfrak{S}''=-\frac{c^2}{8\pi\omega}e^{i[\vec{k''}-\vec{k''^*}]\vec{r}}\left[\Re\vec{k''^*}(\mathbf{E}_0'',\mathbf{E}_0''^*)+2\left[\Im\vec{k''}\times[\Im\mathbf{E}_0''\times\Re\mathbf{E}_0'']\right]\right] \tag{89}$$

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